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Quasi-interpolation based on the ZP-element for the numerical solution of integral equations on surfaces in $\mathbb{R}^3$

Catterina Dagnino[1] · **Sara Remogna**[1]

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Abstract The aim of this paper is to present spline methods for the numerical solution of integral equations on surfaces of $\mathbb{R}^3$, by using optimal superconvergent quasi-interpolants defined on type-2 triangulations and based on the Zwart–Powell quadratic box spline. In particular we propose a modified version of the classical collocation method and two spline collocation methods with high order of convergence. We also deal with the problem of approximating the surface. Finally, we study the approximation error of the above methods together with their iterated versions and we provide some numerical tests.

Keywords Surface integral equation · Spline quasi-interpolation

Mathematics Subject Classification 65R20 · 65D07

1 Introduction

Let

$$\rho(\mathbf{P}_1) - \int_S K(\mathbf{P}_1, \mathbf{P}_2)\rho(\mathbf{P}_2) dS_{\mathbf{P}_2} = \psi(\mathbf{P}_1), \quad \mathbf{P}_1 \in S, \tag{1.1}$$

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✉ Sara Remogna
sara.remogna@unito.it

Catterina Dagnino
catterina.dagnino@unito.it

1 Dipartimento di Matematica, Università di Torino, via C. Alberto, 10, 10123 Turin, Italy

be an integral equation, where S is a connected surface in $\mathbb{R}^3$, described by a sufficiently smooth map $\mathbf{F} : \Omega \to S$, with Ω a polygonal domain in $\mathbb{R}^2$, and the kernel $K(\mathbf{P}_1, \mathbf{P}_2)$ is continuous for $\mathbf{P}_1, \mathbf{P}_2 \in S$.

Therefore, the integral equation (1.1) can be written as

$$\rho(\mathbf{F}(u,v)) - \int_\Omega K(\mathbf{F}(u,v), \mathbf{F}(s,t))\rho(\mathbf{F}(s,t))\,|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)|\; ds\,dt \\ = \psi(\mathbf{F}(u,v)),\;\; (u,v) \in \Omega, \tag{1.2}$$

where

$$|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| \tag{1.3}$$

is the Jacobian of the map $\mathbf{F}(s,t)$.

If we denote by $\mathscr{K} : C(S) \longrightarrow C(S)$ the integral operator defined by

$$\mathscr{K}\rho(\mathbf{F}(u,v)) := \int_\Omega K(\mathbf{F}(u,v), \mathbf{F}(s,t))\rho(\mathbf{F}(s,t))\,|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)|\; ds\,dt,$$

for $(u,v) \in \Omega$, then we can write (1.1) in the following operator form

$$(\mathscr{I} - \mathscr{K})\rho = \psi. \tag{1.4}$$

We remark that (1.4) has a unique solution $\rho \in C(S)$ for any given $\psi \in C(S)$ [3].

A standard technique for numerically solving (1.4) is to replace $\mathscr{K}$ by a finite rank operator and to obtain the approximate solution by solving a system of linear equations. Nyström, Galerkin and collocation methods are the commonly used ones for this purpose. For instance, we recall the collocation ones based on a sequence of linear interpolatory projection operators onto finite dimensional subspaces $\mathscr{X}_{mn}$ of $C(S)$, converging to the identity operator pointwise. A classical choice of $\mathscr{X}_{mn}$ is the space of C^0 piecewise polynomials of a given degree d (usually $d = 2$) on a triangulation of Ω (see [3,7]).

In this paper we propose three collocation methods for (1.4), based on a sequence of optimal superconvergent spline quasi-interpolating operators $\{Q_{mn}\}$, that are not projectors and are defined on the space $\mathscr{X}_{mn} = S_2^1(\Omega, T_{mn})$ of the C^1 quadratic splines on a uniform type-2 triangulation T_{mn} of Ω, with Ω a rectangular domain. We recall [16] that the above quasi-interpolating splines are expressed by means of the scaled/translates of the Zwart–Powell quadratic box spline (ZP-element) (see e.g. [6, Chap. 1], [19, Chap. 2]).

We remark that C^1 quadratic spline spaces on type-2 triangulations have been widely studied (see [1,4,5,8–14,16–20] and references therein), with reference to the dimension, local basis, approximation power, etc. and they have been used in many applications. This paper wants to be a further contribution to the researches on this topic, with regard to the numerical solution of surface integral equations.

In the first proposed method, that we call *spline modified collocation method*, in (1.4) we replace the operator $\mathscr{K}$ by $Q_{mn}\mathscr{K}$ and the right hand side ψ by $Q_{mn}\psi$. As expected, we prove that its convergence rate is of order three and the convergence rate of its iterated version is of order four. Moreover, the approximate solution belongs to $C^1(S)$.

In the other two ones, that we call *spline collocation methods with high order of convergence*, in (1.4) we replace $\mathscr{K}$ by one of the two following finite rank operators

$$\mathscr{K}_{mn,i} := Q_{mn}\mathscr{K} + \mathscr{K}^*_{mn,i} - Q_{mn}\mathscr{K}^*_{mn,i}, \quad i = 1, 2,$$

where $\mathscr{K}^*_{mn,1}$ is the degenerate kernel operator obtained by approximating

$$K(\mathbf{F}(u, v), \mathbf{F}(s, t))\,|(D_s\mathbf{F} \times D_t\mathbf{F})(s, t)|$$

by using Q_{mn} with respect to the variables (s, t) and $\mathscr{K}^*_{mn,2}$ is the Nyström operator based on Q_{mn}. We can establish that, if the kernel is suitably smooth, then the convergence rate of both methods is of order seven and the convergence rate of their iterated version is of order eight. We remark that such methods are defined by a logical scheme similar to that one used in [1] to construct methods for 2D integral equations, based on other quasi-interpolants.

Since with many surfaces S, gaining knowledge of the derivatives of $\mathbf{F}(s, t)$ can be a major inconvenience, both to specify and to program, here we also consider surface approximations of the form $\tilde{S} = Q_{mn}\mathbf{F}(\Omega)$, for which the Jacobians are more easily computed and we investigate its effect on the spline modified collocation method.

Moreover, discrete versions of all our methods are presented. They are based on composite Gaussian cubatures on triangular domains (see [15]).

Finally, we remark that the proposed methods can be generalized to the case of connected piecewise smooth surfaces, i.e. for surfaces S that can be written as $S = S_1 \cup S_2 \cup \cdots \cup S_J$, where each S_j is the continuous image of a rectangular region Ω_j in the plane: $\mathbf{F}_j : \Omega_j \to S_j$, $j = 1, \ldots J$, with $\mathbf{F}_j$, $j = 1, \ldots J$, sufficiently smooth maps.

Here is an outline of the paper. In Sect. 2, we describe the optimal superconvergent spline quasi-interpolant Q_{mn}, defined on the space $S^1_2(\Omega, T_{mn})$ and based on dilation/translation of the ZP-element, proving their properties and providing an application to numerical integration, used in a subsequent section.

Then, in Sect. 3, we propose three collocation methods based on Q_{mn} for surface integral equations. In particular, in Sect. 3.1, we define and study the spline modified collocation method, that is a generalization of the classical one and, in Sect. 3.2, we define and analyse two spline collocation methods with high order of convergence. In Sect. 3.3 we provide numerical tests, illustrating the approximation properties of the proposed schemes. Finally, in Sect. 3.4, we deal with the problem of approximating the surface, considering surface spline approximations for which the Jacobians are more easily evaluated and then we apply the spline modified collocation method of Sect. 3.1, to solve the corresponding surface integral equation.

2 On optimal superconvergent spline quasi-interpolants based on the ZP-element

Let $\Omega = [a, b] \times [c, d]$ be a rectangular domain divided into mn equal squares $\{\Omega_{ij}\}_{i=1,\ j=1}^{m\quad n}$ with a given edge h, $m, n \geq 4$, each of them being subdivided into

4 triangles by its diagonals, obtaining a uniform type-2 triangulation T_{mn} of Ω. We denote by $S_2^1(\Omega, T_{mn})$ the space of C^1 quadratic splines on T_{mn}, whose dimension is $(m+2)(n+2)-1$ ([19] and the reference therein).

This space is generated by the $(m+2)(n+2)$ B-spline functions $\{B_{i,j}, (i,j) \in A_{mn}\}$, where $A_{mn} = \{(i,j), 0 \le i \le m+1, 0 \le j \le n+1\}$, obtained by dilation/translation of the ZP-element. Moreover, in order to obtain a B-spline basis for $S_2^1(\Omega, T_{mn})$ we have to neglect one B-spline from the spanning set ([19] and the reference therein).

In the space $S_2^1(\Omega, T_{mn})$ we consider special optimal quasi-interpolants (abbr. QIs) of the form

$$Q_{mn} f := \sum_{(i,j)\in A_{mn}} \lambda_{i,j}(f) B_{i,j}, \tag{2.1}$$

with $\left\{\lambda_{i,j},\ (i,j) \in A_{mn}\right\}$ a family of local linear functionals defined in this way

$$\lambda_{i,j}(f) := \sum_{(k,l)\in F_{i,j}} \sigma_{i,j}(k,l) f(M_{k,l}), \tag{2.2}$$

where the finite set of points $\left\{M_{k,l},\ (k,l) \in F_{i,j}\right\}$, $F_{i,j} \subset A_{mn}$, lies in some neighbourhood of $\text{supp} B_{i,j} \cap \Omega$ and the $\sigma_{i,j}(k,l)$'s are chosen such that $Q_{mn} f \equiv f$ for all f in $\mathbb{P}_2$ (the space of bivariate polynomials of total degree two) and superconvergence is induced at some specific points, i.e. the vertices, the centers, the midpoints of horizontal and vertical edges of each subsquare of the partition. The coefficient functional expression (2.2) is given in [16] and we recall that $\|Q_{mn}\|_\infty \le 2$.

The points $M_{k,l}$ in (2.2) are the mn centers of the squares, the $2(m+n)$ midpoints of boundary segments and the four vertices of Ω, i.e. $M_{k,l} = (s_k, t_l)$, where

$$\begin{aligned} s_0 &= a,\ s_k = a + \left(k - \tfrac{1}{2}\right) h,\ 1 \le k \le m,\ s_{m+1} = b, \\ t_0 &= c,\ \ t_l = c + \left(l - \tfrac{1}{2}\right) h,\ 1 \le l \le n, \quad t_{n+1} = d. \end{aligned}$$

We underline that the above QIs have the following good properties. They are based on functionals involving only data points inside the domain and, moreover, they are defined by means of the scaled/translates of the ZP-element. From a computational point of view, this is more convenient than the use of other spanning sets, for instance formed by bivariate B-splines with support completely included in Ω [1,10,13,18], that, having different supports, have different expressions in the domain, while the ZP-element is always the same.

We remark that the QIs (2.1) can also be written in quasi-Lagrange form

$$Q_{mn} f := \sum_{(i,j)\in A_{mn}} f(M_{i,j}) L_{i,j}, \tag{2.3}$$

by means of the fundamental functions $L_{i,j}$, obtained as linear combination of the $B_{i,j}$'s and reported in Appendix A.

Now, we need to introduce the following notations:

– $D^\beta = D^{\beta_1 \beta_2} = \frac{\partial^{|\beta|}}{\partial x^{\beta_1} \partial y^{\beta_2}}$, with $|\beta| = \beta_1 + \beta_2$;

– $\|D^{\nu} f\|_{\infty} = \max_{|\beta|=\nu} \left\|D^{\beta} f\right\|_{\infty}$;
– $\omega\left(D^{\nu} f, h\right) = \max\left\{\omega\left(D^{\alpha} f, h\right), |\alpha| = \nu\right\}$, where

$$\omega(f, h) = \max\left\{|f(P_1) - f(P_2)|\,;\, P_1, P_2 \in \Omega, \|P_1 - P_2\| \leq h\right\}$$

is the modulus of continuity of $f \in C(\Omega)$, and $\|\cdot\|$ is the Euclidean norm.

Standard results in approximation theory and other specific ones given in [9] allow us to deduce the following theorem.

Theorem 2.1 *Let $f \in C^{\nu}(\Omega)$, $0 \leq |\alpha| \leq \nu \leq 2$, $|\alpha| = 0, 1$ then*

$$\left\|D^{\alpha}(f - Q_{mn} f)\right\|_{\infty} \leq K_{\alpha,\nu} h^{\nu-|\alpha|} \omega\left(D^{\nu} f, h\right),$$

where the error constant $K_{\alpha,\nu}$ is independent of h and depends only on α and ν.
If, in addition, $f \in C^3(\Omega)$, then

$$\left\|D^{\alpha}(f - Q_{mn} f)\right\|_{\infty} \leq K_{\alpha,3} h^{3-|\alpha|} \left\|D^3 f\right\|_{\infty}.$$

We underline that Q_{mn} has superconvergence properties. In particular, for $f \in C^4(\Omega)$, we have that

$$|(f - Q_{mn} f)(P)| = O(h^4)$$

at specific points P in Ω, that are the vertices, the centers, the midpoints of horizontal and vertical edges of each subsquare of Ω partition.

Moreover, the following theorem holds. We omit its proof, because it can be proved by the same logical scheme given in [1] for another spline QI.

Theorem 2.2 *Let g be a differentiable function with bounded derivatives and $f \in C^4(\Omega)$, then we have*

$$\mathscr{E}(f, g) = \left|\int_{\Omega} (f - Q_{mn} f) g\right| \leq C h^4 \left(\left\|D^3 f\right\|_{\infty} + \left\|D^4 f\right\|_{\infty}\right)$$

where C is a positive constant independent of n, m.

We also prove the following result that we will use later in Sect. 3.4.1.

Lemma 2.1 *Let p be a bivariate cubic polynomial on $\Omega_{ij} := [a + (i-1)h, a + ih] \times [c + (j-1)h, c + jh]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then*

$$\int_{\Omega_{ij}} \frac{\partial}{\partial s}\Big[p(s, t) - Q_{mn} p(s, t)\Big]\, ds\, dt = 0, \tag{2.4}$$

$$\int_{\Omega_{ij}} \frac{\partial}{\partial t}\Big[p(s, t) - Q_{mn} p(s, t)\Big]\, ds\, dt = 0. \tag{2.5}$$

Proof Firstly, we write (2.4) as:

$$\int_{c+(j-1)h}^{c+jh}\int_{a+(i-1)h}^{a+ih}\frac{\partial}{\partial s}\Big[p(s,t)-Q_{mn}p(s,t)\Big]\,ds\,dt = I_1 - I_0,$$

where

$$I_1 = \int_{c+(j-1)h}^{c+jh}\Big[p(a+ih,t)-Q_{mn}p(a+ih,t)\Big]\,dt$$

and

$$I_0 = \int_{c+(j-1)h}^{c+jh}\Big[p(a+(i-1)h,t)-Q_{mn}p(a+(i-1)h,t)\Big]\,dt.$$

The function $p(a+ih,t)$ is a univariate cubic polynomial in the variable t, $t\in[c+(j-1)h, c+jh]$ and $Q_{mn}p(a+ih,t)$ is a univariate quadratic polynomial in the variable t, $t\in[c+(j-1)h, c+jh]$. Thanks to the superconvergence properties of Q_{mn}, $Q_{mn}p(a+ih,t)$ is the quadratic polynomial interpolating $p(a+ih,t)$ at the points $(a+ih, c+(j-1)h)$, $(a+ih, c+(j-\frac{1}{2})h)$ and $(a+ih, c+jh)$. Therefore, we can use the classical Lagrange interpolation error formula, getting

$$\begin{aligned} &p(a+ih,t)-Q_{mn}p(a+ih,t)\\ &\quad= C^*[t-(c+(j-1)h)]\left[t-\left(c+\left(j-\frac{1}{2}\right)h\right)\right][t-(c+jh)], \end{aligned}$$

with C^* a suitable constant independent of h. Then, it is immediate to obtain $I_1=0$.

Similarly, we get $I_0=0$ and therefore (2.4) holds.

Following the same logical scheme, we obtain (2.5). □

Finally, we propose an interesting application of the above superconvergent QIs to numerical integration, getting cubature rules that we will use in Sect. 3.2.

For any function $f\in C(\Omega)$, we consider the numerical evaluation of the integral

$$I(f) = \int_{\Omega} f(s,t)\,ds\,dt$$

by the cubature rule defined by

$$I(Q_{mn}f) = \sum_{(i,j)\in A_{mn}} w_{i,j}\,f(M_{i,j}), \tag{2.6}$$

where the weights are

$$w_{i,j} = \int_{\Omega} L_{i,j}(s,t)\,ds\,dt.$$

Since the fundamental functions $L_{i,j}$'s are linear combinations of B-splines (see Appendix A), the weights $w_{i,j}$ are linear combinations of $\int_\Omega B_{k,l}(s,t)\,ds\,dt$, whose values have been computed in [8]. Therefore, we can get the values $\bar{w}_{i,j}$, such that $w_{i,j} = h^2\bar{w}_{i,j}$, and we report them in Table 1, remarking their symmetry properties.

From Theorem 2.1, we can easily deduce the following result.

Theorem 2.3 *Let $f \in C(\Omega)$ and $E(f) = I(f) - I(Q_{mn}f)$. Then,*

$$|E(f)| \leq \bar{C}\omega(f,h),$$

where $\bar{C}$ is a positive constant independent of m and n.

Moreover if $f \in C^\nu(\Omega)$, $\nu = 1, 2, 3$, then

$$E(f) = O(h^\nu).$$

We remark that the above cubature has precision degree at least 2, because Q_{mn} is exact on $\mathbb{P}_2$. However, since uniform partitions are special cases of the ones with symmetric knots with respect to the center of Ω, Corollary 1 of [14] can be generalized to our case, getting

$$I(f) = I(Q_{mn}f) \quad \text{for } f(s,t) = s^{r_1}t^{r_2},$$

with $0 \leq r_1, r_2 \leq 3$, $r_1 + r_2 = 3$ and $r_1, r_2 = 1, 3$, with $r_1 + r_2 = 4$. Therefore the precision degree of the cubature (2.6) is 3 and, if $f \in C^4(\Omega)$, then

$$E(f) = O(h^4).$$

3 Spline collocation methods for integral equations on surfaces in $\mathbb{R}^3$

In this section we present and analyse three collocation methods based on the sequence $\{Q_{mn}\}$ of spline QI operators defined in Sect. 2.

The first one is a generalization of the classical collocation method. The last two are characterized by high order of convergence, under suitable hypothesis.

3.1 Spline modified collocation method

Approximate the integral equation (1.4) by

$$(\mathscr{I} - Q_{mn}\mathscr{K})\rho_{mn} = Q_{mn}\psi. \tag{3.1}$$

We write the approximated solution ρ_{mn}, belonging to $S_2^1(\Omega, T_{mn})$, as

$$\rho_{mn}(\mathbf{F}(u,v)) = \sum_{\alpha \in A_{mn}} X_\alpha L_\alpha(u,v), \quad \text{with} \quad \alpha = (i,j). \tag{3.2}$$

Table 1 The coefficients $\bar{w}_{i,j}$'s

$j \backslash i$	0	1	2	3	4	5	…	$m-4$	$m-3$	$m-2$	$m-1$	m	$m+1$
$n+1$	$-\frac{15161}{120960}$	$\frac{1187}{7200}$	$\frac{4943}{69120}$	$\frac{17}{192}$	$\frac{10721}{161280}$	$\frac{1}{15}$	…	$\frac{1}{15}$	$\frac{10721}{161280}$	$\frac{17}{192}$	$\frac{4943}{69120}$	$\frac{1187}{7200}$	$-\frac{15161}{120960}$
n	$\frac{1187}{7200}$	$\frac{2343}{2560}$	$\frac{67}{80}$	$\frac{767}{800}$	$\frac{23}{24}$	$\frac{23}{24}$	…	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{767}{800}$	$\frac{67}{80}$	$\frac{2343}{2560}$	$\frac{1187}{7200}$
$n-1$	$\frac{4943}{69120}$	$\frac{67}{80}$	$\frac{68737}{69120}$	$\frac{7451}{7680}$	$\frac{23}{24}$	$\frac{23}{24}$	…	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{7451}{7680}$	$\frac{68737}{69120}$	$\frac{67}{80}$	$\frac{4943}{69120}$
$n-2$	$\frac{17}{192}$	$\frac{767}{800}$	$\frac{7451}{7680}$	$\frac{31}{30}$	$\frac{61}{60}$	$\frac{61}{60}$	…	$\frac{61}{60}$	$\frac{61}{60}$	$\frac{31}{30}$	$\frac{7451}{7680}$	$\frac{767}{800}$	$\frac{17}{192}$
$n-3$	$\frac{10721}{161280}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{61}{60}$	1	1	…	1	1	$\frac{61}{60}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{10721}{161280}$
$n-4$	$\frac{1}{15}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{61}{60}$	1	1	…	1	1	$\frac{61}{60}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{1}{15}$
…	…	…	…	…	…	…		…	…	…	…	…	…
5	$\frac{1}{15}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{61}{60}$	1	1	…	1	1	$\frac{61}{60}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{1}{15}$
4	$\frac{10721}{161280}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{61}{60}$	1	1	…	1	1	$\frac{61}{60}$	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{10721}{161280}$
3	$\frac{17}{192}$	$\frac{767}{800}$	$\frac{7451}{7680}$	$\frac{31}{30}$	$\frac{61}{60}$	$\frac{61}{60}$	…	$\frac{61}{60}$	$\frac{61}{60}$	$\frac{31}{30}$	$\frac{7451}{7680}$	$\frac{767}{800}$	$\frac{17}{192}$
2	$\frac{4943}{69120}$	$\frac{67}{80}$	$\frac{68737}{69120}$	$\frac{7451}{7680}$	$\frac{23}{24}$	$\frac{23}{24}$	…	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{7451}{7680}$	$\frac{68737}{69120}$	$\frac{67}{80}$	$\frac{4943}{69120}$
1	$\frac{1187}{7200}$	$\frac{2343}{2560}$	$\frac{67}{80}$	$\frac{767}{800}$	$\frac{23}{24}$	$\frac{23}{24}$	…	$\frac{23}{24}$	$\frac{23}{24}$	$\frac{767}{800}$	$\frac{67}{80}$	$\frac{2343}{2560}$	$\frac{1187}{7200}$
0	$-\frac{15161}{120960}$	$\frac{1187}{7200}$	$\frac{4943}{69120}$	$\frac{17}{192}$	$\frac{10721}{161280}$	$\frac{1}{15}$	…	$\frac{1}{15}$	$\frac{10721}{161280}$	$\frac{17}{192}$	$\frac{4943}{69120}$	$\frac{1187}{7200}$	$-\frac{15161}{120960}$

Substituting the expressions of Q_{mn} given in (2.3) and ρ_{mn} given in (3.2) into (3.1), we find

$$\sum_{\alpha\in A_{mn}} X_\alpha L_\alpha - \sum_{\alpha\in A_{mn}} \sum_{\beta\in A_{mn}} X_\beta \bar{L}_\beta(M_\alpha) L_\alpha = \sum_{\alpha\in A_{mn}} \psi(\mathbf{F}(M_\alpha)) L_\alpha,$$

with $\bar{L}_\beta = \mathscr{K} L_\beta$. Therefore, by identifying the coefficients of L_α, we obtain

$$X_\alpha - \sum_{\beta\in A_{mn}} X_\beta \bar{L}_\beta(M_\alpha) = \psi(\mathbf{F}(M_\alpha)), \quad \alpha \in A_{mn}.$$

This is a linear system of $(m+2)(n+2)$ equations, that can be written in the form

$$(I - A)\mathbf{X} = \mathbf{a} \tag{3.3}$$

where A is the matrix with entries

$$A_{\alpha\beta} := \bar{L}_\beta(M_\alpha) = \int_\Omega K(\mathbf{F}(M_\alpha), \mathbf{F}(s,t))\, |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)|\, L_\beta(s,t)\, ds\, dt \tag{3.4}$$

and $\mathbf{a}$ is the vector with elements $\mathbf{a}_\alpha := \psi(\mathbf{F}(M_\alpha))$.

The iterated solution is defined by

$$\overline{\rho}_{mn} = \mathscr{K}\rho_{mn} + \psi$$

and it satisfies the equation

$$(\mathscr{I} - \mathscr{K} Q_{mn})\overline{\rho}_{mn} = \psi. \tag{3.5}$$

Therefore, it is necessary to construct $\rho_{mn} = \sum_{\alpha\in A_{mn}} X_\alpha L_\alpha$ and then $\mathscr{K}\rho_{mn} = \sum_{\alpha\in A_{mn}} X_\alpha \bar{L}_\alpha$, in order to finally get $\overline{\rho}_{mn} = \psi + \sum_{\alpha\in A_{mn}} X_\alpha \bar{L}_\alpha$.

We remark that the idea of defining a collocation method by operators that are not projectors has been proposed in [2] for univariate integral equations.

In order to get convergence results, firstly we give the following lemma.

Lemma 3.1 *For m, n large enough, we have*

$$\|\rho - \rho_{mn}\|_\infty \le C_1 \|(\mathscr{I} - Q_{mn})\rho\|_\infty \tag{3.6}$$

and

$$\left\|\rho - \overline{\rho}_{mn}\right\|_\infty \le C_2 \|\mathscr{K}(\mathscr{I} - Q_{mn})\rho\|_\infty, \tag{3.7}$$

where C_1 and C_2 are real constants independent of m and n.

Proof From the standard theory (see e.g. [3]), since $\mathscr{K}$ is a compact operator and Q_{mn} is pointwise convergent to the identity operator, we have that

$$\|(\mathscr{I} - Q_{mn})\mathscr{K}\|_\infty \to 0, \qquad \text{as} \quad m, n \to \infty.$$

Moreover, since $(\mathscr{I}-\mathscr{K})$ is invertible, then $(\mathscr{I}-Q_{mn}\mathscr{K})$ and $(\mathscr{I}-\mathscr{K}Q_{mn})$ are invertible for m, n large enough and

$$\left\|(\mathscr{I}-Q_{mn}\mathscr{K})^{-1}\right\|_\infty \le C_1, \tag{3.8}$$
$$\left\|(\mathscr{I}-\mathscr{K}Q_{mn})^{-1}\right\|_\infty \le C_2. \tag{3.9}$$

where C_1 and C_2 are real constants independent of m and n.

Now, applying the operator Q_{mn} to the Eq. (1.4) and rearranging, we find

$$\rho - Q_{mn}\mathscr{K}\rho = \rho + Q_{mn}\psi - Q_{mn}\rho. \tag{3.10}$$

From (3.1) and (3.10), we get

$$(\mathscr{I}-Q_{mn}\mathscr{K})(\rho-\rho_{mn}) = (\mathscr{I}-Q_{mn})\rho.$$

Then, from (3.8), we obtain (3.6).

Let us consider the two equations (1.4) and (3.5). By the same above arguments, we obtain

$$(\mathscr{I}-\mathscr{K}Q_{mn})(\rho-\overline{\rho}_{mn}) = \mathscr{K}(\mathscr{I}-Q_{mn})\rho.$$

Then, from (3.9), we get (3.7). □

Now we can prove the following result.

Theorem 3.1 *Let $\rho \in C^3(\Omega)$, then*

$$\|\rho-\rho_{mn}\|_\infty = O(h^3). \tag{3.11}$$

Moreover, if $\rho \in C^4(\Omega)$ and $K(\mathbf{P}_1,\cdot) \in C^1(S)$, then

$$\left\|\rho-\overline{\rho}_{mn}\right\|_\infty = O(h^4). \tag{3.12}$$

Proof From (3.6) and Theorem 2.1, we get (3.11).

From (3.7) and Theorem 2.2, (3.12) is proved. □

3.2 Spline collocation methods with high order of convergence

In this section, we propose to approximate $\mathscr{K}$, in (1.4), by one of the following finite rank operators

$$\mathscr{K}_{mn,i} := Q_{mn}\mathscr{K} + \mathscr{K}^*_{mn,i} - Q_{mn}\mathscr{K}^*_{mn,i}, \qquad i = 1, 2, \tag{3.13}$$

where

1. $\mathscr{K}^*_{mn,1}$ is the degenerate kernel operator defined by

$$\begin{aligned}
&\mathscr{K}^*_{mn,1}\rho(\mathbf{F}(u,v)) \\
&:= \int_\Omega Q_{mn}\left(K(\mathbf{F}(u,v),\mathbf{F}(s,t))\left|(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)\right|\right)\rho(\mathbf{F}(s,t))\,ds\,dt \\
&= \sum_{\alpha\in A_{mn}} K(\mathbf{F}(u,v),\mathbf{F}(M_\alpha))\left|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\alpha)\right| \\
&\quad \cdot \int_\Omega L_\alpha(s,t)\rho(\mathbf{F}(s,t))\,ds\,dt, \qquad (3.14)
\end{aligned}$$

2. $\mathscr{K}^*_{mn,2}$ is the Nyström operator based on Q_{mn} and defined by

$$\mathscr{K}^*_{mn,2}\rho(\mathbf{F}(u,v)) := \sum_{\alpha\in A_{mn}} w_\alpha K(\mathbf{F}(u,v),\mathbf{F}(M_\alpha))\left|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\alpha)\right|\rho(\mathbf{F}(M_\alpha)), \qquad (3.15)$$

according to (2.6).

We approximate (1.4) by

$$\rho_{mn,i} - (Q_{mn}\mathscr{K} + \mathscr{K}^*_{mn,i} - Q_{mn}\mathscr{K}^*_{mn,i})\rho_{mn,i} = \psi, \qquad i = 1,2. \qquad (3.16)$$

The iterated solution is defined by

$$\overline{\rho}_{mn,i} = \mathscr{K}\rho_{mn,i} + \psi.$$

Now, we consider the reduction of (3.16) to two systems of $2(m+2)(n+2)$ linear equations.

After some algebra, from (3.14) and (3.16), we can write the approximate solution $\rho_{mn,1}$ as:

$$\begin{aligned}
\rho_{mn,1}(\mathbf{F}(u,v)) = \psi(\mathbf{F}(u,v)) &+ \sum_{\alpha\in A_{mn}} X_\alpha L_\alpha(u,v) \\
&+ \sum_{\alpha\in A_{mn}} Y_\alpha K(\mathbf{F}(u,v),\mathbf{F}(M_\alpha))\left|(D_s\mathbf{F}\times D_t\mathbf{F})(M_\alpha)\right|,
\end{aligned}$$

where the unknowns $\{X_\alpha\}$ and $\{Y_\alpha\}$, $\alpha \in A_{mn}$, are obtained by solving the linear system $(I-R)\mathbf{Z} = \mathbf{d}$, with

$$R := \begin{bmatrix} A & D-B \\ C & E \end{bmatrix}, \quad \mathbf{Z} := \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \mathbf{d} := \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \qquad (3.17)$$

and $A, B, C, D, E \in \mathbb{R}^{(m+2)(n+2)\times(m+2)(n+2)}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^{(m+2)(n+2)}$, whose entries are given by

– $A_{\alpha,\beta} := \bar{L}_\beta(M_\alpha)$, see (3.4),

- $B_{\alpha,\beta} := K(\mathbf{F}(M_\alpha), \mathbf{F}(M_\beta)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)\right|$,
- $C_{\alpha,\beta} := \int_\Omega L_\alpha(s,t) L_\beta(s,t)\, ds\, dt$,
- $D_{\alpha,\beta} := \int_\Omega K(\mathbf{F}(M_\alpha), \mathbf{F}(s,t)) |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| K(\mathbf{F}(s,t), \mathbf{F}(M_\beta)) |(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)|\, ds\, dt$,
- $E_{\alpha,\beta} := \int_\Omega K(\mathbf{F}(s,t), \mathbf{F}(M_\beta)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)\right| L_\alpha(s,t)\, ds\, dt$,
- $\mathbf{b}_\alpha := \mathscr{K}\psi(\mathbf{F}(M_\alpha)) = \int_\Omega K(\mathbf{F}(M_\alpha), \mathbf{F}(s,t)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)\right| \psi(\mathbf{F}(s,t))\, ds\, dt$,
- $\mathbf{c}_\alpha := \int_\Omega \psi(\mathbf{F}(s,t)) L_\alpha(s,t)\, ds\, dt$.

Similarly, from (3.15) and (3.16), we can get that the solution $\rho_{mn,2}$ is

$$\rho_{mn,2}(\mathbf{F}(u,v)) = \psi(\mathbf{F}(u,v)) + \sum_{\alpha \in A_{mn}} X_\alpha L_\alpha(u,v) + \sum_{\alpha \in A_{mn}} w_\alpha Y_\alpha K(\mathbf{F}(u,v), \mathbf{F}(M_\alpha)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(M_\alpha)\right|,$$

where the unknowns $\{X_\alpha\}$ and $\{Y_\alpha\}$, $\alpha \in A_{mn}$, are obtained by solving the linear system $(I - T)\mathbf{Z} = \mathbf{f}$, with

$$T := \begin{bmatrix} A & F - G \\ H & G \end{bmatrix}, \quad \mathbf{Z} := \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} \mathbf{a} \\ \mathbf{e} \end{bmatrix} \tag{3.18}$$

and $F, G, H \in \mathbb{R}^{(m+2)(n+2)\times(m+2)(n+2)}$, $\mathbf{e} \in \mathbb{R}^{(m+2)(n+2)}$, whose entries are given by

- $F_{\alpha,\beta} := w_\beta \int_\Omega K(\mathbf{F}(M_\alpha), \mathbf{F}(s,t)) |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| K(\mathbf{F}(s,t), \mathbf{F}(M_\beta)) |(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)|\, ds\, dt$,
- $G_{\alpha,\beta} := w_\beta K(\mathbf{F}(M_\alpha), \mathbf{F}(M_\beta)) \left|(D_s\mathbf{F} \times D_t\mathbf{F})(M_\beta)\right|$,
- $H_{\alpha,\beta} := L_\beta(M_\alpha)$,
- $\mathbf{e}_\alpha := \psi(\mathbf{F}(M_\alpha))$.

Now, we are able to state the following convergence results.

Lemma 3.2 *For m, n large enough and $i = 1, 2$, it holds*

$$\left\|\rho - \rho_{mn,i}\right\|_\infty \le C_{3,i} \left\|(\mathscr{I} - Q_{mn})(\mathscr{K} - \mathscr{K}^*_{mn,i})\rho\right\|_\infty \tag{3.19}$$

and

$$\left\|\rho - \overline{\rho}_{mn,i}\right\|_\infty \le \left\|(\mathscr{I} - \mathscr{K})^{-1}\right\|_\infty \Big(\left\|\mathscr{K}(\mathscr{I} - Q_{mn})(\mathscr{K} - \mathscr{K}^*_{mn,i})\rho\right\|_\infty + \left\|\mathscr{K}(\mathscr{I} - Q_{mn})(\mathscr{K} - \mathscr{K}^*_{mn,i})\right\|_\infty \left\|\rho - \rho_{mn,i}\right\|_\infty\Big), \tag{3.20}$$

where $C_{3,i}$ are real constants independent of m and n.

Proof From (3.13) and Theorem 2.1, we have

$$\begin{aligned}\left\|\mathscr{K}-\mathscr{K}_{mn,i}\right\|_\infty &= \left\|\mathscr{K}-Q_{mn}\mathscr{K}-\mathscr{K}^*_{mn,i}+Q_{mn}\mathscr{K}^*_{mn,i}\right\|_\infty \\ &= \left\|(\mathscr{I}-Q_{mn})(\mathscr{K}-\mathscr{K}^*_{mn,i})\right\|_\infty \to 0, \qquad \text{as} \quad m,n\to\infty.\end{aligned}$$

Then, $(\mathscr{I}-\mathscr{K}_{mn,i})$ is invertible for m, n large enough and

$$\left\|(\mathscr{I}-\mathscr{K}_{mn,i})^{-1}\right\|_\infty \le C_{3,i}, \tag{3.21}$$

with $C_{3,i}$, $i=1,2$, real constants independent of m and n.

Therefore, by a procedure similar to that one used in [1, Theorem 5], we can write

$$\rho-\rho_{mn,i}=(\mathscr{I}-\mathscr{K}_{mn,i})^{-1}(\mathscr{K}-\mathscr{K}_{mn,i})\rho.$$

Thus, from (3.21), (3.19) follows.

Moreover, since

$$\begin{aligned}\rho-\overline{\rho}_{mn,i} &= \mathscr{K}(\rho-\rho_{mn,i}) \\ &= \mathscr{K}(\mathscr{I}-\mathscr{K})^{-1}(\mathscr{K}-\mathscr{K}_{mn,i})\rho_{mn,i} \\ &= (\mathscr{I}-\mathscr{K})^{-1}\mathscr{K}(\mathscr{I}-Q_{mn})(\mathscr{K}-\mathscr{K}^*_{mn,i})(\rho_{mn,i}\pm\rho),\end{aligned}$$

we can easily get (3.20). □

Theorem 3.2 *Assume that ρ is differentiable with bounded derivatives, $K(\cdot,\cdot)\in C^4(S\times S)$ and $\mathbf{F}\in C^5(\Omega)$. Then*

$$\left\|\rho-\rho_{mn,1}\right\|_\infty = O(h^7) \tag{3.22}$$

and

$$\left\|\rho-\overline{\rho}_{mn,1}\right\|_\infty = O(h^8). \tag{3.23}$$

Proof From (3.19) of Lemma 3.2, with $i=1$ and Theorem 2.1

$$\begin{aligned}\left\|\rho-\rho_{mn,1}\right\|_\infty &\le C_{3,1}\left\|(\mathscr{I}-Q_{mn})(\mathscr{K}-\mathscr{K}^*_{mn,1})\rho\right\|_\infty \\ &\le C_4h^3\left\|D^3[(\mathscr{K}-\mathscr{K}^*_{mn,1})\rho]\right\|_\infty\end{aligned} \tag{3.24}$$

where C_4 is a real constant independent of m and n. Since, for $\beta=(\beta_1,\beta_2)$ and $|\beta|\le 4$,

$$\begin{aligned}&D^\beta\left[(\mathscr{K}-\mathscr{K}^*_{mn,1})\rho(\mathbf{F}(u,v))\right] \\ &= \int_\Omega \rho(\mathbf{F}(s,t))(\mathscr{I}-Q_{mn})\left[\frac{\partial^{|\beta|}K(\mathbf{F}(u,v),\mathbf{F}(s,t))}{\partial u^{\beta_1}\partial v^{\beta_2}}\left|(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)\right|\right]ds\,dt,\end{aligned} \tag{3.25}$$

from Theorem 2.2, with

$$f(s,t) = \frac{\partial^{|\beta|} K(\mathbf{F}(u,v),\mathbf{F}(s,t))}{\partial u^{\beta_1}\partial v^{\beta_2}} \,|(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)| \quad \text{and} \quad g(s,t) = \rho(\mathbf{F}(s,t)),$$

we can get

$$\left\| D^3[(\mathscr{K} - \mathscr{K}^*_{mn,1})\rho]\right\|_\infty = O(h^4). \tag{3.26}$$

Therefore, from (3.24) and (3.26), (3.22) follows.

Now, from (3.20) of Lemma 3.2, with $i = 1$

$$\begin{aligned}\left\|\rho - \overline{\rho}_{mn,1}\right\|_\infty \le \left\|(\mathscr{I} - \mathscr{K})^{-1}\right\|_\infty \Bigg[&\underbrace{\left\|\mathscr{K}(\mathscr{I} - Q_{mn})(\mathscr{K} - \mathscr{K}^*_{mn,1})\rho\right\|_\infty}_{(i)} \\ + \|\mathscr{K}\|_\infty \Bigg(\underbrace{\|(\mathscr{I} - Q_{mn})\mathscr{K}\|_\infty}_{(ii)} + \underbrace{\left\|(\mathscr{I} - Q_{mn})\mathscr{K}^*_{mn,1}\right\|_\infty}_{(iii)} \Bigg) &\left\|\rho - \rho_{mn,1}\right\|_\infty \Bigg],\end{aligned} \tag{3.27}$$

Consider (i) in (3.27). From Theorem 2.2, with

$$\begin{aligned} f(s,t) &= (\mathscr{K} - \mathscr{K}^*_{mn,1})\rho(\mathbf{F}(s,t)), \\ g(s,t) &= K(\mathbf{F}(u,v),\mathbf{F}(s,t))\,|(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)|, \end{aligned}$$

and, from (3.25), with $|\beta| = 3, 4$, we get

$$(i) = O(h^8). \tag{3.28}$$

From Theorem 2.1 we have

$$\|(\mathscr{I} - Q_{mn})\mathscr{K}\rho\|_\infty \le C_5 h^3 \left\|D^3(\mathscr{K}\rho)\right\|_\infty \tag{3.29}$$

with C_5 a real constant independent of m and n. Since, for $\beta = (\beta_1, \beta_2)$ and $|\beta| = 3$,

$$\begin{aligned} D^\beta\left[\mathscr{K}\rho(\mathbf{F}(u,v))\right] = \int_\Omega &\frac{\partial^{|\beta|} K(\mathbf{F}(u,v),\mathbf{F}(s,t))}{\partial u^{\beta_1}\partial v^{\beta_2}} \\ &\times |(D_s\mathbf{F}\times D_t\mathbf{F})(s,t)|\,\rho(\mathbf{F}(s,t))\,ds\,dt, \end{aligned}$$

then, from (3.29),

$$(ii) = O(h^3). \tag{3.30}$$

Similarly, we can easily show that that

$$(iii) = O(h^3). \tag{3.31}$$

Finally, from (3.22), (3.27), (3.28), (3.30) and (3.31) we obtain (3.23). □

Theorem 3.3 *Assume* $\rho \in C^4(S)$, $K(\cdot,\cdot) \in C^4(S \times S)$ *and* $\mathbf{F} \in C^5(\Omega)$. *Then*

$$\left\| \rho - \rho_{mn,2} \right\|_\infty = O(h^7) \tag{3.32}$$

and

$$\left\| \rho - \overline{\rho}_{mn,2} \right\|_\infty = O(h^8). \tag{3.33}$$

Proof From (3.19) of Lemma 3.2, with $i = 2$ and Theorem 2.1

$$\begin{aligned} \left\| \rho - \rho_{mn,2} \right\|_\infty &\leq C_{3,2} \left\| (\mathscr{I} - Q_{mn})(\mathscr{K} - \mathscr{K}^*_{mn,2})\rho \right\|_\infty \\ &\leq C_6 h^3 \left\| D^3[(\mathscr{K} - \mathscr{K}^*_{mn,2})\rho] \right\|_\infty \end{aligned} \tag{3.34}$$

where C_6 is a real constant independent of m and n. Since, for $\beta = (\beta_1, \beta_2)$ and $|\beta| \leq 4$,

$$\begin{aligned} & D^\beta \left[(\mathscr{K} - \mathscr{K}^*_{mn,2})\rho(\mathbf{F}(u,v)) \right] \\ &\quad = \int_\Omega (\mathscr{I} - Q_{mn}) \left[\frac{\partial^{|\beta|} K(\mathbf{F}(u,v), \mathbf{F}(s,t))}{\partial u^{\beta_1} \partial v^{\beta_2}} \left| (D_s\mathbf{F} \times D_t\mathbf{F})(s,t) \right| \rho(\mathbf{F}(s,t)) \right] ds\, dt, \end{aligned}$$

from Theorem 2.2 with

$$f(s,t) = \frac{\partial^{|\beta|} K(\mathbf{F}(u,v), \mathbf{F}(s,t))}{\partial u^{\beta_1} \partial v^{\beta_2}} \left| (D_s\mathbf{F} \times D_t\mathbf{F})(s,t) \right| \rho(\mathbf{F}(s,t)) \quad \text{and} \quad g(s,t) = 1,$$

we can get

$$\left\| D^3[(\mathscr{K} - \mathscr{K}^*_{mn,2})\rho] \right\|_\infty = O(h^4). \tag{3.35}$$

Therefore, from (3.34) and (3.35), (3.32) follows.

Now, from (3.20) of Lemma 3.2, with $i = 2$, following the same logical scheme used in Theorem 3.2, we can get (3.33). □

3.3 Discrete versions and numerical tests

In Sects. 3.1 and 3.2, we discussed spline collocation methods for solving surface integral equations.

In practice, by using the collocation methods (3.1) and (3.16), we have to evaluate many integrals and usually it must be done by suitable numerical integration formulas. Therefore, we have to discretize the proposed methods by introducing convenient cubatures and we denote by ρ^D_{mn}, $\rho^D_{mn,i}$, $i = 1, 2$, the corresponding solutions.

Here, we decide to compute the entries of the matrices and vectors appearing in (3.3), (3.17), (3.18), by using a composite Gaussian cubature on triangular domains (see [15]), implemented by the Matlab function `triquad` (see [21]), with N^2 nodes in each triangle of T_{mn} and with precision degree $2N - 1$. The number of nodes is chosen to preserve the approximation order of the method. Therefore, we choose $N = 2$ for

Table 2 Maximum absolute errors and numerical convergence orders

m	n	E_{mn}	o_{mn}	$E_{mn,1}$	$o_{mn,1}$	$E_{mn,2}$	$o_{mn,2}$
4	8	7.56e−03	–	2.51e−05	–	3.17e−05	–
8	16	8.11e−04	3.22	2.09e−07	6.91	1.29e−07	7.94
16	32	8.21e−05	3.30	1.48e−09	7.14	1.71e−09	6.24
32	64	8.34e−06	3.30	1.12e−11	7.04	1.46e−11	6.87

the spline modified collocation method (3.1) and $N = 4$ for the two spline collocation methods with high order of convergence (3.16).

We test the performances of the proposed methods in the numerical solution of the surface integral equation from [3]

$$\rho(\mathbf{P}_1) - \frac{1}{30}\int_S \rho(\mathbf{P}_2)\frac{\partial}{\partial \mathbf{n}_{\mathbf{P}_2}}\left(\|\mathbf{P}_1 - \mathbf{P}_2\|^2\right) dS_{\mathbf{P}_2} = \frac{1}{30}\psi(\mathbf{P}_1), \ \mathbf{P}_1 \in S, \tag{3.36}$$

where S is the ellipsoidal surface given by

$$x^2 + \left(\frac{4y}{3}\right)^2 + (2z)^2 = 1,$$

$\mathbf{n}_{\mathbf{P}_2}$ is the inner normal to S at $\mathbf{P}_2$ and

$$\mathbf{F}(s,t) = \begin{bmatrix} \sin(s)\cos(t) \\ \frac{3}{4}\sin(s)\sin(t) \\ \frac{1}{2}\cos(s) \end{bmatrix}, \qquad (s,t) \in \Omega = [0,\pi] \times [0, 2\pi].$$

We choose $\rho(\mathbf{P}) = e^{\frac{1}{2}\cos(s)}$ and define ψ accordingly.

For each method we compute the maximum absolute errors

$$E_{mn} = \max_{(u,v)\in G} |\rho(u,v) - \rho^D_{mn}(u,v)|$$
$$E_{mn,i} = \max_{(u,v)\in G} |\rho(u,v) - \rho^D_{mn,i}(u,v)|, \quad i = 1,2,$$

for increasing values of m and n, where G is a uniform grid of 100×100 points in Ω. We also compute the corresponding numerical convergence orders o_{mn}, $o_{mn,i}$, $i = 1, 2$.

The results are shown in Table 2 and we can notice that they agree with the theoretical ones.

3.4 Approximating the surface

As noticed in the Introduction, in (1.2) the evaluation of the Jacobian (1.3) is required. With some surfaces the function $\mathbf{F}$ and its derivatives are easily given and computed.

However, with certain other ones, the knowledge and the evaluation of (1.3) can be a major problem.

For this reason, we consider spline approximations $\tilde{S}$ for the surface S, for which the Jacobians are more easily evaluated.

Let

$$\mathbf{F}(s,t) = \begin{bmatrix} x^1(s,t) \\ x^2(s,t) \\ x^3(s,t) \end{bmatrix}, \quad (s,t) \in \Omega,$$

and

$$\tilde{S} = Q_{mn}\mathbf{F}(\Omega), \quad \text{with} \quad Q_{mn}\mathbf{F}(s,t) = \begin{bmatrix} Q_{mn}x^1(s,t) \\ Q_{mn}x^2(s,t) \\ Q_{mn}x^3(s,t) \end{bmatrix}, \quad (s,t) \in \Omega.$$

$\tilde{S}$ can be represented only by using the values of $\mathbf{F}$ at the points M_α and its derivatives are easily computable, since, in each triangle of T_{mn}, they are the derivatives of a polynomial of total degree two.

Therefore, instead of (1.1), we consider the equation

$$\tilde{\rho}(\mathbf{P}_1) - \int_{\tilde{S}} K(\mathbf{P}_1,\mathbf{P}_2)\tilde{\rho}(\mathbf{P}_2)d\tilde{S}_{\mathbf{P}_2} = \psi(\mathbf{P}_1), \quad \mathbf{P}_1 \in S.$$

that we solve by the spline modified collocation method of Sect. 3.1.

3.4.1 Spline modified collocation method with approximated surface

If we define

$$\tilde{\mathscr{K}}\tilde{\rho}(\mathbf{F}(u,v)) := \int_\Omega K(\mathbf{F}(u,v), Q_{mn}\mathbf{F}(s,t))\tilde{\rho}(Q_{mn}\mathbf{F}(s,t))\,|(D_s Q_{mn}\mathbf{F} \times D_t Q_{mn}\mathbf{F})(s,t)|\;ds\,dt,$$

we have to consider the integral equation

$$(\mathscr{I} - \tilde{\mathscr{K}})\tilde{\rho} = \psi. \tag{3.37}$$

We apply the method of Sect. 3.1 to numerically solve (3.37), obtaining

$$(\mathscr{I} - Q_{mn}\tilde{\mathscr{K}})\tilde{\rho}_{mn} = Q_{mn}\psi,$$

where we require that the approximated solution $\tilde{\rho}_{mn}$ has the form

$$\tilde{\rho}_{mn}(\mathbf{F}(u,v)) = \sum_{\alpha \in A_{mn}} \tilde{X}_\alpha L_\alpha(u,v).$$

This is equivalent to solve the linear system

$$(I - \tilde{A})\tilde{\mathbf{X}} = \mathbf{a}$$

with

$$\tilde{A}_{\alpha\beta} := \tilde{L}_\beta(M_\alpha), \qquad \mathbf{a}_\alpha := \psi(\mathbf{F}(M_\alpha)), \tag{3.38}$$

and

$$\begin{aligned}\tilde{L}_\beta(u, v) &:= \tilde{\mathscr{K}} L_\beta(u, v)\\ &= \int_\Omega K(\mathbf{F}(u, v), Q_{mn}\mathbf{F}(s, t))L_\beta(Q_{mn}\mathbf{F}(s, t))\,|(D_s Q_{mn}\mathbf{F} \times D_t Q_{mn}\mathbf{F})(s, t)|\,ds\,dt.\end{aligned}$$

Concerning the convergence order, we prove the following theorem.

Theorem 3.4 *Let the kernel function $K(\cdot, \cdot) \in C^2(S \times S)$ and $\rho \in C^4(S)$. Then*

$$\|\rho - \tilde{\rho}_{mn}\|_\infty = O(h^3). \tag{3.39}$$

Proof We have

$$\|\rho - \tilde{\rho}_{mn}\|_\infty \le \|\rho - \tilde{\rho}\|_\infty + \|\tilde{\rho} - \tilde{\rho}_{mn}\|_\infty = E_1 + E_2.$$

In order to bound E_2, we can follow the same argument used in Theorem 3.1, obtaining

$$E_2 \le \left\|(\mathscr{I} - Q_{mn}\tilde{\mathscr{K}})^{-1}\right\|_\infty \|(\mathscr{I} - Q_{mn})\tilde{\rho}\|_\infty$$

and, from Theorem 2.1,

$$E_2 = O(h^3). \tag{3.40}$$

Now we focus on E_1. From (1.4) and (3.37), by adding $\tilde{\mathscr{K}}\rho$ on both terms we have

$$(\mathscr{I} - \tilde{\mathscr{K}})(\rho - \tilde{\rho}) = (\mathscr{K} - \tilde{\mathscr{K}})\rho$$

and

$$E_1 \le \left\|(\mathscr{I} - \tilde{\mathscr{K}})^{-1}\right\|_\infty \left\|(\mathscr{K} - \tilde{\mathscr{K}})\rho\right\|_\infty \le C_7 \left\|(\mathscr{K} - \tilde{\mathscr{K}})\rho\right\|_\infty, \tag{3.41}$$

for a suitable real constant C_7 independent of m and n. Then, we write

$$(\mathscr{K} - \tilde{\mathscr{K}})\rho = E_{21} + E_{22} + E_{23} + E_{24} + E_{25}, \tag{3.42}$$

with

$$E_{21} = \int_{\Omega} K(\mathbf{F}(u,v), \mathbf{F}(s,t))\rho(\mathbf{F}(s,t)) \cdot [|(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| - |(D_s Q_{mn}\mathbf{F} \times D_t Q_{mn}\mathbf{F})(s,t)|] \, ds \, dt,$$

$$E_{22} = \int_{\Omega} K(\mathbf{F}(u,v), \mathbf{F}(s,t)) [\rho(\mathbf{F}(s,t)) - \rho(Q_{mn}\mathbf{F}(s,t))] \cdot [|(D_s Q_{mn}\mathbf{F} \times D_t Q_{mn}\mathbf{F})(s,t)| - |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)|] \, ds \, dt,$$

$$E_{23} = \int_{\Omega} K(\mathbf{F}(u,v), \mathbf{F}(s,t)) [\rho(\mathbf{F}(s,t)) - \rho(Q_{mn}\mathbf{F}(s,t))] \cdot |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| \, ds \, dt,$$

$$E_{24} = \int_{\Omega} [K(\mathbf{F}(u,v), \mathbf{F}(s,t)) - K(\mathbf{F}(u,v), Q_{mn}\mathbf{F}(s,t))] \rho(Q_{mn}\mathbf{F}(s,t)) \cdot [|(D_s Q_{mn}\mathbf{F} \times D_t Q_{mn}\mathbf{F})(s,t)| - |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)|] \, ds \, dt,$$

$$E_{25} = \int_{\Omega} [K(\mathbf{F}(u,v), \mathbf{F}(s,t)) - K(\mathbf{F}(u,v), Q_{mn}\mathbf{F}(s,t))] \cdot \rho(Q_{mn}\mathbf{F}(s,t)) \, |(D_s\mathbf{F} \times D_t\mathbf{F})(s,t)| \, ds \, dt.$$

Since Lemma 2.1 holds, we can follow an approach similar to that one proposed in [7, Theorem 1] for collocation methods based on C^0 quadratic piecewise interpolating polynomials, getting

$$E_{21} = O(h^4), \quad E_{22} = O(h^5), \quad E_{23} = O(h^4), \quad E_{24} = O(h^5), \quad E_{25} = O(h^4).$$

Therefore, from (3.41) and (3.42)

$$E_1 = O(h^4) \tag{3.43}$$

and, from (3.40) and (3.43) we obtain (3.39). □

3.4.2 Discrete version and numerical tests

Also in case of spline modified collocation method with approximated surface, we have to consider a discrete version, by introducing a convenient cubature formula and we denote by $\tilde{\rho}_{mn}^D$ the corresponding solution. We decide to compute the entries of the matrix $\tilde{A}$ given in (3.38) by using the same composite Gaussian cubature on triangular domains with four nodes, considered in Sect. 3.3. Therefore the approximation order three is preserved.

Now, we test the proposed method for the numerical solution of the surface integral equation (3.36).

Table 3 Maximum absolute errors and numerical convergence orders

m	n	$\tilde{E}_{mn}$	$\tilde{o}_{mn}$
4	8	1.95e−02	–
8	16	1.46e−03	3.74
16	32	1.18e−04	3.63
32	64	9.88e−06	3.58
64	128	1.09e−06	3.18

We compute the maximum absolute errors

$$\tilde{E}_{mn} = \max_{(u,v)\in G} |\rho(u, v) - \tilde{\rho}^D_{mn}(u, v)|$$

for increasing values of m and n, where G is a uniform grid of 100×100 points in Ω. We also compute the numerical convergence orders $\tilde{o}_{mn}$.

The results are shown in Table 3 and we can notice that they agree with the theoretical ones.

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Appendix A

Here we report the expression of the fundamental functions associated with Q_{mn} defined in (2.1), for $m, n \geq 8$. They are obtained from the coefficient functionals given in [16]. For the pairs (i, j) with $i = 4, \ldots, m-3$ and $j = 4, \ldots, n-3$

$$L_{i,j} = \frac{3}{2}B_{i,j} - \frac{1}{8}(B_{i,j-1} + B_{i,j+1} + B_{i-1,j} + B_{i+1,j}).$$

The other $L_{i,j}$'s have particular definitions. In the neighbourhood of the point (a, c) we have

$$L_{0,0} = \tfrac{1403}{504}B_{0,0} - \tfrac{4}{15}B_{1,1},$$

$$L_{1,0} = \tfrac{131}{60}B_{1,0} - \tfrac{173}{300}B_{0,1} - \tfrac{1}{12}B_{2,1},$$

$$L_{2,0} = -\tfrac{397}{1440}B_{0,0} - \tfrac{2}{15}B_{1,1} + \tfrac{12}{5}B_{2,0} - \tfrac{1}{12}B_{3,1} - \tfrac{7}{30}B_{2,1} + \tfrac{9}{40}B_{1,0},$$

$$L_{3,0} = -\tfrac{1}{12}B_{4,1} + \tfrac{3}{20}B_{0,1} + \tfrac{12}{5}B_{3,0} - \tfrac{1}{12}B_{2,1} - \tfrac{7}{30}B_{3,1},$$

$$L_{4,0} = \tfrac{11}{224}B_{0,0} + \tfrac{12}{5}B_{4,0} - \tfrac{1}{120}B_{1,0} - \tfrac{7}{30}B_{4,1} - \tfrac{1}{12}B_{3,1} - \tfrac{1}{12}B_{5,1},$$

$$L_{1,1} = -\frac{63}{32}B_{0,0} - \frac{13}{40}(B_{1,0} + B_{0,1}) + \frac{33}{20}B_{1,1} - \frac{1}{4}(B_{2,0} + B_{0,2}),$$

$$L_{2,1} = -\frac{47}{60}B_{1,0} - \frac{9}{8}B_{2,0} - \frac{1}{4}B_{3,0} - \frac{1}{20}B_{1,1} + \frac{13}{8}B_{2,1} + \frac{1}{8}B_{0,2} - \frac{1}{24}B_{1,2} - \frac{1}{8}B_{2,2},$$

$$L_{3,1} = \frac{3}{50}B_{1,0} - \frac{1}{4}B_{2,0} - \frac{9}{8}B_{3,0} - \frac{1}{4}B_{4,0} - \frac{7}{40}B_{0,1} + \frac{1}{40}B_{1,1} + \frac{13}{8}B_{3,1} - \frac{1}{8}B_{3,2},$$

$$L_{2,2} = \frac{317}{288}B_{0,0} + \frac{1}{4}(B_{0,1} + B_{1,0}) + \frac{1}{8}(B_{3,0} + B_{0,3}) - \frac{1}{15}B_{1,1} - \frac{1}{6}(B_{1,2} + B_{2,1}) \\ -\frac{1}{24}(B_{1,3} + B_{3,1}) - \frac{1}{8}(B_{2,3} + B_{3,2}) + \frac{3}{2}B_{2,2},$$

$$L_{3,2} = -\frac{37}{160}B_{0,0} + \frac{1}{8}B_{2,0} + \frac{1}{8}B_{4,0} - \frac{1}{24}B_{2,1} - \frac{1}{24}B_{4,1} - \frac{1}{6}B_{3,1} - \frac{1}{40}B_{0,2} \\ +\frac{1}{40}B_{1,2} - \frac{1}{8}B_{2,2} - \frac{1}{8}B_{4,2} + \frac{3}{2}B_{3,2} - \frac{1}{8}B_{3,3},$$

$$L_{3,3} = -\frac{1}{40}(B_{3,0} + B_{0,3}) + \frac{1}{40}(B_{3,1} + B_{1,3}) - \frac{1}{8}(B_{3,2} + B_{2,3}) + \frac{3}{2}B_{3,3} \\ -\frac{1}{8}(B_{3,4} + B_{4,3}).$$

Along the lower edge, for $i = 5, \dots, m-4$, we have:

$$L_{i,0} = \frac{12}{5}B_{i,0} - \frac{7}{30}B_{i,1} - \frac{1}{12}(B_{i-1,1} + B_{i+1,1}),$$

and for $i = 4, \dots, m-3$:

$$L_{i,1} = -\frac{9}{8}B_{i,0} - \frac{1}{4}(B_{i-1,0} + B_{i+1,0}) + \frac{13}{8}B_{i,1} - \frac{1}{8}B_{i,2},$$

$$L_{i,2} = \frac{1}{8}(B_{i-1,0} + B_{i+1,0}) - \frac{1}{6}B_{i,1} - \frac{1}{24}(B_{i-1,1} + B_{i+1,1}) + \frac{3}{2}B_{i,2} \\ -\frac{1}{8}(B_{i-1,2} + B_{i+1,2}) - \frac{1}{8}B_{i,3},$$

$$L_{i,3} = -\frac{1}{40}B_{i,0} + \frac{1}{40}B_{i,1} - \frac{1}{8}B_{i,2} + \frac{3}{2}B_{i,3} - \frac{1}{8}(B_{i-1,3} + B_{i+1,3}) - \frac{1}{8}B_{i,4}.$$

Taking into account the coefficient functional symmetries, analogous formulas exist for the three other edges and vertices of Ω.

We remark that in case $m, n < 8$ the fundamental functions have particular expressions, always obtained from the coefficient functionals given in [16].

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