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Structured perturbation analysis for an infinite size quasi-Toeplitz matrix equation with applications

Hyun-Min Kim[1] · Jie Meng[2]

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Abstract

This paper is concerned with the generalized Sylvester equation $AXB + CXD = E$, where A, B, C, D, E are infinite size matrices with a quasi Toeplitz structure, that is, a semi-infinite Toeplitz matrix plus an infinite size compact correction matrix. Under certain conditions, an equation of this type has a unique solution possessing the same structure as the coefficient matrix does. By separating the analysis on the Toeplitz part with that on the correction part, we provide perturbation results that cater to the particular structure in the coefficient matrices. We show that the Toeplitz part is well-conditioned if the whole problem, without considering the structure, is well-conditioned. Perturbation results that are illustrated through numerical examples are applied to equations arising in the analysis of a Markov process and the 2D Poisson problem.

Keywords Sylvester matrix equation · Quasi Toeplitz matrix · Infinite matrix · Structured perturbation analysis

Mathematics Subject Classification 15A24 · 65F35 · 15B05 · 47A10

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✉ Jie Meng
jiemeng@dm.unipi.it

Hyun-Min Kim
hyunmin@pusan.ac.kr

1 Department of Mathematics, Finance · Fishery · Manufacture Industrial Mathematics Center on Big Data, Pusan National University, Busan 46241, Republic of Korea

2 Dipartimento di Matematica, Università, di Pisa, Pisa 56127, Italy

1 Introduction

The main purpose of this work is to provide perturbation results in a way appropriate to the particular structure in the coefficient matrices of the generalized Sylvester matrix equation

$$AXB + CXD = E, \tag{1.1}$$

where A, B, C, D, E are infinite size matrices of the form $S = T(s) + E_s$, where $T(s)$ is a semi-infinite Toeplitz matrix associated with a function $s(z) = \sum_{i=-\infty}^{\infty} s_i z^i$ and $E_s = (e_{ij})$ is a compact operator on the space ℓ^∞, the usual Banach space of sequences $x = (x_i)_{i\in\mathbb{N}}$ with $\|x\|_\infty := \sup_i |x_i| < \infty$. Here the function $s(z)$ is defined on the unit circle $\mathbb{T}$ and satisfies $\sum_{i\in\mathbb{Z}} |s_i| < \infty$. Matrices of this kind are called quasi Toeplitz (QT) matrices and have been widely studied in [2–6].

Matrix equations with quasi Toeplitz coefficient matrices have been drawing attentions recently [2,4,7,17]. Among those equations, the most widely studied is the quadratic matrix equation $A_1X^2 + A_0X + A_{-1} = X$, which is encountered in a 2-dimensional Quasi-Birth-Death (QDB) stochastic process and the coefficients are infinite tridiagonal matrices with a QT structure. In [7] when applying Newton's method to this quadratic equation, a generalized Sylvester equation, which has coefficients with QT structure, is involved at each step. Numerical algorithms, which fully exploit the QT structure, for the solution of the Sylvester equation was proposed in [17].

The generalized Sylvester Eq. (1.1) in a matrix setting (finite dimensional) has been widely investigated, see [1,9,10,13,18,20] and the references therein. When the coefficients are bounded linear operators on a Hilbert space, sufficient conditions for the existence of solutions were derived in [15]. Equations of type (1.1) with infinite size Toeplitz like coefficients were first studied in [17], where the Stein, Lyapunov and Sylvester type equations were proved to have the same structure as in the coefficient matrices and efficient iteration methods, such as ADI iteration and rational Krylov methods, were proposed. Our goal here is to provide perturbation analysis to equations of type (1.1) with the emphasis on respecting structure in the coefficient matrices.

For matrix equations with QT coefficients, the Toeplitz part of the solution can be computed at a low cost by algorithms based on evaluation and interpolation. An efficient efficient way for computing the solution, when it is an infinite size quasi Toeplitz matrix, is to separate the computation into two parts, that is, the computation of the Toeplitz part and the computation of the correction part, see [7,17]. Accordingly, we investigate the perturbation analysis on the Toeplitz part and on the correction part separately. The existing results on perturbation analysis as in [11,12,19], where the analysis is restricted to equations with finite coefficients and ignores the structures of the matrices, are not appropriate for Eq. (1.1). Motivated by this, we provide perturbation results in a way appropriate to the infinite size QT matrices.

This paper is organized as follows: in Sect. 2 we recall some properties concerning semi-infinite quasi Toeplitz matrices and show that Eq. (1.1) has a unique solution with QT structure under certain conditions; in Sect. 3, we carry out a structure-preserving perturbation analysis and provide sharp perturbation bounds, while in Sect. 4 we extend the perturbation results to an equation with a more general form; in Sect. 5, we apply the perturbation results to equations encountered in solving a quadratic matrix equation arising in a Markov process and in the study of a 2D Poisson problem.

2 Existence of QT solutions

2.1 Preliminaries

We start with a review of some properties of the quasi Toeplitz (QT) matrices. Let ℓ^∞ stand for the usual Banach space of sequences $x = (x_i)_{i\in\mathbb{N}}$ with $\|x\|_\infty := \sup_i |x_i| < \infty$. Denote by $\mathcal{L}_\infty$ the set of matrices $A = (a_{i,j})$ such that $y \to Ax$, where $y = \sum_{j=1}^{\infty} a_{i,j}x_j$, defines a bounded linear operator from ℓ^∞ to ℓ^∞ [7]. Recall that $\mathcal{L}_\infty$ is a Banach space with the induced norm $\|A\|_\infty := \sup_{\|x\|_\infty=1} \|Ax\|_\infty$, which is verified to be $\|A\|_\infty = \sup_i \sum_{j=1}^{\infty} |a_{i,j}|$.

Consider the complex valued functions $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$, where $a(z)$ is defined on the unit circle $\mathbb{T}$, the Wiener algebra $\mathcal{W}$ is defined as the set $\mathcal{W} = \{a(z) = \sum_{i\in\mathbb{Z}} a_i z^i : \|a\|_w = \sum_{i\in\mathbb{Z}} |a_i| < \infty\}$. It is known that $\mathcal{W}$ is a Banach algebra, then $\|ab\|_w \le \|a\|_w\|b\|_w$ for any $a, b \in \mathcal{W}$ and the normed space is complete [8].

For $a \in \mathcal{W}$, we denote by $T(a)$ the semi-infinite Toeplitz matrix associated with the function $a(z)$, that is, $(T(a))_{i,j} = a_{j-i}$ for $i, j \in \mathbb{Z}^+$. Denote by $\mathcal{KL}_\infty$ the set of matrices $E = (e_{i,j}) \in \mathcal{L}_\infty$ such that $\lim_i v_i = 0$, where $v_i = \sum_{j=1}^{\infty} |e_{i,j}|$. It was proved in [5, Lemma 2.9 and Theorem 2.11] that $\mathcal{KL}_\infty$ is a closed subset of $\mathcal{K}(\ell^\infty)$, which is the set of compact operators from ℓ^∞ to ℓ^∞, so that $\mathcal{KL}_\infty$ is a Banach space with the induced norm $\|\cdot\|_\infty$. Consider the class

$$\mathcal{QT}_\infty = \{A = T(a) + E_a : a \in \mathcal{W},\ \ E_a \in \mathcal{KL}_\infty\}.$$

If $A \in \mathcal{QT}_\infty$, we call A a quasi Toeplitz (QT) matrix with a Toeplitz part $T(a)$ and a compact correction part E_a. The function $a(z)$ is called the symbol of A and it is invertible if and only if $a(z) \neq 0$ for all $z \in \mathbb{T}$. It follows from [8, Proposition 1.1] that $T(a) \in \mathcal{L}_\infty$, which implies that $A \in \mathcal{L}_\infty$ if $A \in \mathcal{QT}_\infty$.

For $A \in \mathcal{QT}_\infty$, there exist a unique $a \in \mathcal{W}$ and $E_a \in \mathcal{KL}_\infty$ such that $A = T(a) + E_a$. Indeed, suppose there is $a_1 \in \mathcal{W}$ and $E_{a_1} \in \mathcal{KL}_\infty$ such that $A = T(a_1) + E_{a_1}$, then we have $T(a - a_1) + (E_a - E_{a_1}) = 0$, which yields $T(a - a_1) = E_{a_1} - E_a \in \mathcal{KL}_\infty$. It follows from [8, Corollary 1.13] that $T(a - a_1) = 0$ so that $a = a_1$ and hence $E_a = E_{a_1}$.

Observe that the quasi Toeplitz matrices can be defined as $A = T(a) + E_a$, where $a \in \mathcal{W}$ and E_a is a compact operator on ℓ^p for $1 \le p \le \infty$, see [17] for example. In this paper, we only consider the case where $p = \infty$.

We refer the readers to [2–4,6] for more details about the QT matrices.

The following properties of matrices in $\mathcal{QT}_\infty$ can be found in [8].

Lemma 2.1 [8, Proposition 1.3] *Let $T(a)$ and $T(b)$ be two semi-infinite Toeplitz matrices with $a, b \in \mathcal{W}$. Then*

$$T(a)T(b) = T(ab) - H(a^-)H(b^+),$$

where $H(a^-)$ *and* $H(b^+)$ *are Hankel matrices such that* $(H(a^-))_{i,j} = (a_{-(i+j)+1})_{i,j\in\mathbb{Z}^+}$ *and* $(H(b^+))_{i,j} = (b_{i+j-1})_{i,j\in\mathbb{Z}^+}$*. Moreover,* $H(a^-)H(b^+)$ *is a compact operator on* ℓ^p *for any* $p \in [1,\infty]$.

Lemma 2.2 [8, Theorem 1.14] *If* $a \in \mathcal{W}$*, then* $\|T(a)\|_1 = \|T(a)\|_\infty = \|a\|_w$.

It can be seen from [8, Page 27] that if $a \in \mathcal{W}$ and $E_a \in \mathcal{L}_\infty$ is a compact operator, then $\|T(a)\|_\infty \le \|T(a) + E_a\|_\infty$. Together with Lemma 2.2, we have

Lemma 2.3 *If* $A = T(a) + E_a \in \mathcal{QT}$*, then* $\|a\|_w \le \|A\|_\infty$.

Denote by $\mathcal{B}(\ell^\infty)$ the set of all bounded linear operators from ℓ^∞ to itself. Recall that the spectrum of $A \in \mathcal{B}(\ell^\infty)$ is the set $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \ \textit{is not invertible}\}$. The essential spectrum of $A \in \mathcal{B}(\ell^\infty)$ is the set $\sigma_{ess}(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is not Fredholm$\}$, where $A \in \mathcal{B}(\ell^\infty)$ is said to be Fredholm if it is an invertible operator modulo compact operators. We know from [8] that œ$_{\text{ess}}(A) \subset \sigma(A)$ and œ$_{\text{ess}}(A)$ is invariant under compact perturbations.

In what follows, we denote by $\sigma(A)$ and $\sigma_{ess}(A)$, respectively, the spectrum and the essential spectrum of an operator A in the space $\mathcal{L}_\infty$ unless otherwise specified.

2.2 Existence of solutions

Note that if $X \in \mathcal{QT}_\infty$ then $X \in \mathcal{L}_\infty$, we show that under certain conditions Eq. (1.1) has a unique solution in $\mathcal{L}_\infty$ and then prove that the solution is a QT matrix if the coefficients $A, B, C, D, E \in \mathcal{QT}_\infty$.

Observe that Eq. (1.1) is solvable in $\mathcal{L}_\infty$ if and only if the operator $\mathcal{G} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ defined by $\mathcal{G}(X) = AXB + CXD$ is invertible in $\mathcal{L}_\infty$. It follows from [14, Theorem 4] that $\sigma(\mathcal{G}) \subset \sigma(A)\sigma(B) + \sigma(C)\sigma(D)$ if $AC = CA$ and $BD = DB$. Here we denote by $\sigma(A)\sigma(B)$ the set $\{\lambda_A\lambda_B : \lambda_A \in \sigma(A), \lambda_B \in \sigma(B)\}$ and by $\sigma(A)\sigma(B) + \sigma(C)\sigma(D)$ we denote the set $\{\lambda_A\lambda_B + \lambda_C\lambda_D : \lambda_A\lambda_B \in \sigma(A)\sigma(B), \lambda_C\lambda_D \in \sigma(C)\sigma(D)\}$. Hence, $\mathcal{G}$ is invertible if $AC = CA$, $BD = DB$, and $\sigma(A)\sigma(B) \cap \sigma(-C)\sigma(D) = \emptyset$. We have

Theorem 2.1 *Suppose* $A, B, C, D, E \in \mathcal{L}_\infty$ *are such that* $AC = CA$*,* $BD = DB$ *and* $\sigma(A)\sigma(B) \cap \sigma(-C)\sigma(D) = \emptyset$*, then Eq.* (1.1) *has a unique solution* $X \in \mathcal{L}_\infty$.

If $\sigma(A)\sigma(B)\cap\sigma(-C)\sigma(D) = \emptyset$ holds, then $0 \notin \sigma(A)\sigma(B)\cap\sigma(-C)\sigma(D)$, which implies $0 \notin \sigma(A)\sigma(B)$ or $0 \notin \sigma(-C)\sigma(D)$. If $0 \notin \sigma(A)\sigma(B)$, then both A and B are invertible, so are C and D if $0 \notin \sigma(-C)\sigma(D)$. In what follows, without loss of generality, A and B are assumed to be invertible. We show that the commutativity property $AC = CA$, $BD = DB$ can be removed under a stronger condition and the solution of Eq. (1.1) can be described in an explicit way.

Theorem 2.2 *Suppose* A *and* B *are invertible, let* $\alpha = \|A^{-1}\|_\infty\|B^{-1}\|_\infty$ *and* $\beta = \|C\|_\infty\|D\|_\infty$*. If* $\alpha\beta < 1$*, then Eq.* (1.1) *has a unique solution* $X \in \mathcal{L}_\infty$ *and*

$$X = \sum_{k=0}^{\infty}(-1)^k \check{A}^k \check{E} \check{B}^k, \tag{2.1}$$

where $\check{A} = A^{-1}C$, $\check{B} = DB^{-1}$ *and* $\check{E} = A^{-1}EB^{-1}$.

Proof Set $\check{A} = A^{-1}C$, $\check{B} = DB^{-1}$ and $\check{E} = A^{-1}EB^{-1}$, observe that X solves Eq. (1.1) if and only if X is a solution of equation

$$X + \check{A}X\check{B} = \check{E}. \tag{2.2}$$

It suffices to prove that Eq. (2.2) has a unique solution in $\mathcal{L}_\infty$.

Note that $\|\check{A}\|_\infty\|\check{B}\|_\infty \le \alpha\beta < 1$, then for any $\lambda \in \sigma(-\check{A})\sigma(\check{B})$, we have $|\lambda| < 1$, which implies $1 \notin \sigma(-\check{A})\sigma(\check{B})$. According to Theorem 2.1, the map $\mathcal{F} : X \to X+\check{A}X\check{B}$ is invertible in $\mathcal{L}_\infty$, this implies that Eq. (2.2) has a unique solution $X \in \mathcal{L}_\infty$.

We prove that the series in (2.1) is convergent in $\mathcal{L}_\infty$, it is then easy to check that X in (2.1) solves Eq. (2.2) and hence is a solution of Eq. (1.1). Indeed, we have

$$\begin{aligned}\left\|\sum_{k=0}^{\infty}(-1)^k\check{A}^k\check{E}\check{B}^k\right\|_\infty &\le \sum_{k=0}^{\infty}(\alpha\beta)^k\|A^{-1}\|_\infty\|B^{-1}\|_\infty\|E\|_\infty\\ &\le \frac{\alpha\|E\|_\infty}{1-\alpha\beta} < \infty.\end{aligned} \tag{2.3}$$

The proof is completed. □

In what follows, we use α and β to denote $\|A^{-1}\|_\infty\|B^{-1}\|_\infty$ and $\|C\|_\infty\|D\|_\infty$, respectively. Observe that the assumption $\alpha\beta < 1$ can be relaxed to be $\|\check{A}\|_\infty\|\check{B}\|_\infty < 1$. Under the latter one it is easy to prove that Eq. (2.2) has a unique solution in $\mathcal{L}_\infty$ by applying the same technique as in the proof of Theorem 2.2.

Remark 2.1 Since $\alpha\beta < 1$, there is $\epsilon > 0$ such that ϵ is sufficiently small and $\beta \le \alpha^{-1}-\epsilon$. This implies that $\sigma(-C)\sigma(D)$ is contained in the disk $\{z : |z| \le \alpha^{-1}-\epsilon\}$. On the other hand, it follows from $\|A^{-1}\|_\infty\|B^{-1}\|_\infty = \alpha$ that $\sigma(A^{-1})\sigma(B^{-1})$ is inside the disk $\{z : |z| \le \alpha\}$. Then $\sigma(A)\sigma(B)$ is outside the disk $\{z : |z| \le \alpha^{-1}\}$. This way, we have $\sigma(A)\sigma(B) \cap \sigma(-C)\sigma(D) = \emptyset$.

Concerning the case where the coefficients are QT matrices, if there is a disc of radius $r < 1$ such that $\sigma(\check{A}), \sigma(\check{B}) \subseteq B(0,r)$, it can be seen from [17] that Eq. (2.2) has a unique solution $X \in \mathcal{QT}_\infty$ so that Eq. (1.1) has a unique solution in the $\mathcal{QT}_\infty$ class. Here with the condition $\alpha\beta < 1$ and a slightly different technique, we prove that the unique solution of Eq. (1.1) belongs to the $\mathcal{QT}_\infty$ class.

Theorem 2.3 *For* $A, B, C, D, E \in \mathcal{QT}_\infty$, *under the assumptions in Theorem 2.2, Eq. (1.1) has a unique solution* $X \in \mathcal{QT}_\infty$.

Proof For $A, B, C, D, E \in \mathcal{QT}_\infty$ such that the conditions of Theorem 2.2 are satisfied, by relying on Lemma 2.1 and (2.1) we will prove that X can be written as $X = T(x) + E_x$, where

$$x(z) = \sum_{k=0}^{\infty}(-1)^k c(z)^k a(z)^{-k-1} e(z) b(z)^{-k-1} d(z)^k, \tag{2.4}$$

and $x(z) \in \mathcal{W}$, $E_x \in \mathcal{KL}_\infty$ so that $X \in \mathcal{QT}_\infty$. To show $x \in \mathcal{W}$, we have

$$\|x\|_w \leq \sum_{k=0}^{\infty}(\|a^{-1}\|_w\|b^{-1}\|_w\|c\|_w\|d\|_w)^k\|a^{-1}\|_w\|b^{-1}\|_w\|e\|_w. \tag{2.5}$$

In view of Lemma 2.3, we have

$$\|a^{-1}\|_w\|b^{-1}\|_w\|c\|_w\|d\|_w \leq \alpha\beta < 1 \text{ and } \|a^{-1}\|_w\|b^{-1}\|_w \leq \alpha,$$

it follows from (2.5) that

$$\|x\|_w \leq \frac{\alpha\|e\|_w}{1-\alpha\beta}, \tag{2.6}$$

from which we obtain $x \in \mathcal{W}$. It remains for us to prove that $E_x \in \mathcal{KL}_\infty$. To this end, set $X_n = \sum_{k=0}^{n}(-1)^k\check{A}^k\check{E}\check{B}^k$, we have

$$\begin{aligned}\|X_n - X\|_\infty &\leq \sum_{k=n+1}^{\infty}(\|A^{-1}\|_\infty\|B^{-1}\|_\infty\|C\|_\infty\|D\|_\infty)^k\|A^{-1}\|_\infty\|B^{-1}\|_\infty\|E\|_\infty\\ &\leq \frac{(\alpha\beta)^{n+1}\alpha\|E\|_\infty}{1-\alpha\beta},\end{aligned}$$

which implies $\lim_n \|X_n - X\|_\infty = 0$ as $\alpha\beta < 1$ and hence $(\alpha\beta)^{n+1} \to 0$ as $n \to \infty$.

It is clear that $X_n \in \mathcal{QT}_\infty$ and $X_n = T(x_n) + E_n$, where

$$x_n(z) = \sum_{k=0}^{n}(-1)^k c(z)^k a(z)^{-k-1}e(z)b(z)^{-k-1}d(z)^k \in \mathcal{W}$$

and $E_n \in \mathcal{KL}_\infty$. We prove that $\lim_{n\to\infty}\|E_n - E_x\|_\infty = 0$ so that $E_x \in \mathcal{KL}_\infty$ since $\mathcal{KL}_\infty$ is a Banach space. Note that

$$\begin{aligned}\|x - x_n\|_w &= \sum_{k=n+1}^{\infty}\|c^k a^{-k-1}eb^{-k-1}d^k\|_w\\ &\leq \sum_{k=n+1}^{\infty}(\|a^{-1}\|_w\|b^{-1}\|_w\|c\|_w\|d\|_w)^k\|a^{-1}\|_w\|b^{-1}\|_w\|e\|_w\\ &\leq \frac{(\alpha\beta)^{n+1}\alpha\|e\|_w}{1-\alpha\beta},\end{aligned} \tag{2.7}$$

which implies $\lim_{n\to\infty}\|T(x_n - x)\|_\infty = \lim_{n\to\infty}\|x_n - x\|_w = 0$.

From $E_n - E_x = X_n - X - T(x_n - x)$, we have $\|E_n - E_x\|_\infty \leq \|X_n - X\|_\infty + \|T(x_n - x)\|_\infty$, which yields $\lim_k \|E_n - E_x\|_\infty = 0$ since $\lim_n \|X_n - X\|_\infty = 0$ and $\lim_{n\to\infty}\|T(x_n - x)\|_\infty = 0$. □

The next theorem shows that the symbol of the solution $X \in \mathcal{QT}_\infty$ has an explicit expression.

Theorem 2.4 *For $A = T(a) + E_a$, $B = T(b) + E_b$, $C = T(c) + E_c$, $D = T(d) + E_d$, $E = T(e) + E_e \in \mathcal{QT}_\infty$, under the assumptions in Theorem 2.2, let $X = T(x) + E_x$ be the unique solution of Eq. (1.1), then $x(z) = (a(z)b(z) + c(z)d(z))^{-1}e(z)$.*

Proof In view of Lemma 2.1, we know

$$AXB + CXD - E = T(x(ab + cd) - e) + F,$$

where $F \in \mathcal{KL}_\infty$ is a compact correction. Since $AXB + CXD - E = 0$, we must have

$$x(z)(a(z)b(z) + c(z)d(z)) = e(z). \tag{2.8}$$

We prove that $a(z)b(z) + c(z)d(z)$ is invertible, that is, $a(z)b(z) + c(z)d(z) \neq 0$ for all $z \in \mathbb{T}$. It follows from [8, Corollary 1.10] that $a(\mathbb{T}) = \sigma_{ess}(T(a))$ and since the essential spectrum is invariant under compact perturbations, we have $a(\mathbb{T}) = \sigma_{ess}(T(a)) = \sigma_{ess}(A) \subseteq \sigma(A)$. The same results hold for matrices B, C and D, that is, $b(\mathbb{T}) \subseteq \sigma(B)$, $c(\mathbb{T}) \subseteq \sigma(C)$ and $d(\mathbb{T}) \subseteq \sigma(D)$. Suppose $a(z)b(z) + c(z)d(z) = 0$ for some $z \in \mathbb{T}$, that is, $a(z)b(z) = -c(z)d(z)$, this shows that $\sigma(A)\sigma(B) \cap \sigma(-C)\sigma(D) \neq \emptyset$, which is impossible since $\sigma(A)\sigma(B) \cap \sigma(-C)\sigma(D) = \emptyset$ in view of Remark 2.1. Hence, $a(z)b(z) + c(z)d(z) \neq 0$ for all $z \in \mathbb{T}$ so that $x(z) = (a(z)b(z) + c(z)d(z))^{-1}e(z)$. □

Remark 2.2 Since the decomposition of a QT matrix into the sum of a Toepltiz part and a correction part is unique, we know from Theorem 2.4 and (2.4) that $(a(z)b(z) + c(z)d(z))^{-1}e(z) = \sum_{k=0}^{\infty}(-1)^k c(z)^k a(z)^{-k-1} e(z) b(z)^{-k-1} d(z)^k$.

3 Structured perturbation analysis

Consider the perturbed equation

$$(A + \Delta_A)(X + \Delta_X)(B + \Delta_B) + (C + \Delta_C)(X + \Delta_X)(D + \Delta_D) = E + \Delta_E, \tag{3.1}$$

where $\Delta_A, \Delta_B, \Delta_C, \Delta_D, \Delta_E \in \mathcal{QT}_\infty$ are such that $\Delta_A = T(\delta_a) + E_{\delta_a}$, $\Delta_B = T(\delta_b) + E_{\delta_b}$, $\Delta_C = T(\delta_c) + E_{\delta_c}$, $\Delta_D = T(\delta_d) + E_{\delta_d}$, $\Delta_E = T(\delta_e) + E_{\delta_e}$, and $X + \Delta_X$ is the perturbed solution.

Suppose the assumptions in Theorem 2.2 are satisfied by the perturbed Eq. (3.1), it follows from Theorem 2.3 that $\Delta_X \in \mathcal{QT}_\infty$.

Taking the difference of Eq. (3.1) with (1.1) and omitting the second and higher order terms in the perturbations, it yields

$$A\Delta_X B + C\Delta_X D \doteq \Delta_W, \tag{3.2}$$

where $\Delta_W = \Delta_E - \Delta_A XB - AX\Delta_B - \Delta_C XD - CX\Delta_D$. Then (3.2) can be written as

$$\mathcal{G}(\Delta_X) \doteq \Delta_W,$$

where $\doteq$ means equality up to higher order terms with respect to the perturbations and $\mathcal{G} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ is defined by $\mathcal{G}(Y) = AYB + CYD$. Under the assumptions in Theorem 2.2, $\mathcal{G}$ is invertible and

$$\Delta_X \doteq \mathcal{G}^{-1}(\Delta_W) = \sum_{k=0}^{\infty} (-1)^k \check{A}^k \Delta_{\check{W}} \check{B}^k, \tag{3.3}$$

where $\check{A} = A^{-1}C$, $\check{B} = DB^{-1}$ and $\Delta_{\check{W}} = A^{-1}\Delta_W B^{-1}$. It follows from (3.3) that

$$\begin{aligned}\|\Delta_X\|_\infty &\dot{\leq} \sum_{k=0}^{\infty} (\|A^{-1}\|_\infty \|B^{-1}\|_\infty \|C\|_\infty \|D\|_\infty)^k \|A^{-1}\|_\infty \|B^{-1}\|_\infty \|\Delta_W\|_\infty \\ &\leq \frac{\alpha\|\Delta_W\|_\infty}{1-\alpha\beta}.\end{aligned}$$

On the other hand, we have from (2.3) that $\|X\|_\infty \leq \frac{\alpha\|E\|_\infty}{1-\alpha\beta}$. Set

$$\omega = \max\{\|A\|_\infty, \|B\|_\infty, \|C\|_\infty, \|D\|_\infty\}, \quad \gamma = \max\left\{1, \frac{\alpha\omega\|E\|_\infty}{1-\alpha\beta}\right\}, \tag{3.4}$$

we have

$$\|\Delta_W\|_\infty \leq \gamma(\|\Delta_A\|_\infty + \|\Delta_B\|_\infty + \|\Delta_C\|_\infty + \|\Delta_D\|_\infty + \|\Delta_E\|_\infty),$$

and

$$\|\Delta_X\|_\infty \dot{\leq} \frac{\alpha\gamma}{1-\alpha\beta}(\|\Delta_A\|_\infty + \|\Delta_B\|_\infty + \|\Delta_C\|_\infty + \|\Delta_D\|_\infty + \|\Delta_E\|_\infty). \tag{3.5}$$

Inequality (3.5) provides a perturbation bound to the whole solution without taking the QT structure into account and $\frac{\alpha\gamma}{1-\alpha\beta}$ provides an upper bound to the conditioning of the whole problem. In real applications, when the solution of an equation with QT coefficients is also a QT matrix, in some implementation issues, it is feasible to approximate the Toeplitz part and the correction part of the solution separately. For example, in [17] the Toeplitz part is computed by algorithms based on evaluation and interpolation while the correction part is computed separately by Krylov or ADI methods. Accordingly, it is convenient to separate the perturbation analysis into two parts, that is, the analysis of the Toeplitz part and the analysis of the correction part.

3.1 Toeplitz part

Write $\Delta_X = T(\delta_x) + E_{\delta_x}$, in view of Theorem 2.4, $x + \delta_x$ is the unique solution of the equation

$$((a+\delta_a)(b+\delta_b) + (c+\delta_c)(d+\delta_d))(x+\delta_x) = e + \delta_e. \tag{3.6}$$

Taking the difference of Eq. (3.6) with (2.8) yields

$$(ab+cd)\delta_x \doteq \delta_e - xb\delta_a - ax\delta_b - xd\delta_c - cx\delta_d.$$

According to Theorem 2.4, $a(z)b(z) + c(z)d(z)$ is invertible, we have

$$\delta_x \doteq \frac{\delta_e - xb\delta_a - ax\delta_b - xd\delta_c - cx\delta_d}{ab+cd}, \tag{3.7}$$

from which we obtain

$$\begin{aligned}\|\delta_x\|_w \dot{\leq} \|(ab+cd)^{-1}\|_w \big(\|\delta_e\|_w + (\|\delta_a\|_w\|b\|_w \\ +\|\delta_b\|_w\|a\|_w + \|\delta_c\|_w\|d\|_w + \|\delta_d\|_w\|c\|_w)\|x\|_w\big).\end{aligned} \tag{3.8}$$

Observe that $\|x\|_w = \|(ab+cd)^{-1}e\|_w$ in view of Theorem 2.4. Together with (3.8), we have

$$\|\delta_x\|_w \dot{\leq} \mu\|(ab+cd)^{-1}\|_w(\|\delta_a\|_w + \|\delta_b\|_w + \|\delta_c\|_w + \|\delta_d\|_w + \|\delta_e\|_w), \tag{3.9}$$

where

$$\mu = \max\{1, \nu\|(ab+cd)^{-1}e\|_w\}, \;\; \nu = \max\{\|a\|_w, \|b\|_w, \|c\|_w, \|d\|_w\}. \tag{3.10}$$

It turns out from (3.9) that $\mu\|(ab+cd)^{-1}\|_w$ provides an upper bound to the conditioning of the Toeplitz part. In view of Remark 2.2, we have $\|(ab+cd)^{-1}e\|_w \leq \frac{\alpha\|e\|_w}{1-\alpha\beta}$. Then, according to Lemma 2.3, (3.4) and (3.10), we deduce that $\mu \leq \gamma$. On the other hand, it can be easily checked that $(ab+cd)^{-1} = \sum_{k=0}^{\infty}(ab)^{-k-1}(cd)^k$ so that $\|(ab+cd)^{-1}\|_w \leq \frac{\alpha}{1-\alpha\beta}$, from which we have $\mu\|(ab+cd)^{-1}\|_w \leq \frac{\alpha\mu}{1-\alpha\beta} \leq \frac{\alpha\gamma}{1-\alpha\beta}$. Recall that $\frac{\alpha\gamma}{1-\alpha\beta}$ is an upper bound to the conditioning of the whole problem, it is interesting to observe that if $\frac{\alpha\gamma}{1-\alpha\beta}$ is a small number, that is, the whole problem is well-conditioned, then the Toeplitz part is well-conditioned.

3.2 Correction part

Concerning the perturbation bound of the correction part, it follows from $E_{\delta_x} = \Delta_X - T(\delta_x)$ and Lemma 2.2 that $\|E_{\delta_x}\|_\infty \leq \|\Delta_X\|_\infty + \|\delta_x\|_w$. Moreover, in view of

Lemma 2.3 and (3.9), we have

$$\|\delta_x\|_w \dot{\leq} \mu \|(ab+cd)^{-1}\|_w (\|\Delta_A\|_\infty + \|\Delta_B\|_\infty + \|\Delta_C\|_\infty + \|\Delta_D\|_\infty + \|\Delta_E\|_\infty).$$

Together with (3.5), we derive an upper bound for E_{δ_x}

$$\begin{aligned} \|E_{\delta_x}\|_\infty \dot{\leq} &\frac{\alpha\gamma}{1-\alpha\beta}(\|\Delta_A\|_\infty + \|\Delta_B\|_\infty + \|\Delta_C\|_\infty + \|\Delta_D\|_\infty + \|\Delta_E\|_\infty) + \|\delta_x\|_w \\ &\leq \kappa(\|\Delta_A\|_\infty + \|\Delta_B\|_\infty + \|\Delta_C\|_\infty + \|\Delta_D\|_\infty + \|\Delta_E\|_\infty), \end{aligned} \tag{3.11}$$

where $\kappa = \frac{\alpha\gamma}{1-\alpha\beta} + \mu\|(ab+cd)^{-1}\|_w$, μ and γ are defined in (3.10) and (3.4) respectively.

Inequality (3.11) provides an upper perturbation bound to the correction part. It is interesting to observe from (3.11) that the perturbations on the Toeplitz part of the coefficients may make a difference to the perturbation results on the correction part. This coincides with the fact that, when the computations are separated, the Toeplitz part that has been previously computed appears in the computation of the correction part, see [17] for example.

Set

$$\delta = \max\left\{\frac{\|\delta_a\|_w}{\|E_{\delta_a}\|_\infty}, \frac{\|\delta_b\|_w}{\|E_{\delta_b}\|_\infty}, \frac{\|\delta_c\|_w}{\|E_{\delta_c}\|_\infty}, \frac{\|\delta_d\|_w}{\|E_{\delta_d}\|_\infty}, \frac{\|\delta_e\|_w}{\|E_{\delta_e}\|_\infty}\right\}. \tag{3.12}$$

If $\delta < \infty$, it follows from (3.11) that

$$\begin{aligned} \|E_{\delta_x}\|_\infty &\dot{\leq} \kappa(1+\delta)(\|E_{\delta_a}\|_\infty + \|E_{\delta_b}\|_\infty + \|E_{\delta_c}\|_\infty + \|E_{\delta_d}\|_\infty + \|E_{\delta_e}\|_\infty) \\ &\leq \frac{2\alpha\gamma(1+\delta)}{1-\alpha\beta}(\|E_{\delta_a}\|_\infty + \|E_{\delta_b}\|_\infty + \|E_{\delta_c}\|_\infty + \|E_{\delta_d}\|_\infty + \|E_{\delta_e}\|_\infty), \end{aligned} \tag{3.13}$$

where the last inequality follows from $\|(ab+cd)^{-1}\|_w \leq \frac{\alpha}{1-\alpha\beta}$ and $\mu \leq \gamma$ so that $\kappa \leq \frac{\alpha(\mu+\gamma)}{1-\alpha\beta} \leq \frac{2\alpha\gamma}{1-\alpha\beta}$.

If $\delta < \infty$, we can see from (3.13) that $\frac{2\alpha\gamma(1+\delta)}{1-\alpha\beta}$ may provide some insights to the conditioning of the correction part. When δ is a small number, the perturbations on the Toeplitz part make a very small difference to the perturbations on the correction part and in this case the correction part is well-conditioned if the whole problem is well-conditioned.

3.3 A special case

We consider a special case of Eq. (1.1), which is

$$AX + XA = E, \tag{3.14}$$

where $A = T(a) + E_a \in \mathcal{QT}_\infty$ and $E = T(e) + E_e \in \mathcal{QT}_\infty$. An equation of this type arises in the study of 2D Poisson problem on an unbounded domain, see [17, Example 5.1]. The assumptions in Theorem 2.2 are not satisfied by Eq. (3.14) since $\|A^{-1}\|_\infty \|A\|_\infty \geq 1$ for all invertible A. Hence, the structured perturbation results may not apply.

Let

$$\mathcal{S} = \{\lambda \in \mathbb{C} : \|(A + \lambda I)^{-1}\|_\infty \|A - \lambda I\|_\infty < 1\}.$$

If $\mathcal{S} \neq \emptyset$, then Eq. (3.14) has a unique solution X which satisfies $X \in \mathcal{L}_\infty \cap \mathcal{QT}_\infty$. Indeed, consider the equation

$$(A + \lambda I)X + X(A - \lambda I) = E. \tag{3.15}$$

Suppose $\mathcal{S}$ is nonempty, that is, there is $\lambda \in \mathcal{S}$ such that $\|(A + \lambda I)^{-1}\|_\infty \|A - \lambda I\|_\infty < 1$, by Remark 2.1, we have $\sigma(A + \lambda I) \cap \sigma(\lambda I - A) = \emptyset$, which implies $\sigma(A) \cap \sigma(-A) = \emptyset$, so that Eq. (3.14) has a unique solution $X \in \mathcal{L}_\infty$ in view of Theorem 2.1. On the other hand, observe that X also solves Eq. (3.15) so that $X \in \mathcal{QT}_\infty$, since it follows from Theorem 2.3 that Eq. (3.15) has a unique solution $X_\lambda \in \mathcal{QT}_\infty$.

We have the following properties for the set $\mathcal{S}$.

Lemma 3.1 *If $\mathcal{S}$ is nonempty, then $a(z) \neq 0$ for all $z \in \mathbb{T}$, where $a(z)$ is the symbol of matrix A.*

Proof Suppose by contradiction that $a(z_0) = 0$ for some $z_0 \in \mathbb{T}$. Let $\lambda \in \mathcal{S}$ be such that $\|(A + \lambda I)^{-1}\|_\infty \|A - \lambda I\|_\infty < 1$. Observe that $a(z) + \lambda$ is invertible on the unit circle $\mathbb{T}$ since $a(\mathbb{T}) + \lambda = \sigma_{ess}(A + \lambda I) \subset \sigma(A + \lambda I)$ and $0 \notin \sigma(A + \lambda I)$ so that $a(z) + \lambda \neq 0$ for all $z \in \mathbb{T}$. In view of Lemma 2.3, we have $\|(a + \lambda)^{-1}\|_w \|a - \lambda\|_w \leq \|(A + \lambda I)^{-1}\|_\infty \|A - \lambda I\|_\infty < 1$, which is impossible since $\|(a + \lambda)^{-1}\|_w \|a - \lambda\|_w \geq \|(a + \lambda)^{-1}\|_\infty \|a - \lambda\|_\infty \geq |(a(z_0) + \lambda)^{-1}||a(z_0) - \lambda| = 1$. □

In what follows, we suppose that the set $\mathcal{S}$ is nonempty so that Eq. (3.14) has a unique solution in the $\mathcal{QT}_\infty$ class. Concerning the perturbed Eq. of (3.14), we have

$$(A + \Delta_A)(X + \Delta_X) + (X + \Delta_X)(A + \Delta_A) = E + \Delta_E, \tag{3.16}$$

where $\Delta_A = T(\delta_a) + E_{\delta_a} \in \mathcal{QT}_\infty$ and $\Delta_E = T(\delta_e) + E_{\delta_e} \in \mathcal{QT}_\infty$.

Suppose

$$\|(A + \Delta_A + \lambda I)^{-1}\|_\infty \|A + \Delta_A - \lambda I\|_\infty < 1 \tag{3.17}$$

for some $\lambda \in \mathcal{S}$. This assumption is shown to be feasible by numerical experiment. Then, the perturbed Eq. (3.16) has a unique solution $X + \Delta_X \in \mathcal{QT}_\infty$.

Taking difference of Eq. (3.16) with (3.14) and omitting the second and higher order terms in perturbations, we have

$$A\Delta_X + \Delta_X A \doteq \Delta_E - X\Delta_A - \Delta_A X. \tag{3.18}$$

Now we apply the perturbation results to Eq. (3.14). For the Toeplitz part, since $a(z)$ is invertible in view of Lemma 3.1, we have

$$\delta_x = \frac{1}{2}a^{-1}(\delta_e - 2x\delta_a),$$

from which we obtain

$$\|\delta_x\|_w \le \frac{1}{2}\|a^{-1}\|_w(\|\delta_e\|_w + 2\|x\|_w\|\delta_a\|_w). \tag{3.19}$$

On the other hand, it is easy to get $x(z) = \frac{1}{2}a(z)^{-1}e(z)$ so that $\|x\|_w \le \frac{1}{2}\|a^{-1}e\|_w$. Let $\xi = \max\{1, \|a^{-1}e\|_w\}$, it follows from (3.19) that

$$\|\delta_x\|_w \le \frac{\xi\|a^{-1}\|_w}{2}(\|\delta_e\|_w + \|\delta_a\|_w). \tag{3.20}$$

Concerning the correction part, observe that Eq. (3.18) is equivalent to

$$(A + \lambda I)\Delta_X + \Delta_X(A - \lambda I) \doteq \Delta_E - X\Delta_A - \Delta_A X \tag{3.21}$$

for some $\lambda \in \mathcal{S}$ such that (3.17) is satisfied. Set $\tilde{\alpha} = \|(A + \lambda I)^{-1}\|_\infty$ and $\tilde{\beta} = \|A - \lambda I\|_\infty$, then $\tilde{\alpha}\tilde{\beta} < 1$. We obtain from (3.21) that

$$\Delta_X = \sum_{k=0}^{\infty}(-1)^k(A + \lambda I)^{-k-1}(\Delta_E - X\Delta_A - \Delta_A X)(A - \lambda I)^k,$$

which yields

$$\begin{aligned}\|\Delta_X\|_\infty &\le \frac{\tilde{\alpha}}{1 - \tilde{\alpha}\tilde{\beta}}(\|\Delta_E\|_\infty + 2\|X\|_\infty\|\Delta_A\|_\infty)\\ &\le \frac{\tilde{\alpha}\tilde{\tau}}{1 - \tilde{\alpha}\tilde{\beta}}(\|\Delta_E\|_\infty + \|\Delta_A\|_\infty),\end{aligned} \tag{3.22}$$

where $\tilde{\tau} = \max\{1, 2\|X\|_\infty\}$. It follows

$$\begin{aligned}\|E_{\delta_x}\|_\infty &\le \frac{\tilde{\alpha}\tilde{\tau}}{1 - \tilde{\alpha}\tilde{\beta}}(\|\Delta_E\|_\infty + \|\Delta_A\|_\infty) + \|\delta_x\|_w\\ &\le \left(\frac{\tilde{\alpha}\tilde{\tau}}{1 - \tilde{\alpha}\tilde{\beta}} + \frac{\xi\|a^{-1}\|_w}{2}\right)(\|\Delta_E\|_\infty + \|\Delta_A\|_\infty),\end{aligned} \tag{3.23}$$

where the last inequality follows from (3.20) and Lemma 2.3.

Observe that the perturbation bounds (3.22) and (3.23) shed light on how the whole solution and the correction part behave when the coefficients are perturbed by small perturbations. However they can be inaccurate since they are quite related to the value of some $\lambda \in \mathcal{S}$.

Here we provide some insight on how to choose λ if $A = T(a)$, where $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i \in \mathcal{W}$. This is a special case but has practical interest, see Example 5.2. Observe that we may choose λ such that the value $\|(A+\lambda I)^{-1}\|_\infty \|A-\lambda I\|_\infty$ is far away from 1. For any $\theta \in \mathbb{C}$ such that $A+\theta I$ is invertible, we have $\|(A+\theta I)^{-1}\|_\infty \|A-\theta I\|_\infty \geq \|(a+\theta)^{-1}\|_w \|a-\theta\|_w \geq \frac{\|a-\theta\|_w}{\|a+\theta\|_w}$. This inequality provides a good way to choose λ, that is, we may choose λ such that $\lambda = \arg\min_\theta \frac{\|a-\theta\|_w}{\|a+\theta\|_w}$, and one can see that $\lambda = a_0$. Now let $\lambda = a_0$ and $B = A - \lambda I$, suppose $|a_0| > \sum_{i\neq 0} |a_i|$, we have $|\lambda| > \|B\|_\infty$ and

$$
\begin{aligned}
\|(A+\lambda I)^{-1}\|_\infty \|A-\lambda I\|_\infty &= \|(2\lambda I + B)^{-1}\|_\infty \|B\|_\infty \\
&\leq \frac{1}{2|\lambda|} \sum_{i=0}^{\infty} \frac{\|B\|_\infty^i}{(2|\lambda|)^i} \|B\|_\infty \\
&= \frac{\|B\|_\infty}{2|\lambda| - \|B\|_\infty} < 1.
\end{aligned}
$$

From the above analysis, we can see that $\lambda = a_0$ can be used in practice if $A = T(a)$ and the symbol a satisfies $|a_0| > \sum_{i\neq 0} |a_i|$.

4 A generalization

The perturbation results for the generalized Sylvester Eq. (1.1) are here applied to a more general type equation

$$
\sum_{k=1}^{n} A_k X B_k = C, \tag{4.1}
$$

where $A_k, B_k, k = 1, \ldots, n$ $(n \geq 3)$, and C belong to the $\mathcal{QT}_\infty$ class. Observe that Eq. (4.1) has a unique solution $X \in \mathcal{L}_\infty$ if the elementary operator $\mathcal{T} : X \to \sum_{k=1}^n A_k X B_k$ is invertible in $\mathcal{L}_\infty$. It was proved in [14, Theorem 4] that if $A_i A_j = A_j A_i$ and $B_i B_j = B_j B_i$ for $\forall i, j \in \{1, \ldots, n\}$, then the spectrum of $\mathcal{T}$ satisfies

$$
\sigma(\mathcal{T}) \subset \left\{ \sum_{k=1}^{n} \alpha_k \beta_k : \alpha_k \in \sigma(A_k), \beta_k \in \sigma(B_k), 1 \leq k \leq n \right\},
$$

which implies that $\mathcal{T}$ is invertible in $\mathcal{L}_\infty$ if $0 \notin \sum_{k=1}^n \sigma(A_k)\sigma(B_k)$.

Concerning matrices in the $\mathcal{QT}_\infty$ class, it would be natural to ask if the set of commutative QT matrices is nonempty. We show by an interesting example that there are cases where the QT matrices are commutative. Consider the set $\mathcal{C} = \{A_k = T(a_k) - H(a_k^+)$: $a_k(z) = \sum_{i\in\mathbb{Z}} a_i^{(k)} z^i \in \mathcal{W}$ and $a_i^{(k)} \in \mathbb{R}$ satisfies $a_i^{(k)} = a_{-i}^{(k)}$ for all $i \in \mathbb{Z}$, $H(a_k^+)$ is a Hankel matrix such that $H(a_k^+)_{i,j} = (a_{i+j-1}^{(k)})_{i,j\in\mathbb{Z}^+}\}$. Observe that both $T(a_k)$ and $H(a_k^+)$ are real symmetric and $H(a_k^+) = H(a_k^-)$, where $H(a_k^-)$ is a Hankel matrix such that $H(a_k^-)_{i,j} = (a^{(k)}_{-i-j+1})_{i,j\in\mathbb{Z}^+}$. For any $A_k, A_\ell \in \mathcal{C}$, in view of Lemma 2.1, we have

$$A_k A_\ell = T(a_k a_\ell) - T(a_k)H(a_\ell^+) - H(a_k^+)T(a_\ell),$$

and

$$A_\ell A_k = T(a_k a_\ell) - T(a_\ell)H(a_k^+) - H(a_\ell^+)T(a_k).$$

A direct computation yields that $T(a_k)H(a_\ell^+) + H(a_k^+)T(a_\ell) = T(a_\ell)H(a_k^+) + H(a_\ell^+)T(a_k)$ so that $A_k A_\ell = A_\ell A_k$.

4.1 Existence of QT solutions

Suppose there is at least a pair (A_i, B_i), $i = 1, \ldots, n$, such that both A_i and B_i are invertible in $\mathcal{L}_\infty$. We provide an easy-to-check condition under which $\mathcal{T}$ is invertible in $\mathcal{L}_\infty$. In what follows, without loss of generality, we assume A_1 and B_1 are invertible.

Theorem 4.1 *Assume A_1 and B_1 are invertible in $\mathcal{L}_\infty$, $A_i A_j = A_j A_i$ and $B_i B_j = B_j B_i$, $\forall i, j \in \{1, \ldots, n\}$, and*

$$\eta := \|A_1^{-1}\|_\infty \|B_1^{-1}\|_\infty \left(\sum_{k=2}^{n} \|A_k\|_\infty \|B_k\|_\infty \right) < 1,$$

Then Eq. (4.1) has a unique solution in $\mathcal{L}_\infty$ and

$$X = A_1^{-1} C B_1^{-1} + \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=2}^{n} (-1)^k A_1^{-k-1} A_{i_1} \cdots A_{i_k} C B_{i_1} \cdots B_{i_k} B_1^{-k-1}. \quad (4.2)$$

Moreover, if $A_k, B_k, C \in \mathcal{QT}_\infty$ for $k = 1, \ldots, n$, then $X \in \mathcal{QT}_\infty$.

Proof Set $\eta_1 = \|A_1^{-1}\|_\infty \|B_1^{-1}\|_\infty$, we have $\sigma(A_1)\sigma(B_1) \subseteq \{z : |z| \geq \eta_1^{-1}\}$. Observe that $\sum_{k=2}^{n} \|A_k\|_\infty \|B_k\|_\infty < \eta_1^{-1}$, we have $\sum_{k=2}^{n} \sigma(A_k)\sigma(B_k) \subseteq \{z : |z| < \eta_1^{-1}\}$. It follows that $0 \notin \sum_{k=1}^{n} \sigma(A_k)\sigma(B_k)$. Hence the elementary operator $\mathcal{T} : X \to \sum_{k=1}^{n} A_k X B_k$ is invertible and Eq. (4.1) has a unique solution in $\mathcal{L}_\infty$. It is easy to check that X in (4.2) solves Eq. (4.1). We prove that the series in (4.2) converges in $\mathcal{L}_\infty$.

Set $\check{A}_{i_k} = A_1^{-1} A_{i_k}$ and $\check{B}_{i_k} = B_{i_k} B_1^{-1}$ for $k = 1, \ldots, \infty$ and $i_k = 2, \ldots, n$, concerning the commutativity of the coefficients, (4.2) is equivalent to

$$X = A_1^{-1} C B_1^{-1} + \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=2}^{n} (-1)^k \check{A}_{i_1} \cdots \check{A}_{i_k} A_1^{-1} C B_1^{-1} \check{B}_{i_1} \cdots \check{B}_{i_k}. \quad (4.3)$$

Write $Q = A_1^{-1} C B_1^{-1}$, then $Q \in \mathcal{L}_\infty$. Observe that for $j = 1, \ldots, k$, we have

$$\sum_{i_j=2}^{n} \|\check{A}_{i_j}\|_\infty \|\check{B}_{i_j}\|_\infty \le \|A_1^{-1}\|_\infty \|B_1^{-1}\|_\infty \Big(\sum_{i_j=2}^{n} \|A_{i_j}\|_\infty \|B_{i_j}\|_\infty \Big) = \eta < 1.$$

It follows from (4.3) that

$$\begin{aligned}
\|X\|_\infty &\le \|Q\|_\infty + \Big\| \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=2}^{n} \check{A}_{i_1} \cdots \check{A}_{i_k} Q \check{B}_{i_1} \cdots \check{B}_{i_k} \Big\|_\infty \\
&\le \|Q\|_\infty + \sum_{k=1}^{\infty} \Big(\sum_{i_1=2}^{n} \|\check{A}_{i_1}\|_\infty \|\check{B}_{i_1}\|_\infty \Big) \ldots \Big(\sum_{i_k=2}^{n} \|\check{A}_{i_k}\|_\infty \|\check{B}_{i_k}\|_\infty \Big) \|Q\|_\infty \\
&\le \sum_{k=0}^{\infty} \eta^k \|Q\|_\infty = \frac{\|Q\|_\infty}{1-\eta},
\end{aligned} \tag{4.4}$$

from which we obtain $X \in \mathcal{L}_\infty$.

Concerning the case where the coefficients $A_k, B_k, k = 1, \ldots, n$, and C are QT matrices, the proof to show $X \in \mathcal{QT}_\infty$ is analogous to that of Theorem 2.3 and is omitted here. □

Theorem 4.2 *Assume that the assumptions in Theorem 4.1 hold true. For* $A_k = T(a_k) + E_k, B_k = T(b_k) + E_k, C = T(c) + E_c \in \mathcal{QT}_\infty$, $k = 1, \ldots, n$, *let* $X = T(x) + E_x \in \mathcal{QT}_\infty$ *be the unique solution of Eq. (4.1), then*

$$x(z) = \Big(\sum_{k=1}^{n} a_k(z) b_k(z) \Big)^{-1} c(z).$$

Proof We have from

$$\sum_{k=1}^{n} A_k X B_k - C = T\Big(\Big(\sum_{k=1}^{n} a_k b_k \Big) x - c \Big) + F = 0$$

that $\Big(\sum_{k=1}^{n} a_k(z) b_k(z) \Big) x(z) = c(z)$. Since, for any $z \in \mathbb{T}$, it holds $a_k(z) \in \sigma(A_k)$ and $b_k(z) \in \sigma(B_k)$ for $k = 1, \ldots, n$, we have

$$\sum_{k=1}^{n} a_k(z) b_k(z) \in \sum_{k=1}^{n} \sigma(A_k) \sigma(B_k)$$

for any $z \in \mathbb{T}$. Note that $0 \notin \sum_{k=1}^{n} \sigma(A_k)\sigma(B_k)$ in view of the proof of Theorem 4.1, then $\sum_{k=1}^{n} a_k(z) b_k(z) \neq 0$ for all $z \in \mathbb{T}$, from which we have $x(z) = \Big(\sum_{k=1}^{n} a_k(z) b_k(z) \Big)^{-1} x(z)$. □

4.2 Perturbation results

We are ready to extend the perturbation results to Eq. (4.1). Consider the perturbed matrix equation obtained from (4.1) by replacing the coefficients A_k, B_k and C by $A_k+\varDelta_{A_k}$, $B_k+\varDelta_{B_k}$ and $C+\varDelta_C$, respectively, where $\varDelta_{A_k}=T(\delta_{a_k})+E_{\delta_{a_k}}\in\mathcal{QT}_\infty$, $\varDelta_{B_k}=T(\delta_{b_k})+E_{\delta_{b_k}}\in\mathcal{QT}_\infty, k=1,\dots,n$, and $\varDelta_C=T(\delta_c)+E_{\delta_c}\in\mathcal{QT}_\infty$. Denote by $X+\varDelta_X$ the solution of the perturbed equation, we have

$$\sum_{k=1}^{n}(A_k+\varDelta_{A_k})(X+\varDelta_X)(B_k+\varDelta_{B_k})=C+\varDelta_C. \tag{4.5}$$

In what follows, we suppose the assumptions in Theorem 4.1 are satisfied for the perturbed equation, then there is a unique $X+\varDelta_X\in\mathcal{QT}_\infty$ solving Eq. (4.5). Write $\varDelta_X=T(\delta_x)+E_{\delta_x}$, we have for the Toeplitz part

$$\Big(\sum_{k=1}^{n}a_kb_k\Big)\delta_x\doteq\delta_c-\sum_{k=1}^{n}(\delta_{a_k}xb_k+a_kx\delta_{b_k}).$$

In view of the proof of Theorem 4.2, we know $\sum_{k=1}^{n}a_k(z)b_k(z)$ is invertible, it follows

$$\delta_x\doteq\Big(\sum_{k=1}^{n}a_kb_k\Big)^{-1}\Big(\delta_c-\sum_{k=1}^{n}\Big(\delta_{a_k}b_k+a_k\delta_{b_k}\Big)x\Big),$$

which yields

$$\delta_x\dot{\le}\Big\|\Big(\sum_{k=1}^{n}a_kb_k\Big)^{-1}\Big\|_w\Big(\|\delta_c\|_w+\sum_{k=1}^{n}(\|\delta_{a_k}\|_w\|b_k\|_w+\|\delta_{b_k}\|_w\|a_k\|_w)\|x\|_w\Big).$$

Let

$$v_1=\max_k\{\|a_k\|_w\},\quad v_2=\max_k\{\|b_k\|_w\},\quad \theta=\Big\|\Big(\sum_{k=1}^{n}a_kb_k\Big)^{-1}\Big\|_w\|c\|_w,$$

and

$$\hat{\mu}=\max\{1,v_1\theta,v_2\theta\}. \tag{4.6}$$

Note that $\|x\|_w\le\|(\sum_{k=1}^{n}a_kb_k)^{-1}\|_w\|c\|_w$ in view of Theorem 4.2, we obtain

$$\|\delta_x\|_w\dot{\le}\hat{\mu}\Big\|\Big(\sum_{k=1}^{n}a_kb_k\Big)^{-1}\Big\|_w\Big(\sum_{k=1}^{n}\Big(\|\delta_{a_k}\|_w+\|\delta_{b_k}\|_w\Big)+\|\delta_c\|_w\Big). \tag{4.7}$$

Concerning the correction part, write $\Delta_V = \Delta_C - \sum_{k=1}^{n}(\Delta_{A_k} X B_k + A_k X \Delta_{B_k})$, we have $\mathcal{T}(\Delta_X) \doteq \Delta_V$, where $\mathcal{T} : X \to \sum_{k=1}^{n} A_k X B_k$ is the elementary operator. Since $\mathcal{T}$ is invertible in view of the proof of Theorem 4.1, we have $\Delta_X \doteq \mathcal{T}^{-1}(\Delta_V)$, which is

$$\Delta_X \doteq A_1^{-1}\Delta_V B_1^{-1} + \sum_{k=1}^{\infty}\sum_{i_1,\ldots,i_k=2}^{n}(-1)^k A_1^{-k-1}A_{i_1}\cdots A_{i_k}\Delta_V B_{i_1}\cdots B_{i_k}B_1^{-k-1}. \tag{4.8}$$

Set $\eta_1 = \|A_1^{-1}\|_\infty \|B_1^{-1}\|_\infty$, $\eta_2 = \max\{\max_k \|A_k\|_\infty, \max_k \|B_k\|_\infty\}$ and $\eta_3 = \max\left\{1, \frac{\eta_1\eta_2}{1-\eta}\|C\|_\infty\right\}$, we obtain from (4.8) that

$$\begin{aligned}
\|\Delta_X\|_\infty &\dot{\leq} \|A_1^{-1}\|_\infty \|B_1^{-1}\|_\infty \Big(\sum_{k=0}^{\infty}\eta^k\Big)\|\Delta_V\|_\infty \\
&= \frac{\eta_1}{1-\eta}\Big(\|\Delta_C\|_\infty + \sum_{k=1}^{n}\Big(\|\Delta_{A_k}\|_\infty\|B_k\|_\infty + \|\Delta_{B_k}\|_\infty\|A_k\|_\infty\Big)\|X\|_\infty\Big) \\
&\leq \frac{\eta_1\eta_3}{1-\eta}\Big(\|\Delta_C\|_\infty + \sum_{k=1}^{n}\Big(\|\Delta_{A_k}\|_\infty + \|\Delta_{B_k}\|_\infty\Big)\Big),
\end{aligned}$$

where the last inequality follows from $\|X\|_\infty \leq \frac{\eta_1\|C\|_\infty}{1-\eta}$, which is obtained by (4.4).

Set

$$\hat{\delta} = \max\left\{\max_k\left\{\frac{\|\delta_{a_k}\|_w}{\|E_{\delta_{a_k}}\|_\infty}\right\}, \max_k\left\{\frac{\|\delta_{b_k}\|_w}{\|E_{\delta_{b_k}}\|_\infty}\right\}, \frac{\|\delta_c\|_w}{\|E_{\delta_c}\|_\infty}\right\},$$

if $\hat{\delta} < \infty$, we obtain a bound for E_{δ_x}

$$\|E_{\delta_x}\|_\infty \dot{\leq} \hat{\kappa}(1+\hat{\delta})\Big(\sum_{k=1}^{n}(\|E_{\delta_{a_k}}\|_\infty + \|E_{\delta_{b_k}}\|_\infty) + \|E_{\delta_c}\|_\infty\Big),$$

where $\hat{\kappa} = \frac{\eta_1\eta_3}{1-\eta} + \hat{\mu}\|(\sum_{k=1}^{n} a_k b_k)^{-1}\|_w$ and $\hat{u}$ is defined in (4.6).

We may observe that the results we obtained can be used as a basis to solve equation of the form

$$\sum_{k=1}^{n} f_k(A) X g_k(B) = C,$$

where $f_k(\cdot)$ and $g_k(\cdot), k = 1, \ldots, n$, are holomorphic functions in domains containing $\sigma(A)$ and $\sigma(B)$, respectively. An equation of this type in the (finite dimensional) matrix setting was studied in [21]. Concerning the case where $A, B, C \in \mathcal{L}_\infty$ (or $A, B, C \in$

$\mathcal{QT}_\infty$), according to [14, Theorem 5], the spectrum of the operator $\mathcal{R} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ defined by $\mathcal{R}(X) = \sum_{k=1}^{\infty} f_k(A)Xg_k(B)$ satisfies

$$\sigma(\mathcal{R}) \subset \left\{ \sum_{k=1}^{n} f_k(\lambda_a)g_k(\lambda_b) : \lambda_a \in \sigma(A) \text{ and } \lambda_b \in \sigma(B) \right\}.$$

Under certain conditions, it may be feasible to extend the results we have obtained to this equation. We leave this as a future consideration.

5 Numerical experiments

In this section, we apply the perturbation analysis to the generalized Sylvester Eq. (1.1). All the computations are performed in QT arithmetic relying on the operations implemented in cqt-toolbox [6]. The threshold used in the computations is set to 10^{-12}.

Example 5.1 This example is taken from Example 5.2 in [17], where a generalized Sylvester equation arising in solving a quadratic matrix equation $A_1X^2+A_0X+A_{-1} = 0$ by Newton's method [7] is considered. The involved quadratic matrix equation arises in the analysis of a Markov process modeling a two-node Jackson network [16] and the matrices A_1, A_0, A_{-1} are nonnegative matrices with QT structure. When applying Newton's method to the quadratic matrix equation, the computation of the Newton increment H_k at each step is reduced to solve an equation of the form

$$(A_1X_k + A_0 - I)H + A_1HX_k = L(X_k), \tag{5.1}$$

with $L(X_k) = A_1X_k^2 + (A_0 - I)X_k + A_{-1}$.

We consider the generalized Sylvester equation arising in the computation of the second Newton iterates X_2, that is, we consider the equation

$$(A_1X_1 + A_0 - I)H + A_1HX_1 = L(X_1), \tag{5.2}$$

where X_1 is obtained by using the Newton iteration in [7] and the solution H of the generalized Sylvester equation is computed using the Galerkin approach in [17]. We choose the coefficients A_1, A_0, A_{-1} from the 10 problems as in [7, Example 6.2] and perturb the coefficients by using the same technique as in [7].

It can be seen from [7] that $A_1X_1 + A_0 - I$ is invertible in $\mathcal{L}_\infty$. Moreover, $\|(I - A_0 - A_1X)^{-1}A_1\|_\infty < 1$ and $\|X_1\|_\infty \leq 1$, according to Theorem 2.3, Eq. (5.2) has a unique solution $H = T(h) + E_h \in \mathcal{QT}_\infty$. Moreover, the perturbed equation has a unique solution $H + \Delta_H \in \mathcal{QT}_\infty$ with $\Delta H = T(\delta_h) + E_{\delta_h}$.

Table 1 reports the value δ defined in (3.12), the upper bound of the conditioning of the whole solution, which is $\text{cond}_H = \frac{\alpha\gamma}{1-\alpha\beta}$. Recall that cond_H implies that the Toeplitz part is well-conditioned if it is a small number. In this experiment, we observe that $\frac{\alpha\gamma}{1-\alpha\beta}$ provides a good estimate to the conditioning of the whole problem so that

Table 1 Conditioning of the the whole solution of Eq. (5.2) and perturbation bounds: actual perturbations in the Toeplitz part and in the correction part and the related upper bounds

Problem	δ	Cond_H	$\|\delta_h\|_w$	δ_h-bound	$\|E_{\delta_h}\|_\infty$	E_{δ_h}-bound
1	3.7671	4.1267	3.0393e-09	5.8453e-09	1.2534e-09	3.3407e-08
2	2.8789	2.3941	1.4473e-09	3.8391e-09	1.2294e-09	1.1288e-08
3	4.0215	2.3941	1.4741e-09	3.8678e-09	1.4835e-09	1.1227e-08
4	3.0117	2.1509	1.8394e-09	4.1083e-09	1.1593e-09	1.2465e-08
5	3.1677	1.8838	2.0794e-09	4.6513e-09	9.3214e-10	1.4119e-08
6	3.7744	1.8838	2.4189e-09	6.3207e-09	1.2491e-09	1.9116e-08
7	2.3698	2.0364	1.1101e-09	4.3663e-09	1.6565e-10	9.8742e-09
8	12.0710	3.2542	5.9479e-10	1.2838e-09	2.9312e-10	5.8110e-09
9	9.6032	2.4431	1.0662e-09	2.4756e-09	1.1020e-09	9.4847e-09
10	2.7598	2.4431	1.2139e-09	2.8840e-09	1.4515e-09	1.4340e-08

the Toeplitz part is well-conditioned, and together with δ, it implies that the correction part is also well-conditioned.

Denote by δ_h-bound the bound (3.9) for the Toeplitz part and E_{δ_h}-bound the bound (3.11) for the correction part. In Table 1, we compare these two perturbation bounds with the actual perturbation errors $\|\delta_h\|_w$ and $\|E_{\delta_h}\|_\infty$. It can be seen that, with respect to small perturbations on the coefficients, inequalities (3.9) and (3.11) provide sharp and revealing perturbation bounds to the Toeplitz part and to the correction part, respectively.

Example 5.2 This example is taken from Example 5.1 in [17], where, in the solution of a 2D Poisson problem on the positive orthant, equation of the form

$$MX + XM = Q, \tag{5.3}$$

is encountered, where $M = (\Delta t A - \frac{\Delta x^2}{2} I)$ and $Q = \Delta x^2 U_t - \Delta x^2 \Delta t F$. Here, Δx is the discretization step, Δt is the time step, U_t is the solution of the Poisson problem at time t. Both A and F are QT matrices so that the coefficients M and Q are also QT matrices.

We perturb the matrix F by Δ_F, where $\Delta_F = $ `cqt('hankel', dfm)` $+$ `0.1*cqt(dfm,dfm)` with `dfm=rand*10^{-8}*fm`, the construction of `fm` can be found in [17]. Matrix A is perturbed by Δ_A, where $\Delta_A = \mathtt{cqt}([-c1, c2], [-c1, c2])$ with $c1 = \mathtt{rand} * 10^{-8}$ and $c2 = \mathtt{rand} * 10^{-8}$. It can be examined that $M + \lambda I$ is invertible for $\lambda = 1$. Moreover, it holds $\|(M + \lambda I)^{-1}\|_\infty \|M - \lambda I\|_\infty < 1$ and $\|(M + \Delta_M + \lambda I)^{-1}\|_\infty \|M + \Delta_M - \lambda I\|_\infty < 1$. Hence, there is unique $X = T(x) + E_x \in \mathcal{QT}_\infty$ and $X + \Delta_X$ with $\Delta_X = T(\delta x) + E_{\delta_x}$ solving Eq. (5.3) and the corresponding perturbed equation, respectively.

As in [17], we consider the solution of Eq. (5.3) in $[0, 2] \times [0, 2]$ with timesteps $\Delta t = 0.05, 0.10, 0.15, 0.20$, respectively. Table 2 reports two perturbation bounds (3.20) and (3.23), denoted by δ_x-bound for the Toeplitz part and E_{δ_x}-bound for the

Table 2 Perturbation bounds for Eq. (5.3): actual perturbations in the Toeplitz part and in the correction part and the related upper bounds

Δt	$\|\delta_x\|_w$	δ_x-bound	$\|E_{\delta_x}\|_\infty$	E_{δ_x}-bound
0.05	4.6655e-09	3.4053e-08	5.7381e-09	7.2163e-08
0.10	1.2843e-08	7.1753e-08	8.6048e-09	1.5270e-07
0.15	8.1842e-08	1.0451e-07	2.1398e-08	4.0208e-07
0.20	9.0113e-08	9.5174e-08	9.0113e-08	3.7718e-07

correction part, respectively. Comparison with the corresponding actual perturbation errors $\|\delta_x\|_w$ and $\|E_{\delta_x}\|_\infty$ indicates that these two bounds are sharp.

Example 5.3 Consider the equation $X + A_1 X A_2 = A_3$, where $A_1 = T(a_1)$ with $a_1(z) = \frac{1}{2} - \epsilon + \frac{1}{4}(z + z^{-1})$, where $\epsilon > 0$ is a small number, $A_2 = T(a_2)$ with $a_2(z) = \frac{3}{5} + \frac{1}{5}z + \frac{1}{5}z^{-1}$ and $A_3 = \texttt{cqt(1,1)}$.

Observe that matrix A_1 is nearly singular if ϵ is close to 0, so that the computations involving matrix A_1 are usually very sensitive to numerical errors. Choose $\epsilon = 0.006$, we perturbed A_i, $i = 1, 2, 3$, by $\Delta_{A_i} = \texttt{cqt}(c_i, c_i)$, where $c_i = \texttt{rand} * 10^{-8}$. In this special case, the numerical results show that the perturbation bound for the Toeplitz part is 2.7493e-08 and the actual error is 1.4297e-08. For the perturbation bound of the whole problem, the upper bound of the conditioning is shown to be 1.0000e+02, and the perturbation bound is 2.3465e-06, compared with the actual error 1.4297e-08, we can see that the perturbation bound (3.5) is also descriptive of the errors when the problem is not well-conditioned.

6 Conclusion

We have analysed a generalized Sylvester equation with infinite size matrix coefficients in the $\mathcal{QT}_\infty$ class of quasi Toeplitz matrices. Under sufficient conditions, we showed that an equation of this type has a unique solution in the $\mathcal{QT}_\infty$ class. By separating the perturbation analysis of the Toeplitz part from the correction part, we provided perturbation results that cater to the particular QT structure. The perturbation results were also extended to an equation with a more general form.

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Compliance with ethical standards

Conflicts of interest The authors declare no potential conflict of interests.

References

1. Bartels, R.H., Stewart, G.W.: Solution of the matrix equation AX + XB = C. Commun. ACM **15**, 820–826 (1972)
2. Bini, D.A., Massei, S., Meini, B.: Semi-infinite quasi-Toeplitz matrices with applications to QBD stochastic processes. Math. Comput. **87**, 2811–2830 (2018)
3. Bini, D.A., Massei, S., Meini, B.: On functions of quasi Toeplitz matrices. Sb. Math. **208**, 56–74 (2017)
4. Bini, D.A., Massei, S., Meini, B., Robol, L.: On quadratic matrix equations with infinite size coefficients encountered in QBD stochastic processes. Numer. Linear Algebra Appl. **25e**, 2128 (2018)
5. Bini, D.A., Massei, S., Meini, B., Robol, L.: A computational framework for two-dimensional random walks with restarts. SIAM J. Sci. Comput. **42**(4), A2108–A2133 (2020)
6. Bini, D.A., Massei, S., Robol, L.: Quasi-Toeplitz matrix arithmetic: a MATLAB toolbox. Numer. Algorithms **81**, 741–769 (2019)
7. Bini, D.A., Meini, B., Meng, J.: Solving quadratic matrix equations arising in random walks in the quarter plane. SIAM J. Matrix Anal. Appl. **41**, 691–714 (2020)
8. Böttcher, A., Grudsky, S. M, : Spectral Properties of Banded Toeplitz Matrices. SIAM, Philadelphia (2005)
9. Epton, M.A.: Methods for the solution of AXD – BXC = E and its application in the numerical solution of implicit ordinary differential equations. BIT **20**, 341–345 (1980)
10. Gardiner, J.D., Laub, A.J., Amato, J.J., Moler, C.B.: Solution of the Sylvester matrix equation $AXB^T + CXD^T = E$. ACM Trans. Math. Softw.. **18**, 223–231 (1992)
11. Gahinet, P., Laub, A., Kenney, Ch., Hewer, G.: Sensitivity of the stable discrete-time Lyapunov equation. IEEE Trans. Automat. Control **35**, 1209–1217 (1990)
12. Higham, N.J.: Perturbation theory and backward error for $AX - XB = C$. BIT **33**, 124–136 (1993)
13. Hu, Q., Cheng, D.: The polynomial solution to the Sylvester matrix equation. Appl. Math. Lett. **19**, 859–864 (2006)
14. Lumer, G., Rosenblum, M.: Linear operator equations. Proc. Amer. Math. Soc. **10**, 32–41 (1959)
15. Mortad, M.H.: An Operator Theory Problem Book. World Scientific Publishing, Singapore (2018)
16. Motyer, A.J., Taylor, P.G.: Decay rates for quasi-birth-and-death processes with countably many phases and tridiagonal block generators. Adv. Appl. Prob. **38**, 522–544 (2006)
17. Robol, L.: Rational Krylov and ADI iteration for infinite size quasi-Toeplitz matrix equations. Linear Algebra Appl. **604**, 210–235 (2020)
18. Simoncini, V.: Computational Methods for Linear Matrix Equations. SIAM Rev. **58**, 377–441 (2016)
19. Stykel, T.: Numerical solution and perturbation theory for generalized Lyapunov equations. Linear Algebra Appl. **349**, 155–18 (2002)
20. Terán, F. De., Bruno, B., Poloni, F., Robol, L, : Nonsingular systems of generalized Sylvester equations: an algorithmic approach. Numer. Linear Algebra Appl. **26**, e2261 (2019)
21. Wimmer, H., Ziebur, A.D.: Solving the matrix equaiton $\sum_{p=1}^{r} f_p(A)Xg_p(B) = C$. SIAM Rev. **14**, 318–323 (1979)