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A convergent finite difference scheme for the Ostrovsky–Hunter equation with Dirichlet boundary conditions

J. Ridder[1] · A. M. Ruf[2]

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Abstract
We prove convergence of a finite difference scheme to the unique entropy solution of a general form of the Ostrovsky–Hunter equation on a bounded domain with non-homogeneous Dirichlet boundary conditions. Our scheme is an extension of monotone schemes for conservation laws to the equation at hand. The convergence result at the center of this article also proves existence of entropy solutions for the initial-boundary value problem for the general Ostrovsky–Hunter equation. Additionally, we show uniqueness using Kružkov's doubling of variables technique. We also include numerical examples to confirm the convergence results and determine rates of convergence experimentally.

Keywords Ostrovsky–Hunter equation · Short-pulse equation · Vakhnenko equation · Finite difference methods · Monotone scheme · Existence · Uniqueness · Stability · Convergence · Entropy solution · Dirichlet boundary conditions

Mathematics Subject Classification 65M06 · 35M33 · 35L35

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✉ A. M. Ruf
adrianru@math.uio.no

J. Ridder
jur436@psu.edu

[1] Department of Mathematics, Pennsylvania State University, University Park, USA

[2] Department of Mathematics, University of Oslo, Oslo, Norway

1 Introduction

We consider the initial-boundary value problem

$$u_t + f(u)_x = \gamma \int_0^x u(y,t)\,\mathrm{d}y, \tag{1a}$$
$$u(x,0) = u_0(x), \tag{1b}$$
$$u(0,t) = \alpha(t), \tag{1c}$$
$$u(1,t) = \beta(t), \tag{1d}$$

with $f \in \mathscr{C}^2(\mathbb{R})$ and $\gamma > 0$. Equation (1a) is derived by integrating the nonlinear evolution equation

$$(u_t + f(u)_x)_x = \gamma u, \tag{2}$$

in space. This equation was posed by Ostrovsky [27] and Hunter [18] with $f(u) = \frac{1}{2}u^2$ as a model for small-amplitude long waves on a shallow rotating fluid and is referred to as the Ostrovsky–Hunter equation [3,7,23], short wave equation [18], Vakhnenko equation [25,28,34–36], Ostrovsky–Vakhnenko equation [4,24] and reduced Ostrovsky equation [27,29,33]. If $f(u) = -\frac{1}{6}u^3$, Eq. (2) is known as the short pulse equation, which was introduced by Schäfer and Wayne [32] as a model for the propagation of ultra-short light pulses in silica optical fibers (see also [1,22]). In the present paper, however, we will consider an arbitrary flux $f \in \mathscr{C}^2(\mathbb{R})$ and will refer to Eq. (1a) with general f as Ostrovsky–Hunter equation.

In order to derive Eq. (1a), we integrate Eq. (2) in space to get

$$\begin{aligned} u_t + f(u)_x &= \gamma P, \\ P_x &= u. \end{aligned} \tag{3}$$

The function P must then be further specified by an additional constraint, e.g. $P(-\infty, t) = 0$ (which leads to $P = \int_{-\infty}^x u$; see [8]) or $\int P = 0$ (implying $P = \int_{-\infty}^x u - \int_{-\infty}^\infty u$ on the real line or $P = \int_0^x u - \int_0^1 u$ in the unit interval; see [6,18,23,33]). Here we will consider the unit interval and choose $P(0,t) = 0$, which gives

$$P[u](x,t) = \int_0^x u(t,y)\,\mathrm{d}y\,. \tag{4}$$

Concerning the initial and boundary data, we will assume

$$u_0 \in \mathrm{B.V.}(0,1) \text{ and } \alpha, \beta \in \mathrm{B.V.}(0,T). \tag{5}$$

Coclite et al. developed a global well-posedness analysis utilizing the concept of entropy solutions defined in a distributional sense [see (6) in Definition 1 below] on the domains $\mathbb{R}\times\mathbb{R}^+$ and $\mathbb{R}^+\times\mathbb{R}^+$ in [7–12,30] and on $[0,1]\times\mathbb{R}^+$ with non-homogeneous Dirichlet boundary conditions in [13]. Their proofs are based on a vanishing viscosity regularization and a compensated compactness argument.

In this paper, we aim to show existence of entropy solutions (as defined in Definition 1 below) to the initial-boundary value problem (1) by proving the convergence of a finite difference scheme. We will base our construction of the numerical scheme on the classical theory of monotone schemes for conservation laws and use central differences for the nonlocal source term. In order to get compactness of the scheme, we will employ Helly's theorem together with appropriate a priori bounds of the piecewise constant interpolation. Then, we will show convergence towards the entropy solution using discrete versions of the entropy conditions in the interior of the domain and at the boundary. Furthermore, we prove uniqueness of entropy solutions by showing L^1 stability using Kružkov's 'doubling of variables' technique.

Without convergence proof, numerical methods for Eq. (2) are used in [15,18,23], including Fourier pseudo-spectral methods and a finite difference scheme based on the Engquist–Osher scheme. So far the only rigorous numerical analysis of the Ostrovsky–Hunter equation is performed by Coclite et al. [6]. The authors, however, consider the case of periodic boundary conditions and initial data with zero mean. The present paper directly extends these results to the setting of non-periodic boundary conditions. Although we follow the general strategy of [6], the non-periodicity complicates matters throughout. In particular, we will present new versions of Harten's lemma and Kružkov's 'doubling of variables' technique that properly address the contributions of the boundary terms.

We will consider entropy solutions of (1) based on the following definition:

Definition 1 (*Entropy solution*) A function $u \in \mathscr{C}([0,T];\mathrm{L}^1(0,1)) \cap \mathrm{L}^\infty((0,1)\times(0,T))$ is called an entropy solution of the Ostrovsky–Hunter equation (1) if for all entropy pairs (η, q), i.e. convex functions $\eta \in \mathscr{C}^2(\mathbb{R})$, and q such that $q' = \eta' f'$,

$$\int_0^T \int_0^1 (\eta(u)\phi_t + q(u)\phi_x + \gamma\eta'(u)P[u]\phi)\,\mathrm{dx}\,\mathrm{dt} + \int_0^1 \eta(u_0(x))\phi(x,0)\,\mathrm{dx} \\ - \int_0^1 \eta(u(x,T))\phi(x,T)\,\mathrm{dx} \geq 0, \tag{6}$$

for all nonnegative $\phi \in \mathscr{C}_c^\infty((0,1)\times\mathbb{R})$, and

$$q(u_0^\tau(t)) - q(\alpha(t)) - \eta'(\alpha(t))(f(u_0^\tau(t)) - f(\alpha(t))) \leq 0 \\ \leq q(u_1^\tau(t)) - q(\beta(t)) - \eta'(\beta(t))(f(u_1^\tau(t)) - f(\beta(t))) \tag{7}$$

holds for a. e. $t \in (0,T)$. Here $P[u]$ is as in (4) and u_0^τ and u_1^τ denote the strong traces of u at the boundary $x=0$ respectively $x=1$.

Remark 1 Note that by an approximation argument, cf. [17, pp. 57–58], a function $u \in \mathscr{C}([0,T];\mathrm{L}^1(0,1))$ is an entropy solution if and only if inequalities (6) and (7) hold for all Kružkov entropy pairs,

$$\eta(u,k) = |u-k|, \qquad q(u,k) = \mathrm{sign}(u-k)(f(u)-f(k)), \qquad k \in \mathbb{R}.$$

Remark 2 This is the usual definition of entropy solutions of Eq. (1). However, regarding the entropy boundary condition instead of working with the original condition due to Bardos et al. [2], we will use the entropy boundary condition (7) introduced by Dubois and LeFloch [14]. Due to the regularizing effect of the P Eq. (4) we have that $u \in \mathrm{L}^\infty((0,1)\times(0,T))$ implies $P[u] \in \mathrm{L}^\infty(0,T;\mathrm{W}^{1,\infty}(0,1))$. Therefore, if $u \in \mathrm{L}^\infty((0,1)\times(0,T))$ satisfies the entropy condition (6), then [5, Theorem 1.1] assures the existence of strong traces u_0^τ, u_1^τ and hence boundary entropy condition (7) is well-defined.

The paper is organized as follows. In Sect. 2 we specify the numerical scheme under consideration. Section 3 contains discrete a priori bounds which are used to show compactness of the scheme. In the next section we will develop discrete entropy inequalities both in the interior and at the boundary which will lead to our first main result, the convergence of the numerical solutions to an entropy solution, see Theorem 1 in Sect. 4. Our second main result, the L^1 stability and thus uniqueness of entropy solutions, is shown in Sect. 5, Theorem 2, using Kružkovs 'doubling of variables' technique. Finally, the last section provides some numerical experiments.

2 The numerical scheme

We discretize the domain $[0,1]\times[0,T]$ using $(N+1)\cdot(M+2)$ grid points with $\Delta x = 1/N$ and $\Delta t = \frac{T}{M+1}$, such that for $j=0,\dots,N$ and $n=0,\dots,M+1$,

$$u_j^n \approx u(x_j,t^n), \qquad \text{where } x_j = j\Delta x \text{ and } t^n = n\Delta t.$$

As a shorthand notation for the sequence $(u_j^n)_{j=0}^N$ we will write u^n. We will also frequently use the notation $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ for the interval in space, $I^n = [t^n, t^{n+1})$ for the interval in time and $I_j^n = I_j \times I^n$ for the rectangle in $[0,1]\times[0,T]$. Here, we fix the convention that $x_{j+\frac{1}{2}} = (j+\frac{1}{2})\Delta x$, $j=0,\dots,N-1$, as well as $x_{-\frac{1}{2}} = x_0 = 0$ and $x_{N+\frac{1}{2}} = x_N = 1$. In order to get from the sequence u^n to a function on $[0,1]\times[0,T]$ we define the piecewise constant interpolation

$$u_{\Delta t}(x,t) = u_j^n, \qquad \text{for } (x,t)\in I_j^n.$$

The discrete initial datum u^0 is constructed from $u_0 \in \mathrm{B.V.}(0,1)$ via

$$u_j^0 = \frac{1}{\Delta x}\int_{I_j} u_0(x)\,\mathrm{d}x, \quad \text{for } j=0,\dots,N.$$

Then, the numerical scheme we want to employ reads as follows: For $n \geq 0$ we set

$$\begin{cases} u_0^{n+1} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \alpha(s)\,\mathrm{ds}, \\ u_N^{n+1} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \beta(s)\,\mathrm{ds}, \\ u_j^{n+1} = u_j^n - \lambda\left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n\right) + \gamma\,\Delta t\,P_j^n \qquad \text{if } j = 1,\ldots,N-1, \end{cases} \tag{8}$$

where P_j^n is the following approximation to the integral of u,

$$P_j^n = \Delta x\left(\frac{1}{2}u_0^n + \sum_{i=1}^{j-1} u_i^n + \frac{1}{2}u_j^n\right),$$

and the flux at $(x_{j+\frac{1}{2}}, t^n)$ is approximated by

$$F_{j+\frac{1}{2}}^n = F(u_j^n, u_{j+1}^n), \tag{9}$$

where the discrete flux F is a Lipschitz continuous function in two variables. We will assume that F can be written in the form

$$F(u, v) = F_1(u) + F_2(v),$$

where $F_1, F_2 \in \mathcal{C}^1(\mathbb{R})$, and that F is consistent with f and monotone in the sense that

$$F(u, u) = f(u) \quad \text{and} \quad F_1' \geq 0,\; F_2' \leq 0. \tag{10}$$

Furthermore, we will assume

$$\max_u \lambda(F_1'(u) - F_2'(u)) \leq 1, \tag{11}$$

where $\lambda = \frac{\Delta t}{\Delta x}$. Two examples for discrete flux functions with the assumed properties are the Lax–Friedrichs flux, i.e.

$$F_1(u) = \frac{1}{2}f(u) + \frac{1}{2\lambda}u, \qquad F_2(v) = \frac{1}{2}f(v) - \frac{1}{2\lambda}v,$$

and the Engquist–Osher flux, i.e.

$$F_1(u) = \int_0^u \max(f'(z), 0)\,\mathrm{dz} + f(0), \qquad F_2(v) = \int_0^v \min(f'(z), 0)\,\mathrm{dz} + f(0),$$

which satisfy (10) and (11) provided that the grid satisfies the CFL condition

$$\max_u |f'(u)|\lambda \leq 1.$$

Note that, using our scheme, we can recover a discrete version of (3), since

$$D_+^t u_j^n + D_- F_{j+\frac{1}{2}}^n = \gamma P_j^n$$
$$D_- P_j^n = \frac{1}{2}\left(u_j^n + u_{j-1}^n\right),$$

were we used the following difference operators:

$$D_+^t a^n = \frac{1}{\Delta t}\left(a^{n+1} - a^n\right) \qquad \text{and} \qquad D_- a_j = \frac{1}{\Delta x}\left(a_j - a_{j-1}\right).$$

3 Discrete a priori estimates

In this section we aim to prove compactness of the scheme using Helly's theorem. This requires an L^∞ bound, a BV bound and a bound on the discrete time derivative of the numerical solution. These bounds are similar to the ones in [6], but the boundary conditions lead to additional terms.

Lemma 1 (L^∞ bound) *For $n\Delta t \leq T$, the solution u^n of the numerical scheme* (8) *satisfies*

$$\|u^n\|_\infty \leq e^{\gamma T}\left(\left\|u^0\right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty\right).$$

Proof For $j = 1, \ldots, N-1$ we define $v_j^n = \|u^n\|_\infty$. Then $v_j^n \geq u_j^n$ for all $j = 1, \ldots, N-1$ and thus, by monotonicity and consistency of the scheme (10)–(11),

$$\begin{aligned}
u_j^n &- \lambda(F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)) \\
&= u_j^n - \lambda(F_1(u_j^n) - F_2(u_j^n)) + \lambda F_1(u_{j-1}^n) - \lambda F_2(u_{j+1}^n) \\
&\leq v_j^n - \lambda(F_1(v_j^n) - F_2(v_j^n)) + \lambda F_1(v_{j-1}^n) - \lambda F_2(v_{j+1}^n) \\
&= v_j^n - \lambda(F(v_j^n, v_{j+1}^n) - F(v_{j-1}^n, v_j^n)) \\
&= v_j^n.
\end{aligned}$$

Hence, we have

$$|u_j^{n+1}| \leq \|u^n\|_\infty + \gamma\Delta t |P_j^n|.$$

for $j = 1, 2, \ldots, N-1$. Because $N\Delta x = 1$, also P_j^n is bounded:

$$\left|P_j^n\right| \leq \Delta x \sum_{i=0}^{j} |u_i^n| \leq N\Delta x \left\|u^n\right\|_\infty = \left\|u^n\right\|_\infty. \tag{12}$$

Regarding the boundary terms, clearly

$$|u_0^{n+1}| \leq \|\alpha\|_\infty \qquad \text{as well as} \qquad |u_N^{n+1}| \leq \|\beta\|_\infty.$$

Thus, we have

$$\begin{aligned}\left\|u^n\right\|_\infty &\le (1+\gamma\Delta t)^n\left(\left\|u^0\right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty\right)\\ &\le e^{\gamma n\Delta t}\left(\left\|u^0\right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty\right)\\ &\le e^{\gamma T}\left(\left\|u^0\right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty\right)\end{aligned}$$

for $n\Delta t \le T$. □

The next lemma is a version of Harten's lemma [16] on bounded domains that additionally uses the L^∞ bound from Lemma 1 to estimate the contribution of the source term to the total variation.

Lemma 2 (B.V. bound) *For $n\Delta t \le T$, the solution u^n of the numerical scheme (8) satisfies*

$$\begin{aligned}|u^n|_{\mathrm{B.V.}(0,1)} \le C_T\Big(&|u^0|_{\mathrm{B.V.}(0,1)} + |\alpha|_{\mathrm{B.V.}(0,1)} + |\beta|_{\mathrm{B.V.}(0,1)}\\ &+ \left\|u^0\right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty\Big)\end{aligned}$$

where C_T denotes a constant depending on γ and T.

Proof For $n = 0, \ldots, M$, we have

$$\begin{aligned}|u^{n+1}|_{\mathrm{B.V.}(0,1)} &= \sum_{j=0}^{N-1}|u_{j+1}^{n+1} - u_j^{n+1}|\\ &= |u_1^{n+1} - u_0^{n+1}| + \sum_{j=1}^{N-2}|u_{j+1}^{n+1} - u_j^{n+1}| + |u_N^{n+1} - u_{N-1}^{n+1}|. \qquad (13)\end{aligned}$$

The scheme (8) can then be written in conservative form, i.e. for $j = 1, \ldots, N-1$ we have

$$u_j^{n+1} = u_j^n + C_{j+\frac{1}{2}}^n(u_{j+1}^n - u_j^n) - D_{j-\frac{1}{2}}^n(u_j^n - u_{j-1}^n) + \gamma\Delta t P_j^n,$$

where

$$\begin{aligned}C_{j+\frac{1}{2}}^n &= \lambda\frac{f(u_j^n) - F_{j+\frac{1}{2}}^n}{u_{j+1}^n - u_j^n} \quad \text{for } j = 1, \ldots, N-1\\ D_{j+\frac{1}{2}}^n &= \lambda\frac{f(u_{j+1}^n) - F_{j+\frac{1}{2}}^n}{u_{j+1}^n - u_j^n} \quad \text{for } j = 0, \ldots, N-2.\end{aligned}$$

Using the consistency of the numerical flux and the mean value theorem, we get

$$C^n_{j+\frac{1}{2}} = \lambda \frac{F(u^n_j, u^n_j) - F(u^n_j, u^n_{j+1})}{u^n_{j+1} - u^n_j}$$
$$= -\lambda F_2'(\xi) \geq 0$$

and similarly

$$D^n_{j+\frac{1}{2}} = \lambda \frac{F(u^n_{j+1}, u^n_{j+1}) - F(u^n_j, u^n_{j+1})}{u^n_{j+1} - u^n_j}$$
$$= \lambda F_1'(\zeta) \geq 0$$

for all j in $\{1, \ldots, N-1\}$ and $\{0, \ldots, N-2\}$ respectively. Furthermore, using the mean value theorem on the difference $F_1 - F_2$ a similar calculation shows that the CFL condition (11) assures $C^n_{j+\frac{1}{2}} + D^n_{j+\frac{1}{2}} \leq 1$ for $j = 1, \ldots, N-2$. Now, regarding the sum on the right hand side of (13), we can estimate

$$|u^{n+1}_{j+1} - u^{n+1}_j| \leq$$
$$\left| u^n_{j+1} - u^n_j + C^n_{j+\frac{3}{2}}(u^n_{j+2} - u^n_{j+1}) - (D^n_{j+\frac{1}{2}} + C^n_{j+\frac{1}{2}})(u^n_{j+1} - u^n_j) + D^n_{j-\frac{1}{2}}(u^n_j - u^n_{j-1}) \right|$$
$$+ \gamma \Delta t |P^n_{j+1} - P^n_j|$$

Regarding the first sum, we get

$$\sum_{j=1}^{N-2} \left| u^n_{j+1} - u^n_j + C^n_{j+\frac{3}{2}}(u^n_{j+2} - u^n_{j+1}) - (D^n_{j+\frac{1}{2}} + C^n_{j+\frac{1}{2}})(u^n_{j+1} - u^n_j) + D^n_{j-\frac{1}{2}}(u^n_j - u^n_{j-1}) \right|$$
$$\leq \sum_{j=1}^{N-2} C^n_{j+\frac{3}{2}} |u^n_{j+2} - u^n_{j+1}| + \sum_{j=1}^{N-2} (1 - D^n_{j+\frac{1}{2}} - C^n_{j+\frac{1}{2}}) |u^n_{j+1} - u^n_j| + \sum_{j=1}^{N-2} D^n_{j-\frac{1}{2}} |u^n_j - u^n_{j-1}|$$
$$= \sum_{j=2}^{N-1} C^n_{j+\frac{1}{2}} |u^n_{j+1} - u^n_j| + \sum_{j=1}^{N-2} (1 - D^n_{j+\frac{1}{2}} - C^n_{j+\frac{1}{2}}) |u^n_{j+1} - u^n_j| + \sum_{j=0}^{N-3} D^n_{j+\frac{1}{2}} |u^n_{j+1} - u^n_j|$$
$$= \sum_{j=1}^{N-2} |u^n_{j+1} - u^n_j| - C^n_{\frac{3}{2}} |u^n_2 - u^n_1| + C^n_{N-\frac{1}{2}} |u^n_N - u^n_{N-1}| - D^n_{N-\frac{3}{2}} |u^n_{N-1} - u^n_{N-2}|$$
$$+ D^n_{\frac{1}{2}} |u^n_1 - u^n_0|$$

On the other hand, regarding the boundary terms in (13), since $D^n_{\frac{1}{2}} \leq 1$ and (12), we find

$$
\begin{aligned}
&|u_1^{n+1} - u_0^{n+1}| \\
&\quad \leq |u_1^n - u_0^{n+1} - \lambda(F_{\frac{3}{2}} - F_{\frac{1}{2}})| + \gamma \Delta t |P_1^n| \\
&\quad \leq |u_0^{n+1} - u_0^n| + |u_1^n - u_0^n - \lambda(F_{\frac{3}{2}} - f(u_1^n) - (F_{\frac{1}{2}} - f(u_1^n)))| + \gamma \Delta t \left\| u^n \right\|_\infty \\
&\quad = |u_0^{n+1} - u_0^n| + |u_1^n - u_0^n + C_{\frac{3}{2}}^n (u_2^n - u_1^n) - D_{\frac{1}{2}}^n (u_1^n - u_0^n)| + \gamma \Delta t \left\| u^n \right\|_\infty \\
&\quad \leq |u_0^{n+1} - u_0^n| + C_{\frac{3}{2}}^n |u_2^n - u_1^n| + (1 - D_{\frac{1}{2}}^n)|u_1^n - u_0^n| + \gamma \Delta t \left\| u^n \right\|_\infty
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&|u_N^{n+1} - u_{N-1}^{n+1}| \\
&\quad \leq |u_N^{n+1} - u_N^n| + (1 - C_{N-\frac{1}{2}}^n)|u_N^n - u_{N-1}^n| + D_{N-\frac{3}{2}}^n |u^n_{N-1-u^n_{N-2}}| \\
&\qquad + \gamma \Delta t \left\| u^n \right\|_\infty
\end{aligned}
$$

Moreover, we will estimate the P term with the help of Lemma (1) as follows

$$
\begin{aligned}
\gamma \Delta t \sum_{j=1}^{N-2} |P_{j+1}^n - P_j^n| &= \gamma \Delta t \frac{\Delta x}{2} \sum_{j=1}^{N-2} |u_{j+1}^n + u_j^n| \\
&\leq \gamma \Delta t (\Delta x N) \left\| u^n \right\|_\infty \\
&\leq \gamma \Delta t e^{\gamma T} \left(\left\| u^0 \right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty \right).
\end{aligned}
$$

In summary we get

$$
\begin{aligned}
|u^{n+1}|_{\text{B.V.}(0,1)} &\leq |u^n|_{\text{B.V.}(0,1)} + |u_0^{n+1} - u_0^n| + |u_N^{n+1} - u_N^n| \\
&\quad + \gamma \Delta t e^{\gamma T} \left(\left\| u^0 \right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty \right).
\end{aligned}
$$

Furthermore, we note that

$$
\sum_{n=0}^{M} |u_0^{n+1} - u_0^n| \leq |\alpha|_{\text{B.V.}(0,T)},
$$

and similarly for the right boundary. Therefore we get

$$
|u^n|_{\text{B.V.}(0,1)} \leq C_T \left(|u^0|_{\text{B.V.}(0,1)} + |\alpha|_{\text{B.V.}(0,1)} + |\beta|_{\text{B.V.}(0,1)} + \left\| u^0 \right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty \right).
$$

□

Lastly, we have a bound on the discrete time derivative of $u_{\Delta t}$.

Lemma 3 (Bound of the time derivative) *For $n\Delta t \leq T$, the solution of the numerical scheme (8) satisfies*

$$\Delta x \sum_{j=0}^{N} \left| D_+^t u_j^n \right| \leq C_\lambda \left(|u^0|_{\mathrm{B.V.}(0,1)} + \left\| u^0 \right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty \right),$$

where C_λ depends on γ, T, the Lipschitz constant of the discrete flux F and λ.

Proof Using the definition of the numerical scheme (8), the Lipschitz continuity of F, the L^∞ bound for P as seen in (12), and the BV and L^∞ bounds of u^n from Lemma 2 and 1, we get

$$\begin{aligned}
\Delta x \sum_{j=0}^{N} |D_+^t u_j^n| &= \Delta x \sum_{j=1}^{N-1} |D_+^t u_j^n| + \Delta x |D_+^t u_0^n| + \Delta x |D_+^t u_N^n| \\
&\leq \Delta x \sum_{j=1}^{N-1} |D_- F(u_j^n, u_{j+1}^n)| + \gamma \Delta x \sum_{j=1}^{N-1} |P_j^n| + \frac{1}{\lambda} \left(|\alpha|_{\mathrm{B.V.}(0,1)} + |\beta|_{\mathrm{B.V.}(0,1)} \right) \\
&\leq C \Delta x \sum_{j=1}^{N-1} \left(|D_- u_j^n| + |D_- u_{j+1}^n| \right) + \gamma \Delta x N \left\| u^n \right\|_\infty \\
&\quad + \frac{1}{\lambda} \left(|\alpha|_{\mathrm{B.V.}(0,1)} + |\beta|_{\mathrm{B.V.}(0,1)} \right) \\
&\leq C \left(|u^n|_{\mathrm{B.V.}(0,1)} + \left\| u^n \right\|_\infty \right) + \frac{1}{\lambda} \left(|\alpha|_{\mathrm{B.V.}(0,1)} + |\beta|_{\mathrm{B.V.}(0,1)} \right) \\
&\leq C_\lambda \left(|u^0|_{\mathrm{B.V.}(0,1)} + |\alpha|_{\mathrm{B.V.}(0,1)} + |\beta|_{\mathrm{B.V.}(0,1)} + \left\| u^0 \right\|_\infty + \|\alpha\|_\infty + \|\beta\|_\infty \right)
\end{aligned}$$

□

With the help of these three bounds we finally can apply a version of Helly's theorem to show compactness of the scheme.

Lemma 4 (Convergence) *Let $u_{\Delta t}$ be the family of solutions of the numerical scheme (8) defined by $u_{\Delta t}(x,t) = u_j^n$ for $(x,t) \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}) \times [t^n, t^{n+1})$. Further, let $\lambda = \frac{\Delta t}{\Delta x}$ be fixed such that the discrete flux satisfies (10) and (11). Then there is a sequence Δt_k and a function $u \in \mathrm{Lip}([0,T]; \mathrm{L}^1(0,1))$ such that $\Delta t_k \to 0$ and $u_{\Delta t_k}$ converges to u in $\mathscr{C}([0,T]; \mathrm{L}^1(0,1))$.*

Proof We want to apply Helly's theorem [17, Theorem A.11]. This requires an L^∞ bound, a bound on the variation in space that is independent of Δt, and L^1 continuity in time as $\Delta t \to 0$. An application of Lemma 1 gives

$$\begin{aligned}
\|u_{\Delta t}(\cdot,t)\|_{\mathrm{L}^\infty(0,1)} &\leq e^{\gamma T} \left(\left\| u^0 \right\|_{\mathrm{L}^\infty(0,1)} + \|\alpha\|_\infty + \|\beta\|_\infty \right) \\
&\leq e^{\gamma T} \left(\|u_0\|_{\mathrm{L}^\infty(0,1)} + \|\alpha\|_\infty + \|\beta\|_\infty \right).
\end{aligned}$$

Furthermore, by using Lemma 2, we find

$$
\begin{aligned}
\|u_{\Delta t}(\cdot+\varepsilon,t)-u_{\Delta t}(\cdot,t)\|_{\mathrm{L}^1(0,1)} &\leq \varepsilon |u_{\Delta t}(\cdot,t)|_{\mathrm{B.V.}(0,1)}\\
&\leq \varepsilon\left(|u^0|_{\mathrm{B.V.}(0,1)}+C\left(\left\|u^0\right\|_\infty+\|\alpha\|_\infty+\|\beta\|_\infty\right)\right)\\
&\leq \varepsilon\left(|u_0|_{\mathrm{B.V.}(0,1)}+C\left(\|u_0\|_\infty+\|\alpha\|_\infty+\|\beta\|_\infty\right)\right)\\
&\to 0, \qquad \text{as } \varepsilon\to 0 \text{ uniformly in } \Delta t.
\end{aligned}
$$

Finally, in order to show continuity in time, we employ Lemma 3. For $t\in[t^n,t^{n+1})$ and $s\in[t^{\overline{n}},t^{\overline{n}+1})$ with $n>\overline{n}$ we find

$$
\begin{aligned}
\int_0^1 |u_{\Delta t}(x,t)-u_{\Delta t}(x,s)|\,\mathrm{dx} &= \Delta x\sum_{j=0}^{N}|u_j^n-u_j^{\overline{n}}|\\
&\leq \Delta x\sum_{l=\overline{n}}^{n-1}\sum_{j=0}^{N}|u_j^{l+1}-u_j^l|\\
&= \Delta t\sum_{l=\overline{n}}^{n-1}\Delta x\sum_{j=0}^{N}|D_+^t u_j^l|\\
&\leq \Delta t(n-\overline{n})C_\lambda\left(|u^0|_{\mathrm{B.V.}(0,1)}+\left\|u^0\right\|_\infty+\|\alpha\|_\infty+\|\beta\|_\infty\right)\\
&= (t^n-t^{\overline{n}})C_\lambda\left(|u^0|_{\mathrm{B.V.}(0,1)}+\left\|u^0\right\|_\infty+\|\alpha\|_\infty+\|\beta\|_\infty\right)\\
&\leq C_\lambda|t-s|+\mathcal{O}(\Delta t).
\end{aligned}
$$

An application of Helly's theorem assures the existence of a sequence $\Delta t_k\to 0$ and a function $u\in \mathrm{Lip}([0,T];\mathrm{L}^1(0,1))$ such that such that $u_{\Delta t_k}$ converges to u in the space $\mathscr{C}([0,T];\mathrm{L}^1(0,1))$ as $k\to\infty$. ☐

4 Convergence towards the entropy solution

In this section we prove that the numerical scheme converges to an entropy solution of the Ostrovsky–Hunter equation. This fact hinges on discrete entropy inequalities for the interior of the domain and the boundary. These inequalities require a discrete version of the entropy flux that is consistent with the numerical flux function (9):

$$
Q^n_{j+\frac{1}{2}}=Q(u_j^n,u_{j+1}^n),\qquad Q^n_{j-\frac{1}{2}}=Q(u_{j-1}^n,u_j^n), \tag{14}
$$

where

$$
Q(u,v)=\int_c^u \eta'(z)F_1'(z)\,\mathrm{d}z+\int_c^v \eta'(z)F_2'(z)\,\mathrm{d}z,
$$

and $c\in\mathbb{R}$ is an arbitrary constant. Note that since F_1 and F_2 are Lipschitz continuous, and if η' is bounded, also Q is Lipschitz continuous in both variables.

We will now derive discrete versions of the entropy conditions (6) and (7). The entropy condition in the interior of the domain has already been proven in [6].

Lemma 5 (Discrete entropy inequalities) *For any convex entropy $\eta \in \mathscr{C}^2(\mathbb{R})$ with entropy flux q given by $q' = \eta' f'$, let $Q^n_{j+\frac{1}{2}}$ and $Q^n_{j-\frac{1}{2}}$ be defined by* (14). *Then the solutions of the scheme satisfies for each n*

$$D^t_+ \eta^n_j + D_- Q^n_{j+\frac{1}{2}} - \gamma \eta^{\prime,n+1}_j P^n_j \leq 0 \tag{15}$$

for $j = 1, \dots, N-1$, as well as

$$Q^n_{\frac{1}{2}} - q(u^n_0) - \eta'(u^n_0)(F^n_{\frac{1}{2}} - f(u^n_0)) \leq 0 \tag{16}$$

and

$$Q^n_{N-\frac{1}{2}} - q(u^n_N) - \eta'(u^n_N)(F^n_{N-\frac{1}{2}} - f(u^n_N)) \geq 0.$$

Proof The first inequality is derived in [6, Lemma 5] (see also [19, Lemma 6.1]). For the second inequality we use a Taylor approximation and the convexity of the flux

$$\begin{aligned}
&Q^n_{\frac{1}{2}} - q(u^n_0) - \eta'(u^n_0)(F^n_{\frac{1}{2}} - f(u^n_0))\\
&\quad = Q(u^n_0, u^n_1) - Q(u^n_0, u^n_0) - \eta'(u^n_0)(F(u^n_0, u^n_1) - F(u^n_0, u^n_0))\\
&\quad = \int_c^{u^n_1} \eta'(z) F_2'(z)\,\mathrm{dz} - \int_c^{u^n_0} \eta'(z) F_2'(z)\,\mathrm{dz} - \eta'(u^n_0)(F_2(u^n_1) - F_2(u^n_0))\\
&\quad = \int_{u^n_0}^{u^n_1} \eta'(z) F_2'(z)\,\mathrm{dz} - \int_{u^n_0}^{u^n_1} \eta'(u^n_0) F_2'(z)\,\mathrm{dz}\\
&\quad = \int_{u^n_0}^{u^n_1} \eta''(\xi)(z - u^n_0) F_2'(z)\,\mathrm{dz}\\
&\quad = \operatorname{sign}(u^n_1 - u^n_0) \int_{\min(u^n_0, u^n_1)}^{\max(u^n_0, u^n_1)} \eta''(\xi)(z - u^n_0) F_2'(z)\,\mathrm{dz}\\
&\quad = \int_{\min(u^n_0, u^n_1)}^{\max(u^n_0, u^n_1)} \eta''(\xi)|z - u^n_0| F_2'(z)\,\mathrm{dz} \leq 0
\end{aligned}$$

The proof of the third inequality can be done analogously. □

Thus far, we only know that a sequence of solutions of the numerical scheme (8) converges to some $u \in \mathscr{C}([0, T]; \mathrm{L}^1(0, 1))$. By passing to the limit in the discrete entropy conditions of Lemma 5 we can now show that u is in fact an entropy solution. To accomplish that we will employ similar techniques as in [6] in regards to the entropy condition and as in [31] in regards to the entropy boundary condition. While the following theorem only provides the convergence of a subsequence of $u_{\Delta t}$, the uniqueness result in Sect. 5 ensures that the whole sequence converges to the unique entropy solution.

Theorem 1 (Convergence towards the entropy solution) *Let $u_0 \in$ B.V.$(0, 1)$ and $\alpha, \beta \in$ B.V.$(0, T)$ and fix $\lambda = \frac{\Delta t}{\Delta x}$ such that the discrete flux in the scheme defined by*

(8) *satisfies the* (10) *and* (11). *Then for any sequence* $(\Delta t_n)_n$ *such that* $\Delta t_n \to 0$, *there is a subsequence* Δt_{n_k} *such that the piecewise constant interpolations* $u_{\Delta t_{n_k}}$ *defined by the scheme* (8) *converge in* $\mathscr{C}([0,T];\mathrm{L}^1(0,1))$ *towards an entropy solution of the Ostrovsky–Hunter equation as* $k \to \infty$.

Proof Let $(u_{\Delta t_{n_k}})$ be a sequence of approximate solutions that converges to u in the space $\mathscr{C}([0,T];\mathrm{L}^1(0,1))$ as $\Delta t_{n_k} \to 0$ (cf. Lemma 4). For simplicity, we will omit any indices on Δt. According to Lemma 5, the function $u_{\Delta t}$ satisfies the discrete entropy and entropy boundary conditions.

First, we show that u satisfies the entropy condition (6). Multiplying the discrete entropy condition (15) by $\Delta t \Delta x \phi_j^n$, where $\phi_j^n = \frac{1}{\Delta t \Delta x}\iint_{I_j^n} \phi(x,t)\,\mathrm{dx}\,\mathrm{dt}$ for some nonnegative test function $\phi \in \mathscr{C}_c^\infty((0,1)\times\mathbb{R})$, and taking the sum over $n = 0,\ldots,M$ and $j = 1,\ldots,N-1$ gives

$$
\begin{aligned}
0 &\geq \Delta t \Delta x \sum_{n=0}^{M}\sum_{j=1}^{N-1}\left(\phi_j^n D_+^t \eta_j^n + \phi_j^n D_- Q_{j+\frac{1}{2}}^n - \gamma \phi_j^n \eta_j^{\prime,n+1} P_j^n\right)\\
&= \Delta x \sum_{j=1}^{N-1}(\phi_j^{M+1}\eta_j^{M+1} - \phi_j^0\eta_j^0) - \Delta t \Delta x \sum_{n=0}^{M}\sum_{j=1}^{N-1}\eta_j^{n+1} D_+^t \phi_j^n \\
&\quad - \Delta t \Delta x \sum_{n=0}^{M}\sum_{j=1}^{N} Q_{j-\frac{1}{2}}^n D_- \phi_j^n - \gamma \Delta t \Delta x \sum_{n=0}^{M}\sum_{j=1}^{N-1}\phi_j^n \eta_j^{\prime,n+1} P_j^n,
\end{aligned} \tag{17}
$$

where we have used that $\phi_0^n = \phi_N^n = 0$ for Δx small enough. As in [6] we can pass to the limit $\Delta t \to 0$ in inequality (17).

More precisely, the continuity of η and the convergence of $u_{\Delta t}$ imply that $\eta(u_{\Delta t})$ converges to $\eta(u)$ in $\mathscr{C}([0,T];\mathrm{L}^1(0,1))$. On the other hand, since both the numerical and continuous entropy fluxes are Lipschitz continuous and $u_{\Delta t}(\cdot,t)$ has bounded variation for all $t \in [0,T]$, we find

$$
\begin{aligned}
&\sum_{n=0}^{M}\sum_{j=1}^{N}\iint_{I_j^n}\left|Q_{j-\frac{1}{2}}^n - q(u(x,t))\right|\mathrm{dx}\,\mathrm{dt}\\
&\quad\leq \sum_{n=0}^{M}\sum_{j=1}^{N}\iint_{I_j^n}\left(|Q_{j-\frac{1}{2}}^n - q(u_j^n)| + |q(u_j^n) - q(u(x,t))|\right)\mathrm{dx}\,\mathrm{dt}\\
&\quad\leq C\sum_{n=0}^{M}\sum_{j=1}^{N}\iint_{I_j^n}\left(|u_{j-1}^n - u_j^n| + |u_j^n - u(x,t)|\right)\mathrm{dx}\,\mathrm{dt}\\
&\quad\leq C_T \Delta x + C\int_0^T\int_0^1 |u_{\Delta t} - u|\,\mathrm{dx}\,\mathrm{dt} \to 0.
\end{aligned}
$$

Finally, the L^1 convergence of $u_{\Delta t}$ implies L^∞ convergence of the P term, since for $x \in I_j$ we have

$$
\begin{aligned}
|P_j^n - P[u](x,t)| &= \left| \Delta x \left(\sum_{i=0}^{j-1} u_i^n + \frac{1}{2} u_j^n \right) - \int_0^x u(y,t)\, \mathrm{dy} \right| \\
&\leq \int_0^x |u_{\Delta t}(y,t) - u(y,t)|\, \mathrm{dy} + C \Delta x \, \|u_{\Delta t}(\cdot,t)\|_{\mathrm{L}^\infty(0,1)} \\
&\leq \|u_{\Delta t}(\cdot,t) - u(\cdot,t)\|_{\mathrm{L}^1(0,1)} + C\Delta x \, \|u_{\Delta t}(\cdot,t)\|_{\mathrm{L}^\infty(0,1)} \to 0.
\end{aligned}
$$

Thus we can pass to the limit $\Delta t \to 0$ in (17) and get

$$
\begin{aligned}
0 \geq \int_0^1 \eta(u(x,T))\phi(x,T)\, \mathrm{dx} - \int_0^1 \eta(u(x,0))\phi(x,0)\, \mathrm{dx} \\
- \int_0^T \int_0^1 \left(\eta(u)\phi_t + q(u)\phi_x + \gamma \eta'(u) P[u]\phi \right) \mathrm{dx}\, \mathrm{dt}
\end{aligned}
$$

and therefore u satisfies the entropy condition in the interior of the domain.

Regarding the entropy boundary condition (7), rearranging (15) yields

$$
Q^n_{j+\frac{1}{2}} \leq Q^n_{j-\frac{1}{2}} - \Delta x D^t_+ \eta^n_j + \gamma \Delta x P^n_j \eta'^{,n+1}_j
$$

Multiplying by $\Delta t \psi^n$, where $\psi^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \psi(s)\, \mathrm{ds}$ for some nonnegative test function $\psi \in \mathscr{C}^1_c([0,T])$, and summing over $n = 0, \ldots, M$, we get

$$
\begin{aligned}
\Delta t \sum_{n=0}^M Q^n_{j+\frac{1}{2}} \psi^n &\leq \Delta t \sum_{n=0}^M Q^n_{j-\frac{1}{2}} \psi^n - \Delta x \Delta t \sum_{n=0}^M D^t_+ \eta^n_j \psi^n \\
&\quad + \gamma \Delta x \Delta t \sum_{n=0}^M P^n_j \eta'^{,n+1}_j \psi^n \\
&= \Delta t \sum_{n=0}^M Q^n_{j-\frac{1}{2}} \psi^n + \Delta x \Delta t \sum_{n=0}^M \underbrace{\eta^{n+1}_j}_{\leq \|\eta'\|_\infty \|u_{\Delta t}\|_\infty + C} \underbrace{D^t_+ \psi^n}_{\|\psi'\|_\infty} \\
&\quad + \gamma \Delta x \Delta t \sum_{n=0}^M P^n_j \eta'^{,n+1}_j \psi^n \\
&\leq \Delta t \sum_{n=0}^M Q^n_{j-\frac{1}{2}} \psi^n + CT\Delta x + \gamma \Delta x \Delta t \sum_{n=0}^M P^n_j \eta'^{,n+1}_j \psi^n.
\end{aligned}
$$

Repeating this argument and using the discrete entropy boundary condition (16) yields

$$
\Delta t \sum_{n=0}^M Q^n_{j+\frac{1}{2}} \psi^n \leq \Delta t \sum_{n=0}^M Q^n_{\frac{1}{2}} \psi^n + jCT\Delta x + \gamma \Delta x \Delta t \sum_{i=1}^j \sum_{n=0}^M P^n_i \eta'^{,n+1}_i \psi^n
$$

$$\leq \Delta t \sum_{n=0}^{M} (q(u_0^n) + \eta'(u_0^n)(F_{\frac{1}{2}}^n - f(u_0^n)))\psi^n + jCT\Delta x$$
$$+ \gamma \Delta x \Delta t \sum_{i=1}^{j} \sum_{n=0}^{M} P_i^n \eta_i^{\prime,n+1} \psi^n. \tag{18}$$

In order to recover the entropy boundary condition (7) we now pass to the limit $\Delta t \to 0$ and then $x \to 0$.

Firstly, since $u_{\Delta t}$ converges to u in $\mathscr{C}([0,T]; \mathrm{L}^1(0,1))$ and thus also in $\mathrm{L}^1((0,1) \times (0,T))$, using the Lipschitz continuity of Q, we find

$$\begin{aligned}
&\sum_{n=0}^{M} \sum_{j=0}^{N} \iint_{I_j^n} |Q_{j+\frac{1}{2}} - q(u(x,t))| \,\mathrm{dx}\,\mathrm{dt} \\
&\leq \sum_{n=0}^{M} \sum_{j=0}^{N} \iint_{I_j^n} (|Q(u_j^n, u_{j+1}^n) \\
&\quad - Q(u(x,t), u_{j+1}^n)| + |Q(u(x,t), u_{j+1}^n) - q(u(x,t))|) \,\mathrm{dx}\,\mathrm{dt} \\
&\leq C \sum_{n=0}^{M} \sum_{j=0}^{n} \iint_{I_j^n} (|u_j^n - u(x,t)| + |u_{j+1}^n - u(x,t)|) \,\mathrm{dx}\,\mathrm{dt} \\
&\leq C \sum_{n=0}^{M} \sum_{j=0}^{n} \iint_{I_j^n} (|u_{\Delta t}(x,t) - u(x,t)| + |u_{\Delta t}(x+\Delta x, t) - u_{\Delta t}(x,t)|) \,\mathrm{dx}\,\mathrm{dt} \\
&\leq C \left(\int_0^T \int_0^1 |u_{\Delta t}(x,t) - u(x,t)| \,\mathrm{dx}\,\mathrm{dt} + T\Delta x \sup_{0 \leq n \leq M+1} |u^n|_{\mathrm{B.V.}(0,1)} \right) \to 0.
\end{aligned}$$

Thus the left hand side of (18) converges to $\int_0^T q(u(x,t))\psi(t)\,\mathrm{dt}$ for almost every $x \in (0,1)$.

Because of the Lipschitz continuity of F and the L^∞ bound in Lemma 1, the piecewise constant interpolation in time of the values $F_{\frac{1}{2}}^n$ is bounded in $\mathrm{L}^\infty(0,T)$. Thus there exists a subsequence such that $F_{\frac{1}{2}}^n \overset{*}{\rightharpoonup} \widetilde{f}_0(t)$ in $\mathrm{L}^\infty(0,T)$ for some $\widetilde{f}_0 \in \mathrm{L}^\infty(0,T)$.

Since $u_0^n = \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \alpha(s)\,\mathrm{ds}$ converges to $\alpha(t)$ for almost all $t \in (0,T)$, the continuity of q, η' and f assures convergence of the remaining terms on the right hand side of (18).

Thus, by passing to the limit $\Delta t \to 0$ in (18), we get

$$\int_0^T q(u(x,t))\psi(t)\,\mathrm{dt} \leq \int_0^T \left(q(\alpha(t)) + \eta'(\alpha(t)) \left(\widetilde{f}_0(t) - f(\alpha(t))\right)\right)\psi(t)\,\mathrm{dt} + CTx$$
$$+\gamma \int_0^x \int_0^T \eta'(u)P[u]\psi(t)\,\mathrm{dt}\,\mathrm{dx}\,.$$

Because $u(x,\cdot)$ is of bounded variation in time, we have strong convergence in $\mathrm{L}^1(0,T)$. The limit can only be the strong trace, i.e. $u(x,\cdot)\to u_0^\tau$, as $x\to 0$. Thus, by passing to the limit $x\to 0$ in the foregoing inequality, we get

$$\int_0^T q(u_0^\tau(t))\psi(t)\,\mathrm{d}t \le \int_0^T \left(q(\alpha(t))+\eta'(\alpha(t))\left(\widetilde{f}_0(t)-f(\alpha(t))\right)\right)\psi(t)\,\mathrm{d}t \tag{19}$$

and since $\psi\in\mathscr{C}_c^1([0,T])$ is arbitrary

$$q(u_0^\tau(t))\le q(\alpha(t))+\eta'(\alpha(t))\left(\widetilde{f}_0(t)-f(\alpha(t))\right)$$

for almost every $t\in(0,T)$. It remains to show that $\widetilde{f}_0(t)=f(u_0^\tau(t))$. By an approximation argument, (19) also holds true for Kružkov entropy pairs $\eta(u)=|u-k|, q(u)=\operatorname{sign}(u-k)(f(u)-f(k))$ with arbitrary $k\in\mathbb{R}$. Choosing $k>\max(u_0^\tau(t),\alpha(t))$ yields

$$-(f(u_0^\tau(t))-f(k))\le -(f(\alpha(t))-f(k))-\left(\widetilde{f}_0(t)-f(\alpha(t))\right)$$

and thus

$$f(u_0^\tau(t))\ge \widetilde{f}_0(t).$$

On the other hand, choosing $k<\min(u_0^\tau(t),\alpha(t))$ gives $f(u_0^\tau(t))\le\widetilde{f}_0(t)$, and therefore $\widetilde{f}_0(t)=f(u_0^\tau(t))$. This proves the entropy boundary condition at $x=0$. The boundary at $x=1$ can be handled similarly. □

5 $\mathbf{L^1}$ stability and uniqueness

We now want to prove L^1 stability of solutions following the 'doubling of variables' method introduced by Kružkov [20].

Theorem 2 (L^1 stability) *If u and v are entropy solutions of the Ostrovsky–Hunter equation with initial datum u_0 and v_0 respectively, then*

$$\|u(\cdot,T)-v(\cdot,T)\|_{\mathrm{L}^1(0,1)}\le e^{\gamma T}\|u_0-v_0\|_{\mathrm{L}^1(0,1)}\,.$$

In particular, this implies that entropy solutions to the initial-boundary value problem are unique.

Proof Let u and v be entropy solutions with initial datum u_0 and v_0 respectively. We will now consider the entropy inequality (6) with Kružkov entropy pairs and a nonnegative test function ϕ with support away from $t=0$ and $t=T$. By taking (6) for u in the variables (x,t) and for v in the variables (y,s) both with the test function $\phi(x,t,y,s)$, integrating each with respect to the respective other two variables and adding them we get

$$\begin{aligned}\int_0^T\int_0^1\int_0^T\int_0^1 \Big(&|u(x,t)-v(y,s)|(\phi_t+\phi_s)+q(u(x,t),v(y,s))(\phi_x+\phi_y)\\ &+\gamma\operatorname{sign}(u(x,t)-v(y,s))(P[u](x,t)-P[v](y,s))\phi\Big)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\ge 0\end{aligned}$$

Now, let $\phi = \psi(\frac{x+y}{2}, \frac{t+s}{2})\omega_\varepsilon(x-y)\omega_{\varepsilon_0}(t-s)$, where $0 \le \psi \le 1$ is a test function to be chosen later and $\omega_{\varepsilon,\varepsilon_0}$ are symmetric standard mollifiers. Then, using [17, Lemma 2.9], we find that the terms not involving P converge towards

$$\int_0^T \int_0^1 \Big(|u-v|\psi_t + q(u,v)\psi_x\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

as $\varepsilon, \varepsilon_0 \to 0$. Regarding the remaining term, we use

$$\begin{aligned}
|P[u](x,t) - P[v](y,s)| &\le |P[u](x,t) - P[v](x,s)| + |P[v](x,s) - P[v](y,s)| \\
&\le \|u(\cdot,t) - v(\cdot,s)\|_{\mathrm{L}^1(0,1)} + |x-y| \cdot \|v(\cdot,s)\|_{\mathrm{L}^\infty(0,1)} \\
&\le \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)} \\
&\quad + \|v(\cdot,t) - v(\cdot,s)\|_{\mathrm{L}^1(0,1)} + |x-y| \cdot \|v(\cdot,s)\|_{\mathrm{L}^\infty(0,1)}.
\end{aligned}$$

Hence, using that weak solutions of bounded variation are Lipschitz continuous in time [17, Theorem 7.10], we find

$$\begin{aligned}
&\int_0^T \int_0^1 \int_0^T \int_0^1 |P[u](x,t) - P[v](y,s)|\phi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)}\,\mathrm{d}t \\
&\qquad + \int_0^T \int_0^T \|v(\cdot,t) - v(\cdot,s)\|_{\mathrm{L}^1(0,1)}\,\omega_{\varepsilon_0}(t-s)\,\mathrm{d}t\,\mathrm{d}s \\
&\qquad + \int_0^T \int_0^1 \int_0^1 |x-y| \cdot \|v(\cdot,s)\|_{\mathrm{L}^\infty(0,1)}\,\omega_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \le \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)}\,\mathrm{d}t \\
&\qquad + C\int_0^T \int_0^T |t-s|\omega_{\varepsilon_0}(t-s)\,\mathrm{d}t\,\mathrm{d}s + \varepsilon\,\|v\|_{\mathrm{L}^\infty((0,1)\times(0,T))} \\
&\quad \le \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)}\,\mathrm{d}t + C_T(\varepsilon_0 + \varepsilon) \\
&\quad \to \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)}\,\mathrm{d}t
\end{aligned}$$

as $\varepsilon, \varepsilon_0 \to 0$. Consequently, u and v satisfy

$$\int_0^T \int_0^1 \Big(|u-v|\psi_t + q(u,v)\psi_x\Big)\,\mathrm{d}x\,\mathrm{d}t + \gamma\int_0^T \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)}\,\mathrm{d}t \ge 0.$$

Let now

$$\chi_{\delta,a}(\xi) = \int_0^\xi \big(\omega_{\delta/2}(\zeta - \delta/2) - \omega_{\delta/2}(\zeta - (a - \delta/2))\big)\,\mathrm{d}\zeta$$

which is a smooth approximation to $\chi_{[0,a]}$. Then we define $\psi(x,t) = \chi_{\delta,1}(x)\chi_{\delta,T}(t)$. Taking $\delta \to 0$, we get

$$\begin{aligned}
&\int_0^1 |u_0(x) - v_0(x)|\, \mathrm{dx} - \int_0^1 |u(x,T) - v(x,T)|\, \mathrm{dx} \\
&\quad + \gamma \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_{\mathrm{L}^1(0,1)}\, \mathrm{dt} \\
&\geq \int_0^T q(u_1^\tau(t), v_1^\tau(t))\, \mathrm{dt} - \int_0^T q(u_0^\tau(t), v_0^\tau(t))\, \mathrm{dt}\,. \qquad (20)
\end{aligned}$$

Note that by choosing

$$k(t) = \begin{cases} u_0^\tau(t) & \text{if } u_0^\tau(t) \in I[\alpha(t), v_0^\tau(t)] \\ \alpha(t) & \text{if } \alpha(t) \in I[v_0^\tau(t), u_0^\tau(t)] \\ v_0^\tau(t) & \text{if } v_0^\tau(t) \in I[u_0^\tau(t), \alpha(t)] \end{cases}$$

in the boundary entropy condition (7) we get

$$\begin{aligned}
q(u_0^\tau(t), v_0^\tau(t)) \leq \frac{1}{2} \big(& q(u_0^\tau(t)) - q(\alpha(t)) - \eta'(\alpha(t))(f(u_0^\tau(t)) - f(\alpha(t))) \\
&+ q(v_0^\tau(t)) - q(\alpha(t)) - \eta'(\alpha(t))(f(v_0^\tau(t)) - f(\alpha(t)))\big) \leq 0
\end{aligned}$$

and similarly $q(u_1^\tau(t), v_1^\tau(t)) \geq 0$ for a.e. t. Thus the right-hand side of (20) is non-negative. An application of Gronwall's lemma finishes the proof. □

6 Numerical experiments

In this section we want to conduct two numerical experiments to illustrate our results. Here, we choose $f(u) = u^2/2$ and $\gamma = 1$. Our first numerical experiment uses a well-studied travelling wave solution of the Ostrovsky–Hunter equation with initial datum given by the 'corner wave':

$$u_0(x) = \begin{cases} \frac{1}{6}(x - \frac{1}{2})^2 + \frac{1}{6}(x - \frac{1}{2}) + \frac{1}{36}, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{6}(x - \frac{1}{2})^2 - \frac{1}{6}(x - \frac{1}{2}) + \frac{1}{36}, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

The 'corner wave' consists of two parabolas forming a sharp corner at $x = \frac{1}{2}$ (cf. Fig. 1). The travelling wave solution is

$$u_{\mathrm{ex}}(x,t) = u_0\left(x - \frac{t}{36} - \left\lfloor x - \frac{t}{36} \right\rfloor\right)$$

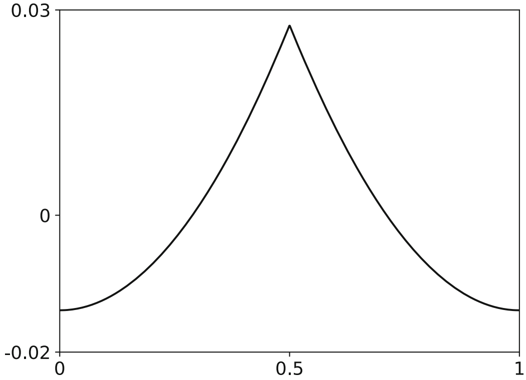

Fig. 1 Initial datum for both numerical experiments

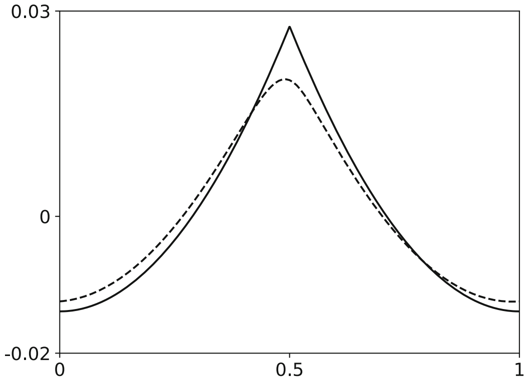

Fig. 2 Explicit and numerical solution for Experiment 1 at $T = 36$

which returns to its initial state after a period of $T = 36$. The 'corner wave' is the limit case of a family of smooth travelling wave solutions that has been investigated by several authors [3,18,27,29,33]. In this section we will not consider $P(0) = 0$, but $\int_0^1 P = 0$, which gives

$$(P[u])(x,t) = \int_0^x u(y,t)\,\mathrm{dy} - \int_0^1 \int_0^y u(z,t)\,\mathrm{dz}\,.$$

This is motivated by the fact that the latter choice limits the growth of the L^∞ norm of the solution for our experiments. Figure 2 shows the exact entropy solution and a numerical solution both at $T = 36$. The numerical solution is calculated by the Lax–Friedrichs method with boundary conditions set as the explicit solution at $x = 0$ and $x = 1$ respectively and a grid discretization parameter of $\Delta x = 2^{-7}$.

For this and all subsequent numerical experiments we use[1] $\Delta t/\Delta x = 25$. Additionally, for the first experiment the known exact entropy solution is used to calculate the error:

$$\mathrm{err}^1_{\mathrm{L}^1}(\Delta t) = \|u_{\Delta t}(\cdot, 36) - u_{\mathrm{ex}}(\cdot, 36)\|_{\mathrm{L}^1(0,1)}\,.$$

[1] Here, we have $\|f'(u_0)\|_{\mathrm{L}^\infty(0,1)} = 1/36$ and therefore $\lambda = \Delta t/\Delta x$ should satisfy $\lambda \leq 36$. However, since the L^∞ bound from Lemma 1 allows for some growth of $\|u^n\|_\infty$ choosing a smaller λ can be neccessary.

Table 1 L^1 errors and convergence rates for Experiment 1

Δx	Lax–Friedrichs	Rate	Engquist–Osher	Rate
2^{-6}	$2.84 \cdot 10^{-3}$		$1.39 \cdot 10^{-3}$	
2^{-7}	$1.72 \cdot 10^{-3}$	0.72	$6.92 \cdot 10^{-4}$	1.00
2^{-8}	$9.71 \cdot 10^{-4}$	0.82	$3.61 \cdot 10^{-4}$	0.94
2^{-9}	$5.32 \cdot 10^{-4}$	0.86	$1.90 \cdot 10^{-4}$	0.93
2^{-10}	$2.83 \cdot 10^{-4}$	0.91	$1.01 \cdot 10^{-4}$	0.91

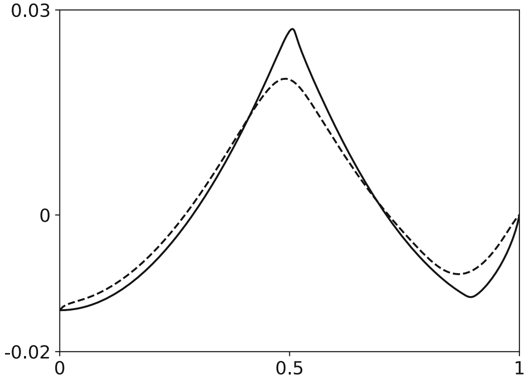

Fig. 3 Numerical solutions for Experiment 2 at $T = 36$ calculated with the Lax–Friedrichs flux and $\Delta x = 2^{-7}$ (dashed) and with the Engquist–Osher flux and $\Delta x^* = 2^{-11}$ (straight)

Table 1 shows the L^1 error between various numerical solutions and the exact solution, as well as the respective experimental convergence rates. Comparing these results to Table 1 in [6], we see that our numerical scheme is consistent with the periodic case.

In our second experiment we use the same initial datum, but set the right boundary datum to zero. Figure 3 displays two numerical solutions, one on a moderate mesh ($\Delta x = 2^{-7}$) calculated with the Lax–Friedrichs flux and one on a fine mesh ($\Delta x^* = 2^{-11}$) calculated with the Engquist–Osher flux. With no explicit entropy solution at hand we consider a numerical solution on a fine grid ($\Delta x^* = 2^{-11}$) in order to calculate the L^1 errors in the second experiment, i.e.,

$$\mathrm{err}^2_{L^1}(\Delta t) = \|u_{\Delta t}(\cdot, 36) - u_{\Delta t^*}(\cdot, 36)\|_{L^1(0,1)} .$$

Here, $u_{\Delta t}$ and $u_{\Delta t^*}$ are always calculated based on the same numerical method. Finally, in Table 2 we compare the L^1 errors between various numerical solutions and provide the experimental convergence rates. One clearly sees that the Engquist–Osher flux leads to a better approximation in this experiment. This is due to the fact that the homogeneous boundary condition at $x = 1$ constitutes a shock that propagates into the domain and that shocks are resolved better with the Engquist–Osher flux.

For conservation laws in $\mathbb{R}$ without source term the classical result concerning convergence rates in L^1, due to Kuznetsov [21], gives a convergence rate of $O(\Delta x^{1/2})$. The same convergence rate was shown in [6] for the Ostrovsky–Hunter equation with periodic boundary conditions. Although theoretical results estimating the convergence rate in the case of Dirichlet boundary conditions are highly desirable, such results are

Table 2 L^1 errors and convergence rates for Experiment 2

Δx	Lax–Friedrichs	Rate	Engquist–Osher	Rate
2^{-6}	$3.00 \cdot 10^{-3}$		$1.36 \cdot 10^{-3}$	
2^{-7}	$1.90 \cdot 10^{-3}$	0.66	$6.60 \cdot 10^{-4}$	1.04
2^{-8}	$1.16 \cdot 10^{-3}$	0.71	$3.24 \cdot 10^{-4}$	1.03
2^{-9}	$6.88 \cdot 10^{-4}$	0.75	$1.50 \cdot 10^{-4}$	1.11
2^{-10}	$4.05 \cdot 10^{-4}$	0.76	$5.83 \cdot 10^{-5}$	1.36

currently not at hand. However, in the absence of source terms Ohlberger and Vovelle [26] proof a rate of $O(\Delta x^{1/6})$ in a very general setting.

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