

A triple-parameter modified SSOR method for solving singular saddle point problems

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Abstract In this paper, we study a triple-parameter modified SSOR (TMSSOR) method for solving singular saddle point problems. We prove the semi-convergence of the TMSSOR method under suitable restrictions on the iteration parameters, and obtain the local optimal parameters which minimize the pseudo-spectral radii of the associated iteration matrices. Finally, numerical experiments demonstrate the effectiveness of the TMSSOR method for solving singular saddle point problems.

Keywords Singular linear systems · Saddle point problems · TMSSOR method · Semi-convergence

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1 Introduction

We consider the iterative solutions of a consistent linear system with the following form:

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$$\mathcal{A}\mathcal{X} = \begin{pmatrix} A & B \\ -B^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix} \quad (1.1)$$

where $A \in R^{m \times m}$ is positive real, i.e., $v^T Av > 0$, for any nonzero $v \in R^m$, $B \in R^{m \times n}$ is rank-deficient, i.e., $\text{rank}(B) < n$ and $b \in R^m$, $q \in R^n$ with $m \geq n$. We use B^T and B^* to denote the transpose and the conjugate transpose of the matrix B , respectively. Linear system (1.1) is often referred to a saddle point problem, which is important and arises in a wide variety of scientific and engineering applications such as computational fluid dynamics, optimization, optimal control, constrained least-squares problems, and so on, see [1, 6, 7, 9, 12, 13]. For a wider class of (generalized) saddle point problems, the readers can refer to [10, 14, 15, 17, 18, 20–22].

In the case of A being symmetric positive definite and B being of full column rank, a number of efficient iteration methods as well as their numerical properties have been studied. Bai et al. [3–7] proposed Hermitian and skew-Hermitian splitting (HSS) iteration method and developed it to solving standard and generalized saddle point problems. Golub et al. [21] presented SOR-like methods for solving the linear system (1.1). Bai et al. [9, 10] developed SOR-like methods, and presented the generalized SOR method and the parameterized inexact Uzawa method. Wu et al. [26] proposed the modified symmetric SOR (MSSOR) method. Recently, Najari Saberi and Edalatpanah [24] proposed a triple-parameter modified SSOR (TMSSOR) method for solving saddle point problems based on a new splitting and new relaxation parameters.

In the linear system (1.1), when B is rank-deficient, then the coefficient matrix is singular, and we call the linear system (1.1) a singular saddle point problem. Some iteration methods and preconditioning techniques for solving singular saddle point problems are proposed in the recent literature, see, e.g., [2, 16, 23, 27, 28, 30, 31]. Zheng et al. [30] applied parameterized Uzawa methods to solve singular saddle point problems. Li and Huang [23] investigated the semi-convergence of the generalized SSOR methods. Bai et al. [8, 22] studied constraint preconditioners, and for non-Hermitian singular saddle point problems, Zhang and Shen [28] provided some constraint preconditioners to accelerate GMRES. In this paper, we study the TMSSOR method for solving singular saddle point problems.

The rest of this paper is organized as follows. In Sect. 2, we present the TMSSOR method for solving the singular saddle point problem (1.1). In Sect. 3, we demonstrate the semi-convergence of this new method. In Sect. 4, we give the local optimal parameters of the TMSSOR method under certain restrictions on the parameters involved. In Sect. 5, several numerical experiments are given to show the efficiency of the method. Finally, conclusions are made for this paper in Sect. 6.

2 The triple-parameter modified SSOR method

For solving the singular saddle point problem (1.1), we make the following matrix splitting [24]

$$\begin{pmatrix} A & B \\ -B^T & \mathbf{0} \end{pmatrix} = D - L - U \quad (2.1)$$

where

$$D = \begin{pmatrix} \beta A & \mathbf{0} \\ \mathbf{0} & Q \end{pmatrix}, \quad L = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B^T & \alpha Q \end{pmatrix}, \quad U = \begin{pmatrix} (\beta - 1)A & -B \\ \mathbf{0} & (1 - \alpha)Q \end{pmatrix}$$

$Q \in R^{n \times n}$ is symmetric positive definite, and α, β are two real parameters. Let $c = (b^T, -q^T)^T$ and $z^{(i)} = ((x^{(i)})^T, (y^{(i)})^T)^T$ be the i -th approximate solution to (1.1).

We compute the approximate solution $z^{(i+1)}$ as follows.

$$z^{(i+\frac{1}{2})} = L_{\alpha,\beta,\omega} z^{(i)} + \omega(D - \omega L)^{-1} c \tag{2.2}$$

$$z^{(i+1)} = U_{\alpha,\beta,\omega} z^{(i+\frac{1}{2})} + \omega(D - \omega U)^{-1} c \tag{2.3}$$

where

$$\begin{aligned} L_{\alpha,\beta,\omega} &= (D - \omega L)^{-1} [(1 - \omega)D + \omega U] \\ &= \begin{pmatrix} (1 - \frac{\omega}{\beta})I_m & -\frac{\omega}{\beta}A^{-1}B \\ \frac{\omega(\beta - \omega)}{\beta(1 - \alpha\omega)}Q^{-1}B^T & I_n - \frac{\omega^2}{\beta(1 - \alpha\omega)}Q^{-1}B^T A^{-1}B \end{pmatrix} \end{aligned} \tag{2.4}$$

$$\begin{aligned} U_{\alpha,\beta,\omega} &= (D - \omega U)^{-1} [(1 - \omega)D + \omega L] \\ &= \begin{pmatrix} \frac{(1 - \omega)\beta}{\beta + \omega - \beta\omega}I_m - \frac{\omega^2}{(\beta + \omega - \beta\omega)(1 - \omega + \alpha\omega)}A^{-1}BQ^{-1}B^T & -\frac{\omega}{\beta + \omega - \beta\omega}A^{-1}B \\ \frac{\omega}{1 - \omega + \alpha\omega}Q^{-1}B^T & I_n \end{pmatrix} \end{aligned} \tag{2.5}$$

Notice

$$\begin{aligned} D - \omega L &= \begin{pmatrix} \beta A & \mathbf{0} \\ -\omega B^T & (1 - \alpha\omega)Q \end{pmatrix}, \\ D - \omega U &= \begin{pmatrix} (\beta + \omega - \beta\omega)A & \omega B \\ \mathbf{0} & (1 - \omega + \alpha\omega)Q \end{pmatrix} \end{aligned}$$

Assume that $\alpha \neq 0, 1, \beta \neq 1, \omega \neq 0$. Since A is positive real and Q is symmetric positive definite, then we obtain that

$$\det(D - \omega L) = \beta^m (1 - \alpha\omega)^n \det(A)\det(Q) \neq 0$$

and

$$\det(D - \omega U) = (\beta + \omega - \beta\omega)^m (1 - \omega + \alpha\omega)^n \det(A)\det(Q) \neq 0$$

if and only if $\beta(1 - \alpha\omega) \neq 0, (\beta + \omega - \beta\omega)(1 - \omega + \alpha\omega) \neq 0, \beta \neq 0$, i.e., $\omega \neq \left\{ 0, \frac{1}{\alpha}, \frac{1}{1 - \alpha}, \frac{\beta}{\beta - 1} \right\}$. From the above corresponding equations, we obtain the TMSSOR method as follows:

$$z^{(i+1)} = T_{\alpha,\beta,\omega} z^{(i)} + C_{\alpha,\beta,\omega} \tag{2.6}$$

with

$$T_{\alpha,\beta,\omega} = U_{\alpha,\beta,\omega} L_{\alpha,\beta,\omega} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{2.7}$$

Here, $T_{\alpha,\beta,\omega}$ is the iteration matrix of the TMSSOR iteration, where

$$\begin{aligned} T_{11} &= \frac{(1-\omega)(\beta-\omega)}{\beta+\omega-\beta\omega} I_m \\ &\quad - \frac{\omega^2(\beta-\omega)(2-\omega)}{\beta(\beta+\omega-\beta\omega)(1-\omega+\alpha\omega)(1-\alpha\omega)} A^{-1} B Q^{-1} B^T \\ T_{12} &= -\frac{\omega(2-\omega)}{\beta+\omega-\beta\omega} A^{-1} B \\ &\quad + \frac{\omega^3(2-\omega)}{\beta(\beta+\omega-\beta\omega)(1-\omega+\alpha\omega)(1-\alpha\omega)} A^{-1} B Q^{-1} B^T A^{-1} B \\ T_{21} &= \frac{\omega(\beta-\omega)(2-\omega)}{\beta(1-\omega+\alpha\omega)(1-\alpha\omega)} Q^{-1} B^T, \\ T_{22} &= I_n - \frac{\omega^2(2-\omega)}{\beta(1-\omega+\alpha\omega)(1-\alpha\omega)} Q^{-1} B^T A^{-1} B \end{aligned}$$

and

$$\begin{aligned} C_{\alpha,\beta,\omega} &= \omega(D-\omega U)^{-1} [(1-\omega)D + \omega L] (D-\omega L)^{-1} \begin{pmatrix} b \\ -q \end{pmatrix} \\ &\quad + \omega(D-\omega U)^{-1} \begin{pmatrix} b \\ -q \end{pmatrix} \\ &= \omega(2-\omega)(D-\omega U)^{-1} D(D-\omega L)^{-1} \begin{pmatrix} b \\ -q \end{pmatrix} \\ &= \omega(2-\omega) \\ &\quad \times \begin{pmatrix} \frac{1}{\beta+\omega-\beta\omega} A^{-1} b - \frac{\omega^2}{\beta(\beta+\omega-\beta\omega)(1-\omega+\alpha\omega)(1-\alpha\omega)} A^{-1} B Q^{-1} B^T A^{-1} b \\ \frac{\omega}{\beta(1-\omega+\alpha\omega)(1-\alpha\omega)} Q^{-1} B^T A^{-1} b - \frac{1}{(1-\omega+\alpha\omega)(1-\alpha\omega)} Q^{-1} q \end{pmatrix} \end{aligned}$$

Let

$$\begin{aligned} M_{\alpha,\beta,\omega} &= \left(\omega(2-\omega)(D-\omega U)^{-1} D(D-\omega L)^{-1} \right)^{-1} \\ &= \frac{1}{\omega(2-\omega)} (D-\omega L) D^{-1} (D-\omega U) \end{aligned}$$

$$= \frac{1}{\omega(2-\omega)} \times \begin{pmatrix} (\beta + \omega - \beta\omega)A & \omega B \\ \frac{-\omega(\beta + \omega - \beta\omega)}{\beta} B^T & -\frac{\omega^2}{\beta} B^T A^{-1} B + (1 - \alpha\omega)(1 - \omega + \alpha\omega)Q \end{pmatrix}$$

Then $C_{\alpha,\beta,\omega} = M_{\alpha,\beta,\omega}^{-1} \begin{pmatrix} b \\ -q \end{pmatrix}$, and it is easy to see that the TMSSOR method can also be induced by the splitting

$$\mathcal{A} = M_{\alpha,\beta,\omega} - (M_{\alpha,\beta,\omega} - \mathcal{A}) = M_{\alpha,\beta,\omega} - N_{\alpha,\beta,\omega} \tag{2.8}$$

and

$$T_{\alpha,\beta,\omega} = M_{\alpha,\beta,\omega}^{-1} N_{\alpha,\beta,\omega}$$

Obviously, $M_{\alpha,\beta,\omega}$ can be regarded as the preconditioner of the linear system (1.1). It is easy to see the TMSSOR method (2.6) can be written in the following form:

$$\begin{cases} y^{(i+1)} = y^{(i)} + \frac{\omega(2-\omega)}{\beta(1-\alpha\omega)(1-\omega+\alpha\omega)} Q^{-1} [(\beta-\omega)B^T x^{(i)} - \omega B^T A^{-1} B y^{(i)} + \omega B^T A^{-1} b - \beta q] \\ x^{(i+1)} = \frac{(1-\omega)(\beta-\omega)}{\beta+\omega-\beta\omega} x^{(i)} + \frac{\omega(2-\omega)}{\beta+\omega-\beta\omega} A^{-1} b - \frac{\omega}{\beta+\omega-\beta\omega} A^{-1} B [y^{(i+1)} + (1-\omega)y^{(i)}] \end{cases} \tag{2.9}$$

Evidently, when the relaxed parameters satisfy $\alpha = \frac{1}{2}$, $\beta = 1$, the TMSSOR method reduces to the MSSOR method [26]; which is also the GMSSOR method when the two parameters τ and ω of the GMSSOR method [31] are equal.

3 The semi-convergence of the TMSSOR method

In this section, we discuss the semi-convergence of the TMSSOR method. It is well known that if \mathcal{A} is nonsingular, the iteration (2.6) is convergent when the spectral radius of the iteration matrix is less than 1. When \mathcal{A} is singular, so $I - T_{\alpha,\beta,\omega}$ is singular. Then $T_{\alpha,\beta,\omega}$ has an eigenvalue 1, which means its spectral radius cannot be less than 1. So, for the iteration matrix $T_{\alpha,\beta,\omega}$ of the singular linear system (1.1), we need to introduce its pseudo-spectral radius [11] $\nu(T_{\alpha,\beta,\omega})$:

$$\nu(T_{\alpha,\beta,\omega}) = \max\{|\lambda| : \lambda \in \sigma(T_{\alpha,\beta,\omega}), \lambda \neq 1\}$$

If the iteration (2.6) is convergent to a solution of the linear system (1.1), then we call the iteration (2.6) is semi-convergent. It is known that the iteration (2.6) is semi-convergent, if and only if the iteration matrix $T_{\alpha,\beta,\omega}$ is semi-convergent. The semi-convergence of the matrix $T_{\alpha,\beta,\omega}$ can be described as follows [11]:

- (a) The elementary divisors of the iteration matrix $T_{\alpha,\beta,\omega}$ associated with its eigenvalue $\lambda = 1$ are linear, i.e., $rank(I - T_{\alpha,\beta,\omega})^2 = rank(I - T_{\alpha,\beta,\omega})$, or equivalently,

$index(I - T_{\alpha,\beta,\omega}) = 1$, where $index(\cdot)$ denotes the index of the corresponding matrix;

(b) $\nu(T_{\alpha,\beta,\omega}) < 1$.

Theorem 3.1 *Let $T_{\alpha,\beta,\omega}$ be the iteration matrix of the TMSSOR method with $\omega \neq 0, 2$. For any eigenvalue μ of $Q^{-1}B^T A^{-1}B$, if λ satisfies*

$$(\lambda - 1)(1 - \omega + \alpha\omega)(1 - \alpha\omega)[(\beta - \omega)(1 - \omega) - \lambda(\beta + \omega - \omega\beta)] = \lambda\omega^2(2 - \omega)^2\mu \tag{3.1}$$

then, λ is an eigenvalue of $T_{\alpha,\beta,\omega}$. Conversely, for any eigenvalue λ of $T_{\alpha,\beta,\omega}$, and there exists μ which satisfies Eq. (3.1), then μ is an eigenvalue of $Q^{-1}B^T A^{-1}B$.

Proof Let λ be an eigenvalue of $T_{\alpha,\beta,\omega}$, not loss of generality, we assume that $\lambda \neq 0$ and $z = (x^T, y^T)^T$ be the corresponding eigenvector. By

$$T_{\alpha,\beta,\omega} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

we have

$$(1 - \lambda)(D - \omega U)z = \omega(2 - \omega)D(D - \omega L)^{-1} \begin{pmatrix} A & B \\ -B^T & \mathbf{0} \end{pmatrix} z$$

Then

$$(1 - \lambda) \begin{pmatrix} (\beta + \omega - \beta\omega)A & \omega B \\ 0 & (1 - \omega + \alpha\omega)Q \end{pmatrix} z = \omega(2 - \omega) \begin{pmatrix} A & B \\ \frac{(\omega - \beta)}{\beta(1 - \alpha\omega)}B^T & \frac{\omega}{\beta(1 - \alpha\omega)}B^T A^{-1}B \end{pmatrix} z$$

That is, it holds

$$[(1 - \lambda)(\beta + \omega - \beta\omega) - \omega(2 - \omega)]x = \omega(\lambda + 1 - \omega)A^{-1}By \tag{3.2}$$

$$(1 - \lambda)(1 - \omega + \alpha\omega)(1 - \alpha\omega)\beta y - \omega^2(2 - \omega)Q^{-1}B^T A^{-1}By = \omega(2 - \omega)(\omega - \beta)Q^{-1}B^T x \tag{3.3}$$

If $(1 - \lambda)(\beta + \omega - \beta\omega) - \omega(2 - \omega) = 0$, which means $\lambda = \frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}$, then it is easy to see $\mu = 0$ is a zero eigenvalue of $Q^{-1}B^T A^{-1}B$ which satisfies (3.1). If $\lambda \neq \frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}$, then it is easy to see $y \neq 0$. Substituting x of (3.2) to the Eq. (3.3), we obtain

$$[(1 - \lambda)^2(\beta + \omega - \beta\omega) - \omega(2 - \omega)(1 - \lambda)](1 - \omega + \alpha\omega)(1 - \alpha\omega)y = -\lambda\omega^2(2 - \omega)^2Q^{-1}B^T A^{-1}By$$

and

$$Q^{-1}B^T A^{-1}By = \frac{[(1 - \lambda)^2(\beta + \omega - \beta\omega) - \omega(2 - \omega)(1 - \lambda)](1 - \omega + \alpha\omega)(1 - \alpha\omega)}{-\lambda\omega^2(2 - \omega)^2}y$$

For $\lambda \neq 0$ and $\omega \neq 0, 2$, by the above equation it is easy to see that μ is an eigenvalue of $Q^{-1}B^T A^{-1}B$ which satisfies (3.1).

Conversely, for any eigenvalue μ of $Q^{-1}B^T A^{-1}B$, if λ satisfies (3.1), then we can prove similarly that λ is an eigenvalue of $T_{\alpha,\beta,\omega}$, here omitted. □

Remark 3.1 From Theorem 3.1, we know that $\frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}$ is an eigenvalue of $T_{\alpha,\beta,\omega}$. In fact, if $\lambda = \frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega} \neq 0$, then from Eq. (3.2), it holds $A^{-1}By = 0$ for $y = 0$. Also, we have

$$\frac{(1 - \omega + \alpha\omega)(1 - \alpha\omega)\beta}{\beta + \omega - \beta\omega}y - \omega Q^{-1}B^T A^{-1}By = (\omega - \beta)Q^{-1}B^T x$$

Notice B is rank-deficient, then the equation $B^T x = 0$ has nonzero solutions. Therefore, $\frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}$ is an eigenvalue of $T_{\alpha,\beta,\omega}$ with the corresponding eigenvector $(x^T, 0^T)^T$.

To obtain the semi-convergence conditions of the TMSSOR method, we introduce some lemmas which will be useful for our discussion.

Lemma 3.1 [9, 10] *Both roots of the quadratic equation $x^2 - px + q = 0$ are less than one in modulus if and only if $|p - \bar{p}q| < 1 - |q|^2$, where \bar{p} is the complex conjugate of p .*

Lemma 3.2 [31] *Let A be positive real, Q be symmetric positive definite and B be rank-deficient. Then for any nonzero eigenvalue $\mu = \mu_{Re} + \mathbf{i}\mu_{Im}$ of $Q^{-1}B^T A^{-1}B$, where μ_{Re} and μ_{Im} are the real part and imaginary part of μ , respectively, and $\mathbf{i} = \sqrt{-1}$, it holds $\mu_{Re} > 0$.*

Theorem 3.2 *Let A be positive real, Q be symmetric positive definite and B be rank-deficient. Then $\nu(T_{\alpha,\beta,\omega}) < 1$ if the following conditions hold:*

- (C1) For $\omega \in (0, 1) \cup (2, +\infty)$, it holds $\frac{\omega^2}{2(\omega - 1)} < \beta$; for $\omega \in (-\infty, 0) \cup (1, 2)$, it holds $\beta < \frac{\omega^2}{2(\omega - 1)}$;
- (C2) $\omega(2 - \omega)(1 - \alpha\omega)(1 - \omega + \alpha\omega) > 0$;
- (C3) $\frac{\omega(2 - \omega)}{\omega^2 - 2\omega\beta + 2\beta}\mu_{Re}^2 + \frac{\omega^2 - 2\omega\beta + 2\beta}{\omega(2 - \omega)}\mu_{Im}^2 < \frac{2(1 - \alpha\omega)(1 - \omega + \alpha\omega)}{\omega(2 - \omega)}\mu_{Re}$.

Proof Making use of Eq. (3.1) and by some algebra, we have

$$\lambda^2 - \left[2 - \frac{\omega(2-\omega)}{\beta + \omega - \beta\omega} - \frac{\omega^2(2-\omega)^2\mu}{(1-\alpha\omega)(1-\omega + \alpha\omega)(\beta + \omega - \beta\omega)} \right] \times \lambda + 1 - \frac{\omega(2-\omega)}{\beta + \omega - \beta\omega} = 0 \tag{3.4}$$

From the above equation, we observe that $\lambda = 1$ or $\lambda = \frac{(1-\omega)(\beta-\omega)}{\beta + \omega - \beta\omega}$ in the case of $\mu = 0$, and $\lambda \neq 1$ in the case of $\mu \neq 0$.

Case 1 When $\lambda = \frac{(1-\omega)(\beta-\omega)}{\beta + \omega - \beta\omega}$, it holds

$$|\lambda| < 1 \Leftrightarrow \left| 1 - \frac{\omega(2-\omega)}{\beta + \omega - \beta\omega} \right| < 1 \Leftrightarrow 0 < \frac{\omega(2-\omega)}{\beta + \omega - \beta\omega} < 2$$

Notice when $\omega = 1$, the above inequality reduces to $0 < 1 < 2$, which holds true trivially. Excluding $\omega = 1$, the above inequality is equivalent to

$$\begin{cases} \omega(2-\omega) > 0 \\ \beta + \omega - \beta\omega > 0 \\ \omega(2-\omega) < 2(\beta + \omega - \beta\omega) \end{cases} \quad \text{or} \quad \begin{cases} \omega(2-\omega) < 0 \\ \beta + \omega - \beta\omega < 0 \\ \omega(2-\omega) > 2(\beta + \omega - \beta\omega) \end{cases} \tag{3.5}$$

By some algebra we have

$$\begin{cases} 0 < \omega < 1, & \beta > \frac{\omega^2}{2(\omega-1)} \\ 1 < \omega < 2, & \beta < \frac{\omega^2}{2(\omega-1)} \end{cases} \quad \text{or} \quad \begin{cases} \omega < 0, & \beta < \frac{\omega^2}{2(\omega-1)} \\ \omega > 2, & \beta > \frac{\omega^2}{2(\omega-1)} \end{cases} \tag{3.6}$$

i.e.,

$$\begin{cases} \omega \in (0, 1) \cup (2, +\infty) \\ \beta > \frac{\omega^2}{2(\omega-1)} \end{cases} \quad \text{or} \quad \begin{cases} \omega \in (-\infty, 0) \cup (1, 2) \\ \beta < \frac{\omega^2}{2(\omega-1)} \end{cases} \tag{3.7}$$

which means the condition (C1) holds.

Case 2 When $\lambda \neq \left\{ \frac{(1-\omega)(\beta-\omega)}{\beta + \omega - \beta\omega}, 1 \right\}$, by Lemma 3.1, for the nonzero eigenvalue λ of $T_{\alpha,\beta,\omega}$, denote $d = \frac{\omega(2-\omega)}{\beta + \omega - \beta\omega}$, $e = \frac{\omega(2-\omega)}{(1-\alpha\omega)(1-\omega + \alpha\omega)}$. Then it holds

$$|\lambda| < 1 \Leftrightarrow \left\{ |1-d| < 1 \right. \left. \left| 2-d-d\epsilon\mu - [2-d-e\bar{\mu}][1-d] \right| + |1-d|^2 < 1 \right. \tag{3.8}$$

The first inequality of (3.8) is equivalent to $0 < d < 2$, same as in Case 1, we immediately obtain the condition (C1).

After some algebra, the second inequality of (3.8) is equivalent to

$$|d [2 - d + e[(1 - d)\bar{\mu} - \mu]]| < 1 - |1 - d|^2 \Leftrightarrow |d [2 - d + e[(1 - d)\bar{\mu} - \mu]]| < d(2 - d)$$

Notice $0 < d < 2, \mu = \mu_{Re} + \mathbf{i}\mu_{Im}$. Then the above inequality is equivalent to

$$\begin{aligned} |2 - d + e[(1 - d)\bar{\mu} - \mu]| < 2 - d &\Leftrightarrow |2 - d - e[d\mu_{Re} + \mathbf{i}(2 - d)\mu_{Im}]| < 2 - d \\ &\Leftrightarrow [2 - d - ed\mu_{Re}]^2 + [e(2 - d)\mu_{Im}]^2 < (2 - d)^2 \end{aligned}$$

which is also equivalent to

$$[de\mu_{Re}]^2 + [(2 - d)e\mu_{Im}]^2 < 2d(2 - d)e\mu_{Re} \tag{3.9}$$

By Lemma 3.2, we have $\mu_{Re} > 0$, and together with $0 < d < 2$, it is easy to see $e = \frac{\omega(2 - \omega)}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)} > 0$ from (3.9), which is equivalent to the condition (C2).

After some algebra, the inequality (3.9) can be simplified as

$$\frac{d}{2 - d}\mu_{Re}^2 + \frac{2 - d}{d}\mu_{Im}^2 < 2\frac{1}{e}\mu_{Re}$$

i.e.,

$$\frac{\omega(2 - \omega)}{\omega^2 - 2\omega\beta + 2\beta}\mu_{Re}^2 + \frac{\omega^2 - 2\omega\beta + 2\beta}{\omega(2 - \omega)}\mu_{Im}^2 < 2\frac{(1 - \alpha\omega)(1 - \omega + \alpha\omega)}{\omega(2 - \omega)}\mu_{Re}$$

which is the condition (C3). □

Lemma 3.3 [29] *Index*($I - T_{\alpha,\beta,\omega}$) = 1 if and only if, for any $\mathbf{0} \neq Y \in \mathcal{R}(\mathcal{A}), Y \notin \mathcal{N}(\mathcal{A}M_{\alpha,\beta,\omega}^{-1})$.

Theorem 3.3 *Assume A be positive real and Q be symmetric positive definite, and B be of rank-deficient. Then index*($I - T_{\alpha,\beta,\omega}$) = 1.

Proof Let

$$\mathbf{0} \neq Y = \mathcal{A}X = \begin{pmatrix} A & B \\ -B^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} Az_1 + Bz_2 \\ -B^T z_1 \end{pmatrix} \tag{3.10}$$

By Lemma 3.3, to prove *index*($I - T_{\alpha,\beta,\omega}$) = 1, it suffices to prove $\mathcal{A}M_{\alpha,\beta,\omega}^{-1}Y \neq \mathbf{0}$, we give the proof for $\mathcal{A}M_{\alpha,\beta,\omega}^{-1}Y \neq \mathbf{0}$ by contradiction. Suppose $\mathcal{A}M_{\alpha,\beta,\omega}^{-1}Y = \mathbf{0}$.

Notice $\mathcal{N}(\mathcal{A}) = \text{span} \left\{ \begin{pmatrix} \mathbf{0} \\ \varphi \end{pmatrix} \right\}$, where $B\varphi = \mathbf{0}$.

Then there exists a vector φ_0 such that

$$M_{\alpha,\beta,\omega}^{-1}Y = \begin{pmatrix} \mathbf{0} \\ \varphi_0 \end{pmatrix}, \quad B\varphi_0 = \mathbf{0}.$$

So

$$\begin{aligned} Y &= M_{\alpha,\beta,\omega} \begin{pmatrix} \mathbf{0} \\ \varphi_0 \end{pmatrix} \\ &= \frac{1}{\omega(2-\omega)} \begin{pmatrix} (\beta+\omega-\beta\omega)A & \omega B \\ -\frac{\omega(\beta+\omega-\beta\omega)}{\beta}B^T & -\frac{\omega^2}{\beta}B^T A^{-1}B + (1-\alpha\omega)(1-\omega+\alpha\omega)Q \end{pmatrix} \\ &\quad \begin{pmatrix} \mathbf{0} \\ \varphi_0 \end{pmatrix} \\ &= \frac{1}{\omega(2-\omega)} \begin{pmatrix} \mathbf{0} \\ (1-\alpha\omega)(1-\omega+\alpha\omega)Q\varphi_0 \end{pmatrix} \end{aligned} \tag{3.11}$$

Making use of Eqs. (3.10) and (3.11), we have $B\varphi_0 = -BQ^{-1}B^T z_1 = \mathbf{0}$. Noticing Q^{-1} is symmetric positive definite, then $B^T z_1 = \mathbf{0}$, which means $\varphi_0 = -Q^{-1}B^T z_1 = \mathbf{0}$. Hence,

$$Y = M_{\alpha,\beta,\omega} \begin{pmatrix} \mathbf{0} \\ \varphi_0 \end{pmatrix} = \mathbf{0}$$

which contradicts with $Y \neq \mathbf{0}$, so we reach the conclusion that $index(I - T) = 1$.

The proof is completed. □

4 The optimal iteration parameters

In this section, we will discuss the optimal iteration parameters and the corresponding optimal semi-convergence factor for the TMSSOR method. Let $A \in R^{m \times m}$ and $Q \in R^{n \times n}$ be symmetric positive definite. Then all the eigenvalues of the matrix $Q^{-1}B^T A^{-1}B$ are real and nonnegative. Denote the smallest and largest nonzero eigenvalues of the matrix $Q^{-1}B^T A^{-1}B$ by μ_{min} and μ_{max} , respectively. By the proof of Theorem 3.2, we see that the eigenvalues λ of $T_{\alpha,\beta,\omega}$ can be $\lambda = 1$,

$\lambda = \frac{(1-\omega)(\beta-\omega)}{\beta+\omega-\beta\omega}$ or λ can be represented by the following λ_1, λ_2 :

$$\begin{aligned} \lambda_1(\alpha, \beta, \omega, \mu) &= \frac{1}{2} \left(f(\alpha, \beta, \omega, \mu) + \sqrt{f(\alpha, \beta, \omega, \mu)^2 - 4g(\beta, \omega)} \right) \\ \lambda_2(\alpha, \beta, \omega, \mu) &= \frac{1}{2} \left(f(\alpha, \beta, \omega, \mu) - \sqrt{f(\alpha, \beta, \omega, \mu)^2 - 4g(\beta, \omega)} \right) \end{aligned}$$

where

$$f(\alpha, \beta, \omega, \mu) = 2 - \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega} - \frac{\omega^2(2 - \omega)^2\mu}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)(\beta + \omega - \beta\omega)}$$

$$g(\beta, \omega) = 1 - \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega}$$

Consider the following two cases:

1. If $\Delta = f(\alpha, \beta, \omega, \mu)^2 - 4g(\beta, \omega) \leq 0$, then

$$|\lambda_1(\alpha, \beta, \omega, \mu)| = |\lambda_2(\alpha, \beta, \omega, \mu)| = \sqrt{1 - \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega}} = \sqrt{\frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}} \tag{4.1}$$

2. If $\Delta > 0$, then both $\lambda_1(\alpha, \beta, \omega, \mu)$ and $\lambda_2(\alpha, \beta, \omega, \mu)$ are real, and it holds

$$\lambda(\alpha, \beta, \omega, \mu) = \begin{cases} \lambda_1(\alpha, \beta, \omega, \mu) & \text{if } f(\alpha, \beta, \omega, \mu) > 0 \\ -\lambda_2(\alpha, \beta, \omega, \mu) & \text{if } f(\alpha, \beta, \omega, \mu) \leq 0 \end{cases}$$

where $\lambda(\alpha, \beta, \omega, \mu) = \max \{|\lambda_1(\alpha, \beta, \omega, \mu)|, |\lambda_2(\alpha, \beta, \omega, \mu)|\}$. From the Eq. (3.1), we obtain

$$\lambda_1(\alpha, \beta, \omega, \mu)\lambda_2(\alpha, \beta, \omega, \mu) = 1 - \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega} = \frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}$$

So, it is easy to see

$$\lambda(\alpha, \beta, \omega, \mu) \geq \sqrt{\left|1 - \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega}\right|} = \sqrt{\left|\frac{(1 - \omega)(\beta - \omega)}{\beta + \omega - \beta\omega}\right|} \tag{4.2}$$

Hence, the pseudo-spectral radius of the TMSSOR iteration matrix can be defined by:

$$\nu(T_{\alpha, \beta, \omega}) = \max_{0 \neq \mu \in \sigma(Q^{-1}B^T A^{-1}B)} \{|\lambda(\alpha, \beta, \omega, \mu)|\}$$

Denote

$$\lambda_i(\alpha, \beta, \omega) = \max_{0 \neq \mu \in \sigma(Q^{-1}B^T A^{-1}B)} \{|\lambda_i(\alpha, \beta, \omega, \mu)|\}, \quad i = 1, 2$$

Obviously, it holds

$$\nu(T_{\alpha, \beta, \omega}) = \max \{\lambda_1(\alpha, \beta, \omega), \lambda_2(\alpha, \beta, \omega)\} \tag{4.3}$$

By Theorem 3.2, it follows that $0 < \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega} < 2$, $\frac{\omega(2 - \omega)}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)} > 0$ which means

$$\frac{\omega^2(2 - \omega)^2}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)(\beta + \omega - \beta\omega)} > 0 \tag{4.4}$$

From the above equations, it holds $|\lambda_1(\alpha, \beta, \omega)| \geq |\lambda_2(\alpha, \beta, \omega)|$ while $\Delta > 0$ and $f(\alpha, \beta, \omega, \mu) > 0$, and $|\lambda_2(\alpha, \beta, \omega)| \geq |\lambda_1(\alpha, \beta, \omega)|$ while $\Delta > 0$ and $f(\alpha, \beta, \omega, \mu) \leq 0$. So, we have

$$\begin{cases} \lambda_1(\alpha, \beta, \omega) = \frac{1}{2} \left[f(\alpha, \beta, \omega, \mu_{min}) + \sqrt{f(\alpha, \beta, \omega, \mu_{min})^2 - 4g(\beta, \omega)} \right] \\ \lambda_2(\alpha, \beta, \omega) = \frac{1}{2} \left[-f(\alpha, \beta, \omega, \mu_{max}) + \sqrt{f(\alpha, \beta, \omega, \mu_{max})^2 - 4g(\beta, \omega)} \right] \end{cases} \tag{4.5}$$

Now we consider the optimal parameters which minimize the pseudo-spectral radius of the iteration matrix $T_{\alpha,\beta,\omega}$. Generally, it is very difficult to determine the global optimal parameters of the TMSSOR method. So, here, we consider a special case and discuss the local optimal parameters, that is, we assume the three parameters satisfying

$$\alpha = \frac{\omega(2 - \omega)}{\beta + \omega - \beta\omega} \in (0, 1) \tag{4.6}$$

Remark 4.1 By Eqs. (4.1) and (4.2), it is easy to see, to investigate the optimal parameters which minimize $\nu(T_{\alpha,\beta,\omega})$, it suffices to consider the case of $\Delta = f(\alpha, \beta, \omega, \mu)^2 - 4g(\beta, \omega) > 0$. So, from now on, we always assume $\Delta > 0$.

Notice

$$\frac{\omega\alpha(2 - \omega)}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)} > 0$$

Then it is easy to see there exist two variables μ_1 and μ_2 satisfying the following equations:

$$2 - \alpha - \frac{\omega\alpha(2 - \omega)\mu_1}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)} = 2\sqrt{1 - \alpha} \tag{4.7}$$

$$2 - \alpha - \frac{\omega\alpha(2 - \omega)\mu_2}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)} = -2\sqrt{1 - \alpha} \tag{4.8}$$

where $0 \leq \mu_1 \leq \mu_2$, furthermore, by (4.5), it is easy to see $\mu_1, \mu_2 \in [\mu_{min}, \mu_{max}]$. Actually, if $\mu_1 < \mu_{min}$ or $\mu_{max} < \mu_2$, then $-2\sqrt{1 - \alpha} < f(\alpha, \beta, \omega, \mu) < 2\sqrt{1 - \alpha}$, which leads to $\Delta < 0$. So it is in contradiction with $\Delta > 0$.

From (4.7) and (4.8), by some algebra, it holds

$$1 - \alpha = \frac{(\sqrt{\mu_1} - \sqrt{\mu_2})^2}{(\sqrt{\mu_1} + \sqrt{\mu_2})^2} \tag{4.9}$$

$$\frac{\omega(2 - \omega)}{(1 - \alpha\omega)(1 - \omega + \alpha\omega)} = \frac{1}{\sqrt{\mu_1\mu_2}} \tag{4.10}$$

So, $f(\alpha, \beta, \omega, \mu)$ can be rewritten as

$$f(\alpha, \beta, \omega, \mu) = \frac{2(\mu_1 + \mu_2 - 2\mu)}{(\sqrt{\mu_1} + \sqrt{\mu_2})^2} \tag{4.11}$$

Substituting $f(\alpha, \beta, \omega, \mu)$ to (4.5), therefore, we have

$$\begin{cases} \lambda_1(\alpha, \beta, \omega) = \frac{(\sqrt{\mu_2 - \mu_{min}} + \sqrt{\mu_1 - \mu_{min}})^2}{(\sqrt{\mu_1} + \sqrt{\mu_2})^2} \\ \lambda_2(\alpha, \beta, \omega) = \frac{(\sqrt{\mu_{max} - \mu_1} + \sqrt{\mu_{max} - \mu_2})^2}{(\sqrt{\mu_1} + \sqrt{\mu_2})^2} \end{cases} \tag{4.12}$$

For convenience, let $\lambda_1(\alpha, \beta, \omega) = \lambda_1(\mu_1, \mu_2)$, and $\lambda_2(\alpha, \beta, \omega) = \lambda_2(\mu_1, \mu_2)$. It is easy to see that the following results hold true.

$$\begin{cases} \lambda_1(\mu_1, \mu_2) = \lambda_2(\mu_1, \mu_2) & \text{if } \mu_1 + \mu_2 = \mu_{max} + \mu_{min}, \\ \lambda_1(\mu_1, \mu_2) > \lambda_2(\mu_1, \mu_2) & \text{if } \mu_1 + \mu_2 > \mu_{max} + \mu_{min}, \\ \lambda_1(\mu_1, \mu_2) < \lambda_2(\mu_1, \mu_2) & \text{if } \mu_1 + \mu_2 < \mu_{max} + \mu_{min}. \end{cases} \tag{4.13}$$

Then by Eqs. (4.3) and (4.13), it holds

$$v(T_{\alpha, \beta, \omega}) = \begin{cases} \lambda_1(\mu_1, \mu_2) & \text{if } \mu_1 + \mu_2 \geq \mu_{max} + \mu_{min} \\ \lambda_2(\mu_1, \mu_2) & \text{if } \mu_1 + \mu_2 < \mu_{max} + \mu_{min} \end{cases} \tag{4.14}$$

By some algebra, it holds that

$$\begin{cases} \frac{\partial \lambda_1(\mu_1, \mu_2)}{\partial \mu_1} = \frac{\sqrt{\mu_2 - \mu_{min}} + \sqrt{\mu_1 - \mu_{min}}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \cdot \frac{\sqrt{\mu_1 \mu_2} + \mu_{min} - \sqrt{(\mu_1 - \mu_{min})(\mu_2 - \mu_{min})}}{\sqrt{\mu_1(\mu_1 - \mu_{min})}(\sqrt{\mu_1} + \sqrt{\mu_2})^2} \\ \frac{\partial \lambda_2(\mu_1, \mu_2)}{\partial \mu_1} = -\frac{\sqrt{\mu_{max} - \mu_1} + \sqrt{\mu_{max} - \mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \cdot \frac{\sqrt{\mu_1 \mu_2} + \mu_{max} - \sqrt{(\mu_{max} - \mu_1)(\mu_{max} - \mu_2)}}{\sqrt{\mu_1(\mu_{max} - \mu_1)}(\sqrt{\mu_1} + \sqrt{\mu_2})^2} \end{cases}$$

Hence, we have

$$\frac{\partial \lambda_1(\mu_1, \mu_2)}{\partial \mu_1} > 0, \quad \frac{\partial \lambda_2(\mu_1, \mu_2)}{\partial \mu_1} < 0$$

In the same way, we obtain

$$\frac{\partial \lambda_1(\mu_1, \mu_2)}{\partial \mu_2} > 0, \quad \frac{\partial \lambda_2(\mu_1, \mu_2)}{\partial \mu_2} < 0$$

We now declare that $v(T_{\alpha, \beta, \omega})$ has no minimum when $\mu_1 + \mu_2 \neq \mu_{min} + \mu_{max}$. Observe the following two cases:

1. Assume $v(T_{\alpha, \beta, \omega})$ has minimum at $\mu_1 + \mu_2 < \mu_{min} + \mu_{max}$. By (4.14), it holds that $v(T_{\alpha, \beta, \omega}) = \lambda_2(\mu_1, \mu_2)$. Let $\bar{\mu} = \mu_{min} + \mu_{max} - \mu_2$. Then, $\mu_1 < \bar{\mu}$. By the monotone property of the function $\lambda_2(\mu_1, \mu_2)$, we have $\lambda_2(\mu_1, \mu_2) > \lambda_2(\bar{\mu}, \mu_2)$, which contradicts the assumption.
2. Assume $v(T_{\alpha, \beta, \omega})$ has minimum at $\mu_1 + \mu_2 > \mu_{min} + \mu_{max}$. By (4.14), it holds that $v(T_{\alpha, \beta, \omega}) = \lambda_1(\mu_1, \mu_2)$. Let $\bar{\mu} = \mu_{min} + \mu_{max} - \mu_2$. Then, $\mu_1 > \bar{\mu}$. By the monotone property of the function $\lambda_1(\mu_1, \mu_2)$, we have $\lambda_1(\mu_1, \mu_2) > \lambda_1(\bar{\mu}, \mu_2)$, which also contradicts the assumption.

So $v(T_{\alpha,\beta,\omega})$ may have minimum only at $\mu_1 + \mu_2 = \mu_{min} + \mu_{max}$. Now we give the local optimal parameters and the corresponding optimal convergence factor of the TMSSOR method by follows.

Theorem 4.1 *Let A and Q be symmetric positive definite, and B be rank-deficient. Assume Eq. (4.6) is satisfied. Then the optimal parameters of the TMSSOR method are given by*

$$\begin{aligned} \alpha_{opt} &= \frac{4\sqrt{\mu_{max}\mu_{min}}}{(\sqrt{\mu_{max}} + \sqrt{\mu_{min}})^2}, \\ \beta_{opt} &= \frac{(\sqrt{\mu_{max}} + \sqrt{\mu_{min}})^2\omega_{opt}(2 - \omega_{opt}) - 4\omega_{opt}\sqrt{\mu_{max}\mu_{min}}}{4\sqrt{\mu_{max}\mu_{min}}(1 - \omega_{opt})} \end{aligned} \tag{4.15}$$

ω_{opt} is the root of the equation

$$\sqrt{\mu_{max}\mu_{min}}(a_1 + 4a_2 - 16\sqrt{\mu_{min}\mu_{max}})\omega^2 - (1 + 2\sqrt{\mu_{max}\mu_{min}})a_1\omega + a_1 = 0$$

where

$$a_1 = (\sqrt{\mu_{max}} + \sqrt{\mu_{min}})^4, \quad a_2 = (\sqrt{\mu_{max}} - \sqrt{\mu_{min}})^2$$

Furthermore, the corresponding optimal semi-convergence factor of the TMSSOR method is

$$\min_{\alpha,\beta,\omega} v(T_{\alpha,\beta,\omega}) = v(T_{\alpha_{opt},\beta_{opt},\omega_{opt}}) = \frac{\sqrt{\mu_{max}} - \sqrt{\mu_{min}}}{\sqrt{\mu_{max}} + \sqrt{\mu_{min}}} \tag{4.16}$$

Proof From the above analysis, we know that $v(T_{\alpha,\beta,\omega})$ may have minimum only at $\mu_1 + \mu_2 = \mu_{min} + \mu_{max}$. When $\mu_1 + \mu_2 = \mu_{min} + \mu_{max}$, from (4.13) and (4.14), it holds

$$\begin{aligned} v(T_{\alpha,\beta,\omega}) &= \lambda_1(\mu_1, \mu_2) = \lambda_2(\mu_1, \mu_2) \\ &= \frac{\mu_{max} - \mu_{min} + 2\sqrt{-\mu_1^2 + (\mu_{min} + \mu_{max})\mu_1 - \mu_{min}\mu_{max}}}{\mu_{min} + \mu_{max} + 2\sqrt{-\mu_1^2 + (\mu_{min} + \mu_{max})\mu_1}} \end{aligned}$$

By the technique in [19], let $r = -\mu_1^2 + (\mu_{min} + \mu_{max})\mu_1 \geq \mu_{min}\mu_{max}$ and define

$$g(r) = v(T_{\alpha,\beta,\omega}) = \frac{\mu_{max} - \mu_{min} + 2\sqrt{r - \mu_{min}\mu_{max}}}{\mu_{min} + \mu_{max} + 2\sqrt{r}}$$

It is easy to see that $g(r)$ has minimum at $r = \mu_{min}\mu_{max}$, which means that $v(T_{\alpha,\beta,\omega})$ has minimum $\frac{\sqrt{\mu_{max}} - \sqrt{\mu_{min}}}{\sqrt{\mu_{max}} + \sqrt{\mu_{min}}}$ at $\mu_1 = \mu_{min}$ and $\mu_2 = \mu_{max}$. By Eqs. (4.6), (4.9),

(4.10) and together with $\mu_1 = \mu_{min}$, $\mu_2 = \mu_{max}$, after some simple algebra, we obtain the following optimal parameters:

$$\alpha_{opt} = \frac{4\sqrt{\mu_{max}\mu_{min}}}{(\sqrt{\mu_{max}} + \sqrt{\mu_{min}})^2},$$

$$\beta_{opt} = \frac{(\sqrt{\mu_{max}} + \sqrt{\mu_{min}})^2\omega_{opt}(2 - \omega_{opt}) - 4\omega_{opt}\sqrt{\mu_{max}\mu_{min}}}{4\sqrt{\mu_{max}\mu_{min}}(1 - \omega_{opt})}$$

and ω_{opt} is the root of the equation

$$\sqrt{\mu_{max}\mu_{min}}(a_1 + 4a_2 - 16\sqrt{\mu_{min}\mu_{max}})\omega^2 - (1 + 2\sqrt{\mu_{max}\mu_{min}})a_1\omega + a_1 = 0$$

where $a_1 = (\sqrt{\mu_{max}} + \sqrt{\mu_{min}})^4$, $a_2 = (\sqrt{\mu_{max}} - \sqrt{\mu_{min}})^2$. □

Remark 4.2 Theorem 4.1 shows that the TMSSOR method has the same asymptotic convergence rate as the GSOR method [9] at their own optimal points of the iteration parameters. Otherwise, these two methods may exhibit different convergence behavior. The GSOR method has less computational cost per iteration, so, the GSOR method is more effective in practical applications.

5 Numerical experiments

In this section, some numerical experiments are given to compare the performance of the TMSSOR method with GSOR, MSSOR, MSSOR-like [25], GMSSOR methods. We denote the number of iteration steps by IT , elapsed CPU time in seconds by CPU and the norm of residual vectors by RES , respectively.

All the computations are implemented in MATLAB 7.0.19920 (R14) on a PC computer with a 1.86GHz 64-bit processor and 2GB memory. In actual computations, we choose the right-hand vector $(b^T, -q^T)^T \in R^{m+n}$ such that the exact solution of the augmented linear system (1.1) is $((x^*)^T, (y^*)^T)^T = (1, 1, \dots, 1)^T \in R^{m+n}$ and all runs are started from the initial vector $((x^{(0)})^T, (y^{(0)})^T)^T = (0, 0, \dots, 0)^T \in R^{m+n}$. The stopping criterion is taken when the current iteration satisfies $RES \leq 10^{-6}$, where

$$RES := \frac{\sqrt{\|b - Ax^{(k)} - By^{(k)}\|_2^2 + \|q - B^T x^{(k)}\|_2^2}}{\sqrt{\|b - Ax^{(0)} - By^{(0)}\|_2^2 + \|q - B^T x^{(0)}\|_2^2}}$$

with $((x^{(k)})^T, (y^{(k)})^T)^T$ being the final approximate solution.

Example 5.1 Consider the Oseen equation:

$$\begin{cases} -\nu \Delta \boldsymbol{\mu} + (\boldsymbol{\omega} \cdot \nabla) \boldsymbol{\mu} + \nabla p = f, & \text{in } \Omega \\ -\nabla \cdot \boldsymbol{\mu} = 0, & \text{in } \Omega \\ \nabla \cdot \boldsymbol{\omega} = 0. \end{cases} \tag{5.1}$$

where Ω is an open bounded domain in R^2 or R^3 , the vector field μ represents the velocity in Ω , p denotes pressure, and the scalar ν is the viscosity, which is inversely proportional to the Reynolds number. The test problem is a leaky two-dimensional lid-driven cavity problem in the square domain: $\Omega = (0 < x < 1; 0 < y < 1)$, where $\mu = (\mu, \nu)^T$ denotes the velocity field, and $\omega = (a, b)^T$ denotes the wind. The boundary conditions are $\mu = \nu = 0$ on the three walls ($x = 0, y = 0, x = 1$), and $\mu = 1, \nu = 0$ on the moving wall ($y = 1$). We take constant “wind” $a = 1, b = 2$. To discretize (5.1), we use the “market and cell”(MAC) finite difference scheme [20]. Divide Ω into a uniform $l \times l$ grid of cells of width $h = \frac{1}{l}$. The discrete velocities and pressures are defined on a staggered grid in which the discrete values of μ lie in the centers of the cell boundaries orthogonal to the $x - axis$, the discrete values of ν lie in center boundaries orthogonal to the $y - axis$, and the discrete pressures lie in the cell centers. Then we obtain the matrix representation of the Oseen equations (5.1):

$$\begin{pmatrix} A & B \\ -B^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mu \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{5.2}$$

with the matrix blocks of the following form:

$$A = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \in R^{2l(l-1) \times 2l(l-1)}, B = (B_1, B_2) \in R^{2l(l-1) \times l^2},$$

$F_i = \nu A_i + N_i \in R^{l(l-1) \times l(l-1)}$, ($i = 1, 2$). For convenience, let $m = 2l(l - 1)$, $n = l^2$ in this example, and A is positive real, $rank(B) = l^2 - 1$. $\mathcal{N}(B) = span\{e_n\}$, $e_n = (1, 1, \dots, 1)^T \in R^n$. To ensure the (1,1)-block matrix symmetric positive definite, finally we take the test coefficient matrix as follows:

$$\begin{pmatrix} \bar{A} & B^T \\ -B & \mathbf{0} \end{pmatrix}$$

where $\bar{A} = \frac{1}{2}(A + A^T)$. For the preconditioner of the TMSSOR method, notice

$$M_{\alpha, \beta, \omega}^{-1} = t \begin{pmatrix} (1-\omega+\alpha\omega)(1-\alpha\omega)\bar{A}^{-1} - \frac{\omega^2}{\beta}\bar{A}^{-1}BQ^{-1}B^T\bar{A}^{-1} & -\omega\bar{A}^{-1}BQ^{-1} \\ \frac{\omega(\beta+\omega-\beta\omega)}{\beta}Q^{-1}B^T\bar{A}^{-1} & (\beta+\omega-\beta\omega)Q^{-1} \end{pmatrix}$$

and $t = \frac{\omega(2-\omega)}{(\beta+\omega-\beta\omega)(1-\omega+\alpha\omega)(1-\alpha\omega)}$. We compute the \bar{A}^{-1}, Q^{-1} in $M_{\alpha, \beta, \omega}^{-1}$ by the incomplete LU factorization with drop tolerance 0.001. We take the viscosity $\nu = 0.5$ in our experiments and choose the matrix Q by Table 1. For this example, here we consider the following four cases when $l = 8, l = 16, l = 24$ and $l = 32$.

In Fig. 1, we choose Q as Case II in Table 1, and we display the distribution on the eigenvalues of matrix \mathcal{A} and the matrices $M_{\alpha, \beta, \omega}^{-1}\mathcal{A}$ for different iteration parameters.

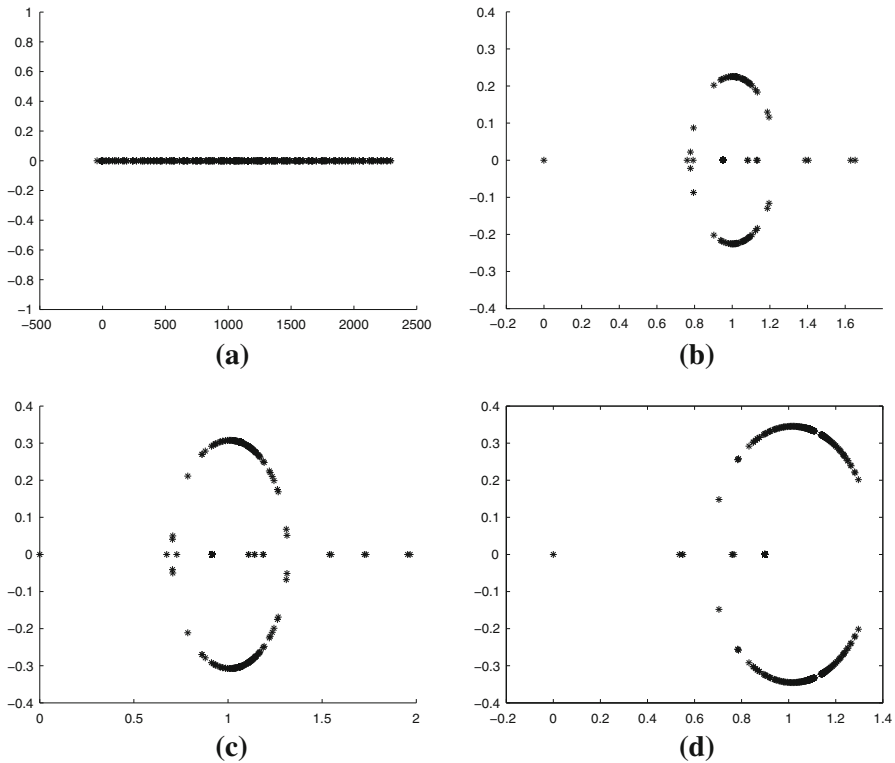


Fig. 1 The distribution of the eigenvalues of \mathcal{A} and $M_{\alpha,\beta,\omega}^{-1}\mathcal{A}$ with different parameters. **a** The distribution of the eigenvalues of \mathcal{A} . **b** The distribution of the eigenvalues of $M_{\alpha,\beta,\omega}^{-1}\mathcal{A}$ with $\omega = 0.426, \alpha = 0.952, \beta = 0.485, l = 8$. **c** The distribution of the eigenvalues of $M_{\alpha,\beta,\omega}^{-1}\mathcal{A}$ with $\omega = 0.477, \alpha = 0.916, \beta = 0.604, l = 16$. **d** The distribution of the eigenvalues of $M_{\alpha,\beta,\omega}^{-1}\mathcal{A}$ with $\omega = 0.501, \alpha = 0.898, \beta = 0.671, l = 24$

Table 1 Choices of matrix Q , with $\widehat{Q} = \text{Diag}(B^T \widehat{A}^{-1} B, B^T B)$

Case no.	Matrix Q	Description
I	$\text{tridiag}(\widehat{Q})$	$\widehat{A} = \text{tridiag}(\widehat{A})$
II	\widehat{Q}	$\widehat{A} = \text{diag}(\widehat{A})$
III	\widehat{Q}	$\widehat{A} = \widehat{A}$

We find that the eigenvalues of the preconditioned matrices are quite clustered. In Tables 2, 3, 4 and 5, we choose Q as Case III in Table 1, and we present the numerical results for the different iteration methods, both as solvers and preconditioners to accelerate the restarted GMRES(k). In Table 6, we choose Q as Case I and Case II in Table 1, respectively, and we list the optimal parameters and the corresponding optimal convergence factors of the different iteration methods. When the optimal parameters are employed, it reveals that the TMSSOR method is effective, especially it has the smallest number of iterations.

Table 2 Numerical results for Case III with $l = 8$

Iteration method	CPU(s)	IT	$RES \times 10^{-7}$
GSOR	0.239	43	8.0632
MSSOR	0.218	22	2.9920
GMSSOR	0.220	18	2.6041
TMSSOR	0.262	9	9.2353
GSOR-GMRES(18)	4.257	31	7.6343
MSSOR-GMRES(18)	7.546	28	8.8796
GMSSOR-GMRES(18)	2.319	27	2.7786
TMSSOR-GMRES(18)	1.030	10	4.4938

Table 3 Numerical results for Case III with $l = 16$

Iteration method	CPU(s)	IT	$RES \times 10^{-7}$
GSOR	0.816	47	9.8942
MSSOR	0.940	30	8.8023
GMSSOR	1.029	30	1.4462
TMSSOR	0.889	14	9.1427
GSOR-GMRES(18)	9.789	58	7.5641
MSSOR-GMRES(18)	10.570	56	6.5821
GMSSOR-GMRES(18)	21.487	34	5.1137
TMSSOR-GMRES(18)	8.424	14	7.8347

Table 4 Numerical results for Case III with $l = 24$

Iteration method	CPU(s)	IT	$RES \times 10^{-7}$
GSOR	2.757	55	9.0771
MSSOR	5.879	38	3.0635
GMSSOR	6.336	43	2.1824
TMSSOR	5.869	15	6.5279
GSOR-GMRES(18)	26.607	40	6.2180
MSSOR-GMRES(18)	28.745	38	5.5814
GMSSOR-GMRES(18)	76.571	25	4.4692
TMSSOR-GMRES(18)	27.534	9	7.0937

Table 5 Numerical results for Case III with $l = 32$

Iteration method	CPU(s)	IT	$RES \times 10^{-7}$
GSOR	8.545	59	8.5286
MSSOR	20.891	40	2.2858
GMSSOR	22.121	46	7.7115
TMSSOR	21.068	18	6.4050
GSOR-GMRES(18)	40.368	39	6.8921
MSSOR-GMRES(18)	43.056	31	4.5180
GMSSOR-GMRES(18)	76.571	25	4.4692
TMSSOR-GMRES(18)	92.758	10	5.5695

Table 6 Numerical results for Case I and Case II

			112	480	1104	1984	
<i>m</i>			112	480	1104	1984	
<i>n</i>			64	256	576	1024	
<i>m + n</i>			176	736	1680	3008	
Case I	GSOR	ω_{opt}	0.9479	0.8876	0.7175	0.6966	
		τ_{opt}	1.0833	1.2018	1.1217	1.0793	
		ρ_{opt}	0.2193	0.2889	0.3188	0.3367	
		IT	16	19	21	22	
		CPU	0.187	0.609	2.246	7.893	
		$RES(10^{-7})$	4.8260	7.6412	4.4871	7.7860	
	GMSSOR	ω_{opt}	0.8412	0.7512	0.7212	0.7542	
		τ_{opt}	0.4476	0.5040	0.5255	0.5112	
		ρ_{opt}	0.2192	0.2888	0.3188	0.3367	
		IT	14	18	20	21	
		CPU	0.223	0.750	5.105	18.186	
		$RES(10^{-7})$	5.2579	8.3271	7.1346	5.8310	
	TMSSOR	ω_{opt}	0.4264	0.4777	0.5014	0.5156	
		α_{opt}	0.9520	0.9166	0.8983	0.8866	
		β_{opt}	0.4854	0.6044	0.6719	0.7177	
		ρ_{opt}	0.2192	0.2888	0.3188	0.3367	
		IT	10	12	13	14	
		CPU	0.502	0.772	5.116	18.109	
	Case II	GSOR	ω_{opt}	0.7050	0.4035	0.3247	0.8709
			τ_{opt}	0.5159	0.3124	0.1940	0.4132
			ρ_{opt}	0.3871	0.4012	0.0946	0.2382
IT			45	55	117	396	
CPU			0.281	0.733	3.807	17.674	
$RES(10^{-7})$			8.4118	8.0730	9.7845	9.7643	
GMSSOR		ω_{opt}	0.6029	0.5788	0.4454	0.4518	
		τ_{opt}	0.4516	0.5569	0.4175	0.4134	
		ρ_{opt}	0.3871	0.4012	0.0946	0.2382	
		IT	14	15	22	20	
		CPU	0.214	0.872	5.783	20.085	
		$RES(10^{-7})$	5.4172	5.6036	5.1872	8.9884	
TMSSOR		ω_{opt}	0.4380	0.5322	0.4061	0.3974	
		α_{opt}	0.8502	0.8390	0.9910	0.9543	
		β_{opt}	0.6526	0.8526	0.4159	0.4609	
		ρ_{opt}	0.3871	0.4012	0.0946	0.2382	
		IT	12	13	21	32	
		CPU	0.224	0.768	5.812	19.688	
		$RES(10^{-7})$	5.1851	2.8272	7.4531	9.6766	

6 Conclusion

In this paper, we propose a triple-parameter modified SSOR iteration method to solve a class of large sparse singular saddle point problems. We prove the semi-convergence of the TMSSOR method under suitable restrictions on the iteration parameters, and obtain the local optimal parameters which minimize the pseudo-spectral radii of the associated iteration matrices, as for the global optimal parameters, it still needs further studies.

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