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Relative perturbation bounds for eigenpairs of diagonalizable matrices

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Abstract In this paper, we present some uniform relative perturbation bounds for eigenvalues and eigenspaces of diagonalizable matrices under additive and multiplicative perturbations. Some existing perturbation bounds can be improved based on the new bounds. Numerical experiments are given to demonstrate the advantage of the new bounds.

Keywords Diagonalizable matrix · Additive perturbation bound · Multiplicative perturbation bound · Eigenpair

Mathematics Subject Classification 65F10 · 65F15 · 65F35

1 Introduction

The numerical analysis for an eigenpair of a matrix plays an important role in science engineering computing, physical science et al. Usually, the numerical analysis for eigenpairs contains two aspects:

– algorithms for computing the eigenvalue and the eigenvector;

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– (absolute and relative) perturbation analysis for additive perturbations and multiplicative perturbations, which can lead to the condition number for computing the eigenvalue and the eigenvector.

There have been many significant results for the perturbation analysis (see, e.g., [20]). The classical bounds for the eigenpair perturbation are the Hoffman–Wielandt theorem for eigenvalues [6] and the sin Θ theorem for eigenspaces [3], respectively. Recently, some new perturbation bounds for eigenpairs have been obtained; see [1,2,4,7,8,12–14,16,19]. Some combined perturbations for matrix decompositions were established; see [2,14,16,17]. In particular, some combined perturbation bounds for eigenpairs of Hermitian matrices were presented; see [14,16]. In order to give combined bounds for eigenpairs of diagonalizable matrices, we first introduce some notations.

Let $\mathbf{C}^{m\times n}$ be the set of $m \times n$ complex matrices, and let $\langle n\rangle = \{1, 2, \ldots, n\}$. By A^* and I we denote the conjugate transpose of a matrix A and the identity matrix, respectively. The Frobenius norm, the spectral norm, the minimum and maximum singular value of a matrix are denoted by $\|\cdot\|_F$, $\|\cdot\|_2$, $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$, respectively.

Let both A and its perturbed matrix $\widetilde{A}$ be $n \times n$ diagonalizable matrices with the following eigendecompositions:

$$AX = X\Lambda \equiv \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \text{and} \quad \widetilde{A}\widetilde{X} = \widetilde{X}\widetilde{\Lambda} \equiv \begin{pmatrix} \widetilde{X}_1 & \widetilde{X}_2 \end{pmatrix} \begin{pmatrix} \widetilde{\Lambda}_1 & 0 \\ 0 & \widetilde{\Lambda}_2 \end{pmatrix}, \tag{1.1}$$

where X and $\widetilde{X} \in \mathbf{C}^{n\times n}$ are nonsingular, X_1 and $\widetilde{X}_1 \in \mathbf{C}^{n\times r}$, $1 \le r \le n$,

$$\Lambda_1 = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r), \quad \Lambda_2 = \operatorname{diag}(\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n), \tag{1.2}$$

$$\widetilde{\Lambda}_1 = \operatorname{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_r), \quad \widetilde{\Lambda}_2 = \operatorname{diag}(\widetilde{\lambda}_{r+1}, \widetilde{\lambda}_{r+2}, \ldots, \widetilde{\lambda}_n), \tag{1.3}$$

and λ_i and $\widetilde{\lambda}_j$ may be complex. Partition

$$X^{-1} = \begin{pmatrix} Y_1^* \\ Y_2^* \end{pmatrix} \quad \text{and} \quad \widetilde{X}^{-1} = \begin{pmatrix} \widetilde{Y}_1^* \\ \widetilde{Y}_2^* \end{pmatrix}, \tag{1.4}$$

where $Y_1, \widetilde{Y}_1 \in \mathbf{C}^{n\times r}$. Let

$$\delta_{ij}^{(l,k)} = \min_{\lambda\in\lambda(\Lambda_i),\widetilde{\lambda}\in\lambda(\widetilde{\Lambda}_j)} \frac{|\lambda - \widetilde{\lambda}|}{|\lambda|^l|\widetilde{\lambda}|^k}, \quad i, j = 1, 2. \tag{1.5}$$

For simplicity, we always use the notation $\delta_{ij} = \delta_{ij}^{(0,0)}$. Another relative gap is given by

$$\rho_{ij}^{(l,k)} = \min_{\lambda\in\lambda(\wedge_i),\widetilde{\lambda}\in\lambda(\widetilde{\wedge}_j)} \frac{|\lambda - \widetilde{\lambda}|}{|\lambda|^l|\widetilde{\lambda}|^k\sqrt{|\widetilde{\lambda}|^2 + |\lambda|^2}}, \quad i, j = 1, 2, \tag{1.6}$$

where k and l are nonnegative real numbers.

Let X_1 and $\widetilde{X}_1 \in \mathbf{C}^{n\times r}(n \geq r)$ have full column rank r. Then the angle matrix $\Theta(X_1, \widetilde{X}_1)$ between X_1 and $\widetilde{X}_1$ is defined by [20]:

$$\Theta(X_1, \widetilde{X}_1) = \arccos((X_1^* X_1)^{-\frac{1}{2}} X_1^* \widetilde{X}_1 (\widetilde{X}_1^* \widetilde{X}_1)^{-1} \widetilde{X}_1^* X_1 (X^* X_1)^{-\frac{1}{2}})^{\frac{1}{2}}.$$

In particular, if both X_1 and $\widetilde{X}_1$ have orthonormal columns, then for any unitarily invariant norm $\|\cdot\|$ we have

$$\|\sin\Theta(X_1, \widetilde{X}_1)\| = \|X_1^* \widetilde{X}_2\| = \|\widetilde{X}_1^* X_2\|,$$

where (X_1, X_2) and $(\widetilde{X}_1, \widetilde{X}_2)$ are $n \times n$ unitary matrices.

Let A and $\widetilde{A} = A + \Delta A$ have the decomposition (1.1)–(1.3). Here we consider the relative perturbation for the eigenpair.

For the relative perturbation of eigenvalues and eigenspaces, Ipsen presented the following general relative bounds

$$\sqrt{\sum_{i=1}^{n} \left|\widetilde{\lambda}_i^{-k} \lambda_{\tau(i)}^{1-l} - \widetilde{\lambda}_i^{1-k} \lambda_{\tau(i)}^{-l}\right|^2} \leq \kappa\left(\widetilde{X}\right) \kappa(X) \|\widetilde{A}^{-k} \Delta A A^{-l}\|_F \tag{1.7}$$

and

$$\|\sin\Theta(\widetilde{X}_1, X_1)\|_F \leq \kappa(\widetilde{Y}_2)\kappa(X_1) \frac{\|\widetilde{A}^{-k} \Delta A A^{-l}\|_F}{\delta_{12}^{(l,k)}}, \tag{1.8}$$

respectively, where $\kappa(B) = \|B\|_2 \|B^{-1}\|_2$ for any nonsingular matrix B or $\kappa(B) = \|B\|_2 \|B^{\dagger}\|_2$ for any matrix B (see Theorem 6.1 and Corollary 3.3 of [8]).

The idea of this paper is to combine (1.7) and (1.8) together into one formula, from which one may deduce some classical perturbation bounds for eigenvalues and eigenspaces, respectively.

The rest of this paper is organized as follows. In Sect. 2, we present the combined bound for eigenvalues and eigenspaces in the additive perturbation case. In Sect. 3, we consider the multiplicative perturbation case, and get the combined bound for the multiplicative perturbation. In Sect. 4, we give some numerical example to show the theoretical results. Some concluding remarks are given in the final section.

2 Relative bounds for additive perturbation

In this section we will get a relative combined perturbation bound for the eigenpair. First of all, we give some lemmas which will be used in this section.

The following Lemma 2.1 was implicitly hidden in the presentation of Sun [18], but formally stated in Lemma 2.2 of Li [9], and also can be found in [13] and [15].

Lemma 2.1 *Suppose that $X = (X_1, X_2) \in \mathbf{C}^{n\times n}$ is a nonsingular matrix, where $X_1 \in \mathbf{C}^{n\times m}$, and its inverse has the block form (1.4). Then for the 2-norm or Frobenius* *norm* $\|\cdot\|$ *and any full column matrix* $\widetilde{X}_1 \in \mathbf{C}^{n\times m}$,

$$\|\sin\Theta(X_1, \widetilde{X}_1)\| \le \|Y_2^{\dagger}\|_2\|\widetilde{X}_1^{\dagger}\|_2\|Y_2^*\widetilde{X}_1\|,$$

where by $M^{\dagger}$ *we denote the Moore–Penrose inverse of a matrix* M.

The following lemma is Theorem 3.2 in [5], which is further generalized to more general case in Proposition 3.1 of [11]; see also the last line of [5] for more details.

Lemma 2.2 [5] *Let* $T \in \mathbf{C}^{n\times n}$ *and* $\Lambda_i = diag(\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \in \mathbf{C}^{n\times n}$, $i = 1, 2, 3, 4$. *Then there exists a permutation* τ *of* $\langle n\rangle$ *such that*

$$\sigma_{min}^2(T)\sum_{i=1}^{n}\left|\lambda_i^{(1)}\lambda_{\tau(i)}^{(2)} - \lambda_i^{(3)}\lambda_{\tau(i)}^{(4)}\right|^2 \le \|\Lambda_1 T\Lambda_2 - \Lambda_3 T\Lambda_4\|_F^2.$$

Now we present a combined perturbation bound for the relative measure.

Theorem 2.3 *Let* A *and its perturbed matrix* $\widetilde{A}$ *be two* $n\times n$ *nonsingular diagonalizable matrices with the eigendecompositions* (1.1)–(1.3). *Then there exists a permutation* τ *of* $\langle r\rangle$ *such that*

$$\frac{\left(\delta_{12}^{(l,k)}\right)^2}{\|\widetilde{Y}_2^{\dagger}\|_2^2\|X_1^{\dagger}\|_2^2}\|\sin\Theta(\widetilde{X}_1, X_1)\|_F^2 + \sigma_{min}^2(\widetilde{Y}_1^* X_1)\sum_{i=1}^{r}|\widetilde{\lambda}_i^{-k}\lambda_{\tau(i)}^{1-l} - \widetilde{\lambda}_i^{1-k}\lambda_{\tau(i)}^{-l}|^2$$
$$\le \|\widetilde{X}^{-1}\widetilde{A}^{-k}\Delta A A^{-l} X_1\|_F^2, \quad (2.1)$$

where $\delta_{12}^{(l,k)}$ *is given by* (1.5).

Proof It is easy to check

$$\widetilde{A}^{-k}(A - \widetilde{A})A^{-l} = \widetilde{A}^{-k}A^{1-l} - \widetilde{A}^{1-k}A^{-l},$$

i.e.,

$$\widetilde{X}\widetilde{\Lambda}^{-k}\widetilde{X}^{-1}X\Lambda^{1-l}X^{-1} - \widetilde{X}\widetilde{\Lambda}^{1-k}\widetilde{X}^{-1}X\Lambda^{-l}X^{-1} = \widetilde{A}^{-k}(A - \widetilde{A})A^{-l}.$$

Hence,

$$\widetilde{\Lambda}^{-k}\widetilde{X}^{-1}X\Lambda^{1-l} - \widetilde{\Lambda}^{1-k}\widetilde{X}^{-1}X\Lambda^{-l} = \widetilde{X}^{-1}\widetilde{A}^{-k}(A - \widetilde{A})A^{-l}X.$$

Let $\Delta A = A - \widetilde{A}$. By (1.4) we have

$$\widetilde{X}^{-1}X_1 = \begin{pmatrix}\widetilde{Y}_1^* X_1\\ \widetilde{Y}_2^* X_1\end{pmatrix}.$$

Then

$$\begin{pmatrix} \widetilde{\Lambda}_1^{-k}\widetilde{Y}_1^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_1^{1-k}\widetilde{Y}_1^* X_1 \Lambda_1^{-l} \\ \widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^* X_1 \Lambda_1^{-l} \end{pmatrix} = \widetilde{X}^{-1}\widetilde{A}^{-k}\Delta A A^{-l} X_1. \tag{2.2}$$

Taking the Frobenius norm on both sides of (2.2) gives

$$\begin{aligned} &\left\| \widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^* X_1 \Lambda_1^{-l} \right\|_F^2 + \left\| \widetilde{\Lambda}_1^{-k}\widetilde{Y}_1^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_1^{1-k}\widetilde{Y}_1^* X_1 \Lambda_1^{-l} \right\|_F^2 \\ &= \left\| \widetilde{X}^{-1}\widetilde{A}^{-k}\Delta A A^{-l} X_1 \right\|_F^2. \end{aligned} \tag{2.3}$$

It is easy to see that

$$\begin{aligned} &\left| \left(\widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^* X_1 \Lambda_1^{-l} \right)_{ij} \right|^2 \\ &= \left(\lambda_j^{1-l}\widetilde{\lambda}_i^{-k} - \lambda_j^{-l}\widetilde{\lambda}_i^{1-k} \right)^2 \left| \left(\widetilde{Y}_2^* X_1 \right)_{ij} \right|^2 \geq \left(\delta_{12}^{(l,k)} \right)^2 \left| (\widetilde{Y}_2^* X_1)_{ij} \right|^2. \end{aligned}$$

Thus

$$\left(\delta_{12}^{(l,k)} \right)^2 \| \widetilde{Y}_2^* X_1 \|_F^2 \leq \left\| \widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^* X_1 \Lambda_1^{-l} \right\|_F^2. \tag{2.4}$$

By Lemma 2.1,

$$\| \sin \Theta(\widetilde{X}_1, X_1) \|_F \leq \| \widetilde{Y}_2^\dagger \|_2 \| X_1^\dagger \|_2 \| \widetilde{Y}_2^* X_1 \|_F,$$

which together with (2.4) gives

$$\frac{\left(\delta_{12}^{(l,k)} \right)^2}{\| \widetilde{Y}_2^\dagger \|_2^2 \| X_1^\dagger \|_2^2} \left\| \sin \Theta(\widetilde{X}_1, X_1) \right\|_F^2 \leq \left\| \widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^* X_1 \Lambda_1^{-l} \right\|_F^2. \tag{2.5}$$

It follows from Lemma 2.2 that there exists a permutation τ of $\langle r \rangle$ such that

$$\sigma_{\min}^2 \left(\widetilde{Y}_1^* X_1 \right) \sum_{i=1}^{r} \left| \widetilde{\lambda}_i^{-k} \lambda_{\tau(i)}^{1-l} - \widetilde{\lambda}_i^{1-k} \lambda_{\tau(i)}^{-l} \right|^2 \leq \left\| \widetilde{\Lambda}_1^{-k}\widetilde{Y}_1^* X_1 \Lambda_1^{1-l} - \widetilde{\Lambda}_1^{1-k}\widetilde{Y}_1^* X_1 \Lambda_1^{-l} \right\|_F^2. \tag{2.6}$$

By (2.3), (2.5) and (2.6) we obtain

$$\begin{aligned} &\frac{\left(\delta_{12}^{(l,k)} \right)^2}{\| \widetilde{Y}_2^\dagger \|_2^2 \| X_1^\dagger \|_2^2} \left\| \sin \Theta(\widetilde{X}_1, X_1) \right\|_F^2 + \sigma_{\min}^2 \left(\widetilde{Y}_1^* X_1 \right) \sum_{i=1}^{r} \left| \widetilde{\lambda}_i^{-k} \lambda_{\tau(i)}^{1-l} - \widetilde{\lambda}_i^{1-k} \lambda_{\tau(i)}^{-l} \right|^2 \\ &\leq \left\| \widetilde{X}^{-1}\widetilde{A}^{-k}\Delta A A^{-l} X_1 \right\|_F^2, \end{aligned}$$

which proves the desired bound. □

Remark 2.1 Some existing bounds can be obtained from (2.1):

– If we take $X_1 = X$ and $\widetilde{X}_1 = \widetilde{X}$, then $\|\sin\Theta(\widetilde{X}_1, X_1)\|_F = 0$. For this case, (2.1) reduces to

$$\sum_{i=1}^{n} \left|\widetilde{\lambda}_i^{-k}\lambda_{\tau(i)}^{1-l} - \widetilde{\lambda}_i^{1-k}\lambda_{\tau(i)}^{-l}\right|^2 \le \frac{\|\widetilde{X}^{-1}\widetilde{A}^{-k}\Delta A A^{-l} X\|_F^2}{\sigma_{min}^2(\widetilde{X}^{-1}X)}.$$

Notice that

$$\sigma_{\min}(\widetilde{X}^{-1}X) = \|(\widetilde{X}^{-1}X)^{-1}\|_2^{-1} \ge \|\widetilde{X}\|_2^{-1}\|X^{-1}\|_2^{-1}$$

and

$$\|\widetilde{X}^{-1}\widetilde{A}^{-k}\Delta A A^{-l} X\|_F \le \|\widetilde{X}^{-1}\|_2\|X\|_2\|\widetilde{A}^{-k}\Delta A A^{-l}\|_F.$$

Immediately, (1.7) follows from (2.1), which implies the proposed bound (2.1) is sharper than the bound (1.7).

– By (2.2) and (2.5) it is easy to get the relative perturbation bound for eigenspaces:

$$\begin{aligned}\|\sin\Theta(\widetilde{X}_1, X_1)\|_F &\le \|\widetilde{Y}_2^{\dagger}\|_2\|X_1^{\dagger}\|_2 \frac{\|\widetilde{Y}_2^{*}\widetilde{A}^{-k}\Delta A A^{-l} X_1\|_F}{\delta_{12}^{(l,k)}}\\ &\le \kappa(\widetilde{Y}_2)\kappa(X_1)\frac{\|\widetilde{A}^{-k}\Delta A A^{-l}\|_F}{\delta_{12}^{(l,k)}},\end{aligned}$$

which is the bound (1.8).

– If A and $\widetilde{A}$ are Hermitian, then X and $\widetilde{X}$ are unitary, and $\sigma_{\min}^2(\widetilde{X}_1^{*}X_1) = 1 - \|\sin\Theta(\widetilde{X}_1, X_1)\|_2^2$. Hence the bound (2.1) can be simplified as follows:

$$\begin{aligned}&\left(\delta_{12}^{(l,k)}\right)^2\|\sin\Theta(\widetilde{X}_1, X_1)\|_F^2 + \left(1 - \|\sin\Theta(\widetilde{X}_1, X_1)\|_2^2\right)\sum_{i=1}^{r}\left|\widetilde{\lambda}_i^{-k}\lambda_{\tau(i)}^{1-l} - \widetilde{\lambda}_i^{1-k}\lambda_{\tau(i)}^{-l}\right|^2\\ &\le \|\widetilde{A}^{-k}\Delta A A^{-l} X_1\|_F^2, \qquad (2.7)\end{aligned}$$

which is a generalization of (2.3) of [16]. Comparing the bound (2.16) in [16] with (2.7), it is difficult to say which is sharper. When $l = k = 0$, this bound reduces to (2.3) of [16]. In particular, by (2.7) we have

$$\|\sin\Theta(\widetilde{X}_1, X_1)\|_F \le \frac{\|\widetilde{A}^{-k}\Delta A A^{-l} X_1\|_F}{\delta_{12}^{(l,k)}}. \qquad (2.8)$$

When $l = k = 0$, the bound (2.8) is the classical $\sin\Theta$ theorem [3]. When $l = k = \frac{1}{2}$, the bound (2.8) is givens as follows:

$$\|\sin\Theta(\widetilde{X}_1, X_1)\|_F \le \frac{\|\widetilde{A}^{-\frac{1}{2}}\Delta A A^{-\frac{1}{2}} X_1\|_F}{\delta_{12}^{(\frac{1}{2},\frac{1}{2})}}. \qquad (2.9)$$

In [4], the authors obtained

$$\|\sin\Theta(\widetilde{X}_1, X_1)\|_F \leq \frac{\|\widetilde{A}^{-\frac{1}{2}}\Delta A A^{-\frac{1}{2}}\|_F}{\delta_{12}^{(\frac{1}{2},\frac{1}{2})}}, \tag{2.10}$$

which can be derived from (2.9). Clearly, the bound (2.9) is sharper than the existing one (2.10).

3 Relative bounds for multiplicative perturbation

In this section, we consider the combined multiplicative perturbation bounds for diagonalizable matrices, i.e., the perturbed matrix $\widetilde{A} = D_1^* A D_2$, where D_1 and D_2 are nonsingular and close to the identity matrix. Assume that A and its perturbed matrix $\widetilde{A}$ are two $n \times n$ nonsingular diagonalizable matrices with the eigendecompositions (1.1)–(1.3).

The proof of the following lemma can be given similar to the proof of Lemma 2.2 in [13].

Lemma 3.1 *Let Λ_1 and $\widetilde{\Lambda}_2$ satisfy $\widetilde{\Lambda}_2^{1-k} B \Lambda_1^{-l} - \widetilde{\Lambda}_2^{-k} B \Lambda_1^{1-l} = \widetilde{\Lambda}_2 E + F \Lambda_1$. Then*

$$\|B\|_F \leq \sqrt{\|E\|_F^2 + \|F\|_F^2}/\rho_{12}^{(l,k)},$$

where Λ_1, $\widetilde{\Lambda}_2$ and $\rho_{12}^{(l,k)}$ are given by (1.2), (1.3) and (1.6), respectively.

Lemma 3.2 *Let Λ_1 and $\widetilde{\Lambda}_1$ satisfy $\widetilde{\Lambda}_1^{1-k} B \Lambda_1^{-l} - \widetilde{\Lambda}_1^{-k} B \Lambda_1^{1-l} = \widetilde{\Lambda}_1 E + F \Lambda_1$, where Λ_1 and $\widetilde{\Lambda}_1$ are given by (1.2) and (1.3), respectively. Then there is a permutation τ of $\langle r \rangle$ such that*

$$\sigma_{\min}^2(B) \sum_{i=1}^{r} \frac{|\lambda_i - \widetilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l} |\widetilde{\lambda}_{\tau(i)}|^{2k} (|\lambda_i|^2 + |\widetilde{\lambda}_{\tau(i)}|^2)} \leq \|E\|_F^2 + \|F\|_F^2.$$

Proof Similar to the proof of Lemma 2.2 in [13], we obtain easily

$$\left| \frac{\lambda_i^{-l} \widetilde{\lambda}_j^{1-k} - \lambda_i^{1-l} \widetilde{\lambda}_j^{-k}}{\sqrt{|\lambda_i|^2 + |\widetilde{\lambda}_j|^2}} \cdot b_{ji} \right|^2 \leq |e_{ji}|^2 + |f_{ji}|^2, \tag{3.1}$$

from which one can deduce that

$$\sum_{i,j=1}^{r} \frac{|\lambda_i - \widetilde{\lambda}_j|^2}{|\lambda_i|^{2l} |\widetilde{\lambda}_j|^{2k} (|\lambda_i|^2 + |\widetilde{\lambda}_j|^2)} |b_{ji}|^2 \leq \|E\|_F^2 + \|F\|_F^2. \tag{3.2}$$

On the other hand, from the result given in the last line of [5], it is known that there is a permutation τ of $\langle r\rangle$ such that

$$\sigma_{\min}^2(B)\sum_{i=1}^{r}\frac{|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l}|\widetilde{\lambda}_{\tau(i)}|^{2k}(|\lambda_i|^2+|\widetilde{\lambda}_{\tau(i)}|^2)}\le\sum_{i,j=1}^{r}\frac{|\lambda_i-\widetilde{\lambda}_j|^2}{|\lambda_i|^{2l}|\widetilde{\lambda}_j|^{2k}(|\lambda_i|^2+|\widetilde{\lambda}_j|^2)}|b_{ji}|^2,$$

which together with (3.2) gives the desired bound. □

The following bound (3.3) is the combined form for eigenvalues and eigenspaces of diagonalizable matrices.

Theorem 3.3 *Let A and its perturbed matrix $\widetilde{A}=D_1^*AD_2$ be two $n\times n$ nonsingular diagonalizable matrices with the eigendecompositions (1.1)–(1.3), where D_i is nonsingular, $i=1,2$. Then there is a permutation τ of $\langle r\rangle$ such that*

$$\begin{aligned}&\frac{\left(\rho_{12}^{(l,k)}\right)^2}{\|\widetilde{Y}_2^\dagger\|_2^2\|X_1^\dagger\|_2^2}\|\sin\Theta(X_1,\widetilde{X}_1)\|_F^2+\sigma_{\min}^2(\widetilde{Y}_1{}^*X_1)\sum_{i=1}^{r}\frac{|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l}|\widetilde{\lambda}_{\tau(i)}|^{2k}(|\lambda_i|^2+|\widetilde{\lambda}_{\tau(i)}|^2)}\\&\le\|\widetilde{X}^{-1}\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}X_1\|_F^2+\|\widetilde{X}^{-1}\widetilde{A}^{-k}(D_1^*-I)A^{-l}X_1\|_F^2.\end{aligned}\tag{3.3}$$

Proof From $\widetilde{A}=D_1^*AD_2$ one may get

$$\widetilde{A}^{1-k}A^{-l}-\widetilde{A}^{-k}A^{1-l}=\widetilde{A}\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}+\widetilde{A}^{-k}(D_1^*-I)A^{-l}A.\tag{3.4}$$

Multiplying by $\widetilde{X}^{-1}$ and X_1 from the left and right of (3.4) respectively gives

$$\begin{aligned}&\widetilde{\Lambda}^{1-k}\widetilde{X}^{-1}X_1\Lambda_1^{-l}-\widetilde{\Lambda}^{-k}\widetilde{X}^{-1}X_1\Lambda_1^{1-l}\\&=\widetilde{\Lambda}\widetilde{X}^{-1}\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}X_1+\widetilde{X}^{-1}\widetilde{A}^{-k}(D_1^*-I)A^{-l}X_1\Lambda_1.\end{aligned}\tag{3.5}$$

Let $Q=\widetilde{X}^{-1}X_1$, $Z=\widetilde{X}^{-1}\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}X_1$ and $\widetilde{Z}=\widetilde{X}^{-1}\widetilde{A}^{-k}(D_1^*-I)A^{-l}X_1$. By (1.4),

$$Q=\begin{pmatrix}\widetilde{Y}_1^*X_1\\\widetilde{Y}_2^*X_1\end{pmatrix}.$$

Let

$$Z=\begin{pmatrix}Z_1\\Z_2\end{pmatrix}\quad\text{and}\quad\widetilde{Z}=\begin{pmatrix}\widetilde{Z}_1\\\widetilde{Z}_2\end{pmatrix}$$

have the same block structure as Q. Rewriting (3.5) in the block form yields

$$\begin{pmatrix}\widetilde{\Lambda}_1^{1-k}\widetilde{Y}_1^*X_1\Lambda_1^{-l}-\widetilde{\Lambda}_1^{-k}\widetilde{Y}_1^*X_1\Lambda_1^{1-l}\\\widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^*X_1\Lambda_1^{-l}-\widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^*X_1\Lambda_1^{1-l}\end{pmatrix}=\begin{pmatrix}\widetilde{\Lambda}_1Z_1+\widetilde{Z}_1\Lambda_1\\\widetilde{\Lambda}_2Z_2+\widetilde{Z}_2\Lambda_1\end{pmatrix}.$$

Or equivalently,

$$\widetilde{\Lambda}_1^{1-k}\widetilde{Y}_1^*X_1\Lambda_1^{-l}-\widetilde{\Lambda}_1^{-k}\widetilde{Y}_1^*X_1\Lambda_1^{1-l}=\widetilde{\Lambda}_1Z_1+\widetilde{Z}_1\Lambda_1\tag{3.6}$$

and

$$\widetilde{\Lambda}_2^{1-k}\widetilde{Y}_2^*X_1\Lambda_1^{-l} - \widetilde{\Lambda}_2^{-k}\widetilde{Y}_2^*X_1\Lambda_1^{1-l} = \widetilde{\Lambda}_2 Z_2 + \widetilde{Z}_2\Lambda_1. \tag{3.7}$$

Applying Lemma 3.1 to (3.7) yields

$$\left(\rho_{1,2}^{(l,k)}\right)^2 \|\widetilde{Y}_2^*X_1\|_F^2 \le \|Z_2\|_F^2 + \|\widetilde{Z}_2\|_F^2. \tag{3.8}$$

It follows from Lemma 2.1 that

$$\frac{1}{\|\widetilde{Y}_2^\dagger\|_2\|X_1^\dagger\|_2}\|\sin\Theta(X_1,\widetilde{X}_1)\|_F \le \|\widetilde{Y}_2^*X_1\|_F\,. \tag{3.9}$$

By (3.8) and (3.9) we get

$$\frac{\left(\rho_{1,2}^{(l,k)}\right)^2}{\|\widetilde{Y}_2^\dagger\|_2^2\|X_1^\dagger\|_2^2}\|\sin\Theta(X_1,\widetilde{X}_1)\|_F^2 \le \|Z_2\|_F^2 + \|\widetilde{Z}_2\|_F^2. \tag{3.10}$$

Applying Lemma 3.2 to (3.6) gives that there is a permutation τ of $\langle r\rangle$ so that

$$\sigma_{\min}^2(\widetilde{Y}_1^*X_1)\sum_{i=1}^r \frac{|\lambda_i - \widetilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l}|\widetilde{\lambda}_{\tau(i)}|^{2k}(|\lambda_i|^2 + |\widetilde{\lambda}_{\tau(i)}|^2)} \le \|Z_1\|_F^2 + \|\widetilde{Z}_1\|_F^2,$$

which together with (3.10) gives that

$$\begin{aligned}
&\frac{\left(\rho^{(l,k)}\right)^2}{\|\widetilde{Y}_2^\dagger\|_2^2\|X_1^\dagger\|_2^2}\|\sin\Theta(X_1,\widetilde{X}_1)\|_F^2 + \sigma_{\min}^2(\widetilde{Y}_1^*X_1)\sum_{i=1}^r \frac{|\lambda_i - \widetilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l}|\widetilde{\lambda}_{\tau(i)}|^{2k}(|\lambda_i|^2 + |\widetilde{\lambda}_{\tau(i)}|^2)}\\
&\le \|Z_2\|_F^2 + \|\widetilde{Z}_2\|_F^2 + \|Z_1\|_F^2 + \|\widetilde{Z}_1\|_F^2 = \|Z\|_F^2 + \|\widetilde{Z}\|_F^2\\
&= \|\widetilde{X}^{-1}\widetilde{A}^{-k}(I - D_2^{-1})A^{-l}X_1\|_F^2 + \|\widetilde{X}^{-1}\widetilde{A}^{-k}(D_1^* - I)A^{-l}X_1\|_F^2.
\end{aligned}$$

This proves the theorem. □

Remark 3.1 From Theorem 3.3 and its proof, we can derive some existing bounds:

– If we take $X_1 = X$ and $\widetilde{X}_1 = \widetilde{X}$, then $\|\sin\Theta(X_1,\widetilde{X}_1)\|_2 = \|\sin\Theta(X_1,\widetilde{X}_1)\|_F = 0$ and the bound (3.3) implies that

$$\begin{aligned}
&\sqrt{\sum_{i=1}^n \frac{|\lambda_i - \tilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l}|\tilde{\lambda}_{\tau(i)}|^{2k}(|\lambda_i|^2 + |\tilde{\lambda}_{\tau(i)}|^2)}}\\
&\quad\le \frac{1}{\sigma_{\min}(\widetilde{X}^{-1}X)}\sqrt{\|\widetilde{X}^{-1}\widetilde{A}^{-k}(I - D_2^{-1})A^{-l}X\|_F^2 + \|\widetilde{X}^{-1}\widetilde{A}^{-k}(D_1^* - I)A^{-l}X\|_F^2}
\end{aligned}$$

$$\leq \|\widetilde{X}X^{-1}\|_2\|X\|_2\|\widetilde{X}^{-1}\|_2\sqrt{\|\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}\|_F^2+\|\widetilde{A}^{-k}(D_1^*-I)A^{-l}\|_F^2}$$
$$\leq \kappa(X)\kappa(\widetilde{X})\sqrt{\|\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}\|_F^2+\|\widetilde{A}^{-k}(D_1^*-I)A^{-l}\|_F^2}.$$

When $l=k=0$, the above bound reduces to the one given by Li [10].

– By (3.10) we may deduce the bound for eigenspaces. In fact,

$$\frac{\left(\rho_{1,2}^{(l,k)}\right)^2}{\|\widetilde{Y}_2^{\dagger}\|_2^2\|X_1^{\dagger}\|_2^2}\|\sin\Theta(X_1,\widetilde{X}_1)\|_F^2$$
$$\leq \|\widetilde{Y}_2^*\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}X_1\|_F^2+\|\widetilde{Y}_2^*\widetilde{A}^{-k}(D_1^*-I)A^{-l}X_1\|_F^2,$$

and thus

$$\|\sin\Theta(X_1,\widetilde{X}_1)\|_F \leq \frac{\kappa(X_1)\kappa(\widetilde{Y}_2)\sqrt{\|\widetilde{A}^{-k}(I-D_2^{-1})A^{-l}\|_F^2+\|\widetilde{A}^{-k}(D_1^*-I)A^{-l}\|_F^2}}{\rho_{12}^{(l,k)}}. \quad (3.11)$$

When $l=k=0$, the above bound is identical to the bound given in Remark 3.3 of [13].

– If A and $\widetilde{A}=D^*AD$ are Hermitian, then X and $\widetilde{X}$ are unitary, and $\sigma_{\min}^2(\widetilde{X}_1^*X_1)=1-\|\sin\Theta(\widetilde{X}_1,X_1)\|_2^2$. Hence the bound (3.3) can be rewritten as the following form:

$$\left(\rho_{1,2}^{(l,k)}\right)^2\|\sin\Theta(\widetilde{X}_1,X_1)\|_F^2+(1-\|\sin\Theta(\widetilde{X}_1,X_1)\|_2^2)\varepsilon(\lambda_i,\widetilde{\lambda}_{\tau(i)})$$
$$\leq \|\widetilde{A}^{-k}(I-D^{-1})A^{-l}X_1\|_F^2+\|\widetilde{A}^{-k}(D^*-I)A^{-l}X_1\|_F^2, \quad (3.12)$$

where by $\varepsilon(\lambda_i,\widetilde{\lambda}_{\tau(i)})$ we denote the sum

$$\sum_{i=1}^{r}\frac{|\lambda_i-\widetilde{\lambda}_{\tau(i)}|^2}{|\lambda_i|^{2l}|\widetilde{\lambda}_{\tau(i)}|^{2k}(|\lambda_i|^2+|\widetilde{\lambda}_{\tau(i)}|^2)}.$$

If $l=k=0$, then the bound (3.12) reduces to (2.2) of [14].

4 Numerical experiments

From Sects. 2 and 3, the perturbation bounds (2.1) and (3.3) not only generalize some existing perturbation bounds but also can be used to produce new perturbation bounds. In this section, we will take some matrices from the University of Florida Sparse Matrix collection to test the new perturbation bounds (2.8) and (3.11), respectively. The test matrices are described in Table 1, where we denote Symmetric Positive Definite and

Table 1 Name, sources and properties of the $n \times n$ test matrices A

Name	Source	n	Nonzeros	Type	$\kappa(A)$
bcsstk14	Structural problem	1806	63454	SPD	1.19e+10
mhd1280b	Electromagnetics	1280	22778	HPD	4.75e+12
SpaceStation9	Optimal Control	1180	19674	Symmetric	1.08e+12
nasa1824	Duplicate structural problem	1824	39208	Symmetric	5.89e+6
plskz362	2D/3D problem	362	1760	Unsymmetric	4.67e+5

Hermite Positive Definite by SPD and HPD, respectively. Note that these test matrices have the large 2-norm condition numbers from the magnitude order 10^5 to 10^{12}.

For all test matrices, the perturbations ΔA in (2.8) and $D_i (i = 1, 2)$ in (3.11) are generated randomly by the MATLAB commands sprandsym and sprand, which are explained in detail as follows:

- $sprandsym(A)$: giving a sparse symmetric random matrix with the same structure as A, whose nonzero entries lie in the interval (0, 1);
- $sprand(m, n, d)$: giving an $m \times n$ sparse random matrix with approximately dmn nonzero entries in the interval (0, 1) each time;
- $sprand(A)$: giving a sparse random matrix with the same structure as A each time, whose nonzero entries locate in the interval (0, 1).

We run the above commands and then generate the additive and multiplicative perturbations with small enough ε, which are given as follows:

- For the bound (2.8), let $\Delta A = \varepsilon B$ with $B = sprandsym(A)$;
- For the bound (3.11), let $D_i = I + \varepsilon E_i$ $(i = 1, 2)$. We distinguish the following cases:
  - For the test matrices from bcsstk14, mhd1280b and nasa1824, let

    $$\alpha = sprand(n, 1, 1e-2) \quad \text{and} \quad E_i = \text{diag}(\alpha)$$

    where diag(α) is a diagonal matrix with the elements of the vector α on the main diagonal;
  - For the test matrix from plskz362, let $E_i = sprand(A)$. For this case, D_i is not necessarily diagonal.

As pointed out in Sects. 2–3, the derived bounds (2.8) and (3.11) are also generalizations of some existing bounds. In Tables 2 and 3 and Fig. 1, we give some comparison results of the bounds (2.8) and (3.11) with different values of l and k. For simplicity, we use notations $\text{Pb}_{\text{new}}(l, k)$ and $\text{Pb}_{\text{old}}(l, k)$ to denote a new bound and an existing one, respectively, which can be derived by the proposed bounds (2.8) or (3.11). In other words, both $\text{Pb}_{\text{new}}(l, k)$ and $\text{Pb}_{\text{old}}(l, k)$ are exactly (2.8) or (3.11) with specific values for l and k. In addition, the notation Pb_{old} is used to denote the existing bound (2.10). Here we choose the quasi-optimal parameters l_* and k_* in the bounds (2.8) and (3.11) by the following methods:

Table 2 The additive perturbation bound (2.8) for SpaceStation9: $\varepsilon = 10^{-5}$, $(l_*, k_*) = (1, 0)$

	$r = 200$	$r = 300$	$r = 400$	$r = 500$
$\mathrm{Pb}_{\mathrm{old}}(0, 0)$	4.40e−2	7.84e−3	8.46e−3	7.72e−3
$\mathrm{Pb}_{\mathrm{old}}$	3.29e+5	2.36e+4	8.90e+3	1.83e+3
$\mathrm{Pb}_{\mathrm{new}}(0.5, 0.5)$	3.21e+2	2.50e+1	1.41e+1	5.37
$\mathbf{Pb_{new}}(l_*, k_*)$	2.38e−2	3.78e−3	3.02e−3	1.72e−3
$\|\sin\Theta(\tilde{X}_1, X_1)\|_F$	**3.36e**−4	**6.68e**−5	**1.17e**−4	**1.26e**−4

Table 3 The multiplicative perturbation bound (3.11) for the matrices A: $r = 300$, $\varepsilon = 10^{-5}$

Name	(l_*, k_*)	$\mathrm{Pb}_{\mathrm{old}}(0, 0)$	$\mathrm{Pb}_{\mathrm{new}}(l_*, k_*)$	$\|\sin\Theta(\tilde{X}_1, X_1)\|_F$
bcsstk14	(0.1, 0.2)	8.85e−4	4.16e−4	1.68e−6
mhd1280b	(0.5, 0.4)	4.85e−4	2.15e−5	4.37e−6
nasa1824	(0.1, 0.2)	1.27e−3	8.36e−4	2.93e−6
plskz362	(0.2, 0.2)	8.03e−4	4.36e−4	1.04e−5

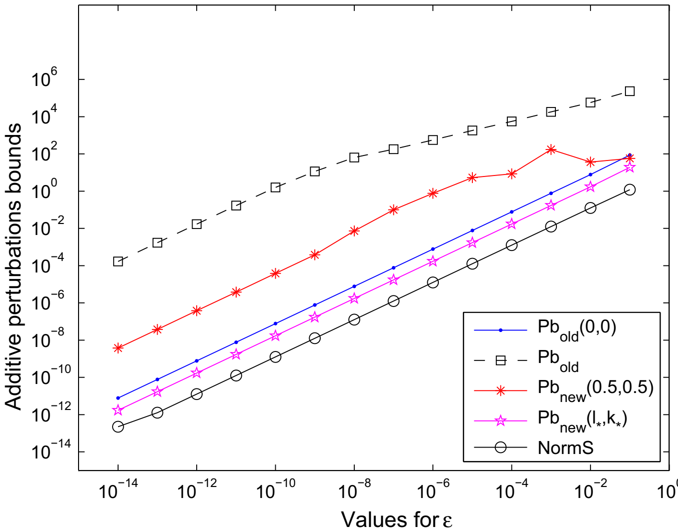

Fig. 1 Additive perturbation bound (2.8) for the test matrix SpaceStation with $r = 500$: $l_* = 1$, $k_* = 0$

– For the additive perturbation bound (2.8), l_* and k_* are chosen by taking $(l, k) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(0.5, 0.5)$, respectively, so that the associated perturbation bound attains the minimum.
– For the multiplicative perturbation bound (3.11), l_* and k_* are obtained experimentally by minimizing the bound (3.11) in the interval $[0, 0.5]$.

By Table 2 we report the numerical results of the bound (2.8) for the test matrices from the optimal control field with the different sizes X_1 and X_2, which are shown by the different r, in which one may see that the perturbation bound (2.8) with the quasi-optimal parameters l_* and k_* is always tighter than the associated existing ones $\mathrm{Pb}_{\mathrm{old}}(0,0)$ and $\mathrm{Pb}_{\mathrm{old}}$. The latter two ones are exactly the $\sin\Theta$ theorem in [3] and the bound in [4], respectively.

In Fig. 1 we denote by NormS the real value of $\|\sin\Theta(\tilde{X}_1, X_1)\|_F$. From Fig. 1 it is known that the new bound $\mathrm{Pb}_{\mathrm{new}}(l_*, k_*)$ is still sharper than the existing ones for the different values of the perturbation ε. Moreover, Fig. 1 implies that the bound (2.8) with $l = k = 0.5$ always outperforms the associated existing one (2.10), which further confirms the theoretical analysis in Sect. 2.

In order to verify the effectiveness of the multiplicative perturbation bound (3.11), we test the different matrices with $r = 300$ and $\varepsilon = 10^{-5}$, where the bound $\mathrm{Pb}_{\mathrm{old}}(0,0)$ is given by Remark 3.3 of [13]. As expected, the quasi-optimal bound $\mathrm{Pb}_{\mathrm{new}}(l_*, k_*)$ from (3.11) is always sharper than the existing bound $\mathrm{Pb}_{\mathrm{old}}(0,0)$.

Remark 4.1 From numerical results given in Tables 2 and 3 and Fig. 1, we can always obtain the sharper new bounds by taking l and k. In particular, we test the additive bound (2.8) and the multiplicative bound (3.11) by a large number of examples, which show that quasi-optimal parameters are given by $(l_*, k_*) = (1, 0)$ or $(0, 1)$ for (2.8) and $|l_* - k_*| \leq 0.1$ for (3.11) when $l, k \in [0, 0.5]$. However, we are not able to present some theoretical results for the optimal parameters in the proposed perturbation bounds. This remains open.

5 Conclusions

In this paper, we have proposed two relative perturbation bounds (2.1) and (3.3), which provide a general framework of relative bounds for additive and multiplicative perturbations for eigenpairs of diagonalizable matrices, respectively. With suitable choices of eigenspaces or the parameters l and k, the new bounds (2.1) and (3.3) not only cover some classical perturbation bounds as their special cases, but also yield some new perturbation bounds, for example, the new bounds (2.8) and (3.11). We have shown that the bounds (2.1) and (3.3) may improve some existing ones. The numerical experiments reveal that new perturbation bounds are sharper than the existing ones. In the future we will consider to get the optimal parameters l and k in the bounds (2.1) and (3.3) for some special structure matrices.

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