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On the convergence of collocation solutions in continuous piecewise polynomial spaces for Volterra integral equations

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Abstract This paper fills an important gap in the convergence analysis of collocation solutions in spaces of continuous piecewise polynomials for Volterra integral equations of the second kind. Our analysis is then extended to Volterra functional integral equations of the second kind with constant delays.

Keywords Volterra integral equations · Collocation solutions · Continuous piecewise polynomials · Convergence · Volterra functional integral equations with constant delays

Mathematics Subject Classification 65R20 · 65Q20

1 Introduction

The convergence analysis of piecewise polynomial collocation solutions for Volterra integral equations (VIEs) of the second kind,

$$u(t) = g(t) + \int_0^t K(t,s)u(s)ds, \quad t \in I := [0,T], \tag{1.1}$$

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with continuous kernel $K(t, s)$ is now largely well understood; see [4–6] and, especially, the surveys [1,3]. However, there has remained an important gap in the convergence analysis of collocation solutions for the second-kind VIE (1.1): it concerns the convergence/divergence of piecewise polynomial collocation solutions for (1.1) that are *globally continuous* on I and correspond to collocation points that do not include the points of the underlying mesh I_h.

It is the aim of this paper to close this gap and to employ the gained insight to establish the analogous convergence analysis for the Volterra functional integral equation (VFIE) with constant delay $\tau > 0$,

$$\begin{cases} u(t) = g(t) + \displaystyle\int_{t-\tau}^{t} K(t, s)u(s)ds, & t \in I, \\ u(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{1.2}$$

The outline of the paper is as follows. In Sects. 2 and 3 we state our main results on the convergence of globally continuous piecewise polynomial collocation solutions for the second-kind VIE (1.1) and the second-kind VFIE (1.2); their proofs are given in Sects. 4 and 5. In Sect. 6, we use a number of examples to illustrate the validity of our results on the attainable order of these collocation solutions. Section 7 concludes with a concluding remark.

2 Continuous collocation solutions for second kind VIEs

2.1 Meshes and collocation spaces

Let $I_h := \{t_n := nh : \ n = 0, 1, \ldots, N \ \ (t_N = T)\}$ be a given mesh on $I = [0, T]$, with $\sigma_n := [t_n, t_{n+1}]$ and mesh diameter $h = T/N$. We seek a collocation solution u_h for (1.1) in the space

$$S_m^{(0)}(I_h) := \left\{v \in C(I) : \ v|_{\sigma_n} \in \pi_m = \pi_m(\sigma_n) \ (0 \le n \le N-1)\right\},$$

where π_m denotes the space of all (real) polynomials of degree not exceeding m. For a prescribed set of collocation points

$$X_h := \{t = t_n + c_i h : \ 0 < c_1 < \cdots < c_m \le 1 \ (0 \le n \le N-1)\}, \tag{2.1}$$

u_h is defined by the collocation equation

$$u_h(t) = g(t) + \int_0^t K(t, s)u_h(s)ds, \quad t \in X_h, \tag{2.2}$$

with $u_h(0) = g(0)$.

Consequently the collocation polynomial can be written as (see [2])

$$u_h'(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \quad s \in (0, 1], \tag{2.3}$$

where $U_{n,i} := u_h'(t_{n,i})$, $t_{n,i} := t_n + c_i h$ and the polynomials

$$L_j(s) := \prod_{k \neq j}^{m} \frac{s - c_k}{c_j - c_k} \quad (j = 1, \ldots, m),$$

denote the Lagrange fundamental polynomials with respect to the (distinct) collocation parameters $\{c_i\}$.

Integrating (2.3), we obtain

$$u_h(t_n + sh) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j}, \quad s \in [0, 1], \tag{2.4}$$

where $\beta_j(s) := \int_0^s L_j(v) dv$.

Therefore, at $t = t_{n,i}$,

$$\begin{aligned} u_h(t_{n,i}) &= u_h(t_n) + h \sum_{j=1}^{m} a_{ij} U_{n,j} = g(t_{n,i}) + \int_0^{t_{n,i}} K(t_{n,i}, s) u_h(s) ds \\ &= g(t_{n,i}) + h \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) \left[u_h(t_l) + h \sum_{j=1}^{m} \beta_j(s) U_{l,j} \right] ds \\ &\quad + h \int_0^{c_i} K(t_{n,i}, t_n + sh) \left[u_h(t_n) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j} \right] ds, \end{aligned} \tag{2.5}$$

where $a_{ij} := \beta_j(c_i)$.

Denote $A := (a_{ij})_{m \times m}$, $e := (1, \ldots, 1)^T$, $G_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T$, $U_n := (U_{n,1}, \ldots, U_{n,m})^T$, $B_n^{(l)} := \left(\int_0^1 K(t_{n,i}, t_l + sh) \beta_j(s) ds \right)$ $(0 \leq l \leq N-1)$, $B_n := \left(\int_0^{c_i} K(t_{n,i}, t_n + sh) \beta_j(s) ds \right)$, $C_n^{(l)} := diag \left(\int_0^1 K(t_{n,i}, t_l + sh) ds \right)$ $(0 \leq l \leq N-1)$, and $C_n := diag \left(\int_0^{c_i} K(t_{n,i}, t_n + sh) ds \right)$, we have

$$\left(hA - h^2 B_n \right) U_n = G_n + (hC_n - I_m) e u_h(t_n) + h \sum_{l=0}^{n-1} \left[C_n^{(l)} e u_h(t_l) + h B_n^{(l)} U_l \right], \tag{2.6}$$

where I_m denotes the identity in $L(\mathbb{R}^m)$.

If $g \in C(I)$ and $K \in C(D)$ $(D := \{(t, s) : \; 0 \leq s \leq t \leq T\})$, (2.6) determines a unique $u_h \in S_m^{(0)}(I_h)$ for all sufficiently small mesh diameters, say $h \in (0, \bar{h})$.

However, the resulting collocation solution will not converge uniformly on I to the exact solution of (1.1) for every choice of the collocation parameters $\{c_i\}$: while the convergence statement

$$\lim_{h\to 0} \|u - u_h\|_\infty = 0 \tag{2.7}$$

holds whenever $c_1 > 0$ and $c_m = 1$ (cf. [3], and [7,8]), this will in general no longer remain true when $c_m < 1$. If (2.7) holds, the order of convergence will not be the same for all $\{c_i\}$.

2.2 The main convergence results

Theorem 2.1 *Assume that $g \in C^{m+2}(I)$, $K \in C^{m+2}(D)$, and $u_h \in S_m^{(0)}(I_h)$ is the collocation solution for the second-kind Volterra integral equation (1.1) defined by the collocation equation (2.2) whose underlying meshes have mesh diameters $h < \bar{h}$. Then (2.7) holds if, and only if, the collocation parameters $\{c_i\}$ satisfy the condition*

$$-1 \le \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \le 1.$$

The corresponding attainable global order of convergence is given by

$$\max_{t\in I} |u(t) - u_h(t)| \le C \begin{cases} h^{m+1}, & \text{if } -1 \le \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1, \end{cases}$$

where the constant C depends on the collocation parameters $\{c_i\}$ but not on h.

3 Continuous collocation solutions for second kind VIEs with constant delay

3.1 Meshes and collocation spaces

It is well known (see for example [2, Ch. 4]) that the constant delay $\tau > 0$ in (1.2) induces the primary discontinuity points $\xi_\mu = \mu\tau$ $(\mu \ge 0)$ at which the regularity of the solution $u(t)$ is, at least for small values of μ, lower than it is in $(\xi_\mu, \xi_{\mu+1})$. Thus, the collocation solution $u_h \in S_m^{(0)}(I_h)$ will attain an order of global convergence equal to that for VIEs (1.1) without delay only if the underlying mesh I_h includes these primary discontinuity points. Assuming for ease of notation that $T = \xi_{M+1}$ for some $M \ge 1$, we choose this so-called constrained mesh to be

$$I_h := \bigcup_{\mu=0}^{M} I_h^{(\mu)}, \quad \text{with } I_h^{(\mu)} := \{t_n^{(\mu)} := \xi_\mu + nh \ (n = 0, 1, \ldots, N)\}, \tag{3.1}$$

where $h = \tau/N$. We set $\sigma_n^{(\mu)} := [t_n^{(\mu)}, t_{n+1}^{(\mu)}]$. The solution u of (1.2) will be approximated by the collocation solution

$$u_h \in S_m^{(0)}(I_h) := \left\{ v \in C(I_h) : \; v|_{\sigma_n^{(\mu)}} \in \pi_m \; (0 \leq n \leq N-1) \right\},$$

using collocation points

$$X_h := \bigcup_{\mu=0}^{M} X_h^{(\mu)},$$
$$\text{with } \; X_h^{(\mu)} := \{ t_{n,i}^{(\mu)} = t_n^{(\mu)} + c_i h : \; i = 1, \ldots, m \; (0 \leq n \leq N-1) \} \tag{3.2}$$

corresponding to prescribed collocation parameters $\{c_i\}$ with $0 < c_1 < \cdots < c_m \leq 1$. Hence, the collocation equation for the subinterval $\sigma_n^{(\mu)}$ is

$$u_h(t) = g(t) + \int_{t-\tau}^{t} K(t,s) u_h(s) ds, \; t \in X_h^{(\mu)} \quad (\mu = 0, 1, \ldots, M). \tag{3.3}$$

If $\mu = 0$, the values of u_h at $t \in [-\tau, 0]$ are determined by the given initial function, i.e., $u_h(t) = \varphi(t)$.

3.2 The main convergence results

Theorem 3.1 *Assume that* $g \in C^{m+2}(I)$, $K \in C^{m+2}(D)$, $\varphi \in C^{m+1}[-\tau, 0]$, *and let* $u_h \in S_m^{(0)}(I_h)$ *be the collocation solution for the second-kind VFIE* (1.2) *determined by the collocation equation* (3.3)*, using constrained meshes* I_h *of the form* (3.1)*. Then* u_h *converges uniformly on* I *to the solution* u *of* (1.2) *if, and only if, the collocation parameters in* (3.2) *satisfy the condition*

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leq 1.$$

The resulting attainable global order of convergence is then given by

$$\max_{t \in I} |u(t) - u_h(t)| \leq C \begin{cases} h^{m+1}, & \text{if } -1 \leq \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1, \end{cases}$$

where the constant C *depends on the collocation parameters* $\{c_i\}$ *but not on* h.

Remark 3.1 The convergence results for the case $0 < c_1 < \cdots < c_m = 1$ follow trivially from the proof of Theorem 2.1 (see also [8] and [3]). A similar conclusion holds in the case of second-kind VFIEs (Theorem 3.1 and its proof).

Remark 3.2 The case $\rho_m = 1$ can happen only when m is even, but as Theorem 2.1 and Theorem 3.1 describe, this case leads to a reduction of the order of convergence.

4 Proof of Theorem 2.1

We assume that $c_m < 1$.

According to the theory of Lagrange interpolation we may write

$$u'(t_n + sh) = \sum_{j=1}^{m} L_j(s)u'(t_{n,j}) + h^m R^1_{m,n}(s), \quad s \in [0, 1], \tag{4.1}$$

where the Peano remainder term and Peano kernel (see [2]) are given by

$$R^1_{m,n}(v) := \int_0^1 K_m(v, z)u^{(m+1)}(t_n + zh)dz$$

and

$$K_m(v, z) := \frac{1}{(m-1)!}\left\{(v - z)_+^{m-1} - \sum_{k=1}^{m} L_k(v)(c_k - z)_+^{m-1}\right\}, \; v \in [0, 1].$$

Here, $(v - z)_+^{m-1} := 0$ for $v < z$ and $(v - z)_+^{m-1} := (v - z)^{m-1}$ for $v \geq z$.

Integration of (4.1) leads to

$$u(t_n + sh) = u(t_n) + h\sum_{j=1}^{m} \beta_j(s)u'(t_{n,j}) + h^{m+1} R_{m,n}(s), \; s \in [0, 1], \tag{4.2}$$

where $R_{m,n}(s) := \int_0^s R^1_{m,n}(v)dv$.

We first consider the case of constant kernel $K(t, s) \equiv 1$. This case already contains all important ideas.

By (2.4) and (4.2), the collocation error $e_h := u - u_h$ on $[t_n, t_{n+1}]$ may be written as

$$e_h(t_n + sh) = e_h(t_n) + h\sum_{j=1}^{m} \beta_j(s)\varepsilon_{n,j} + h^{m+1} R_{m,n}(s), \tag{4.3}$$

where $\varepsilon_{n,i} := u'(t_{n,i}) - u'_h(t_{n,i})$. Particularly,

$$e_h(t_{n,i}) = e_h(t_n) + h\sum_{j=1}^{m} a_{ij}\varepsilon_{n,j} + h^{m+1} R_{m,n}(c_i). \tag{4.4}$$

By (1.1)–(2.2) and using (4.3), it can be shown that

$$
\begin{aligned}
e_h(t_{n,i}) &= \int_0^{t_{n,i}} e_h(s)ds = h\sum_{l=0}^{n-1}\int_0^1 e_h(t_l+sh)ds + h\int_0^{c_i} e_h(t_n+sh)ds \\
&= h\sum_{l=0}^{n-1} e_h(t_l) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{l,j} + hc_i e_h(t_n) + h^2\sum_{j=1}^{m} b_{ij}\varepsilon_{n,j} + h^{m+1}\tilde{R}_{m,n}(c_i),
\end{aligned}
\tag{4.5}
$$

where $\gamma_j(s) := \int_0^s \beta_j(v)dv$, $b_{ij} := \int_0^{c_i} \beta_j(s)ds = \gamma_j(c_i)$, and $\tilde{R}_{m,n}(c_i) := \sum_{l=0}^{n-1} h\int_0^1 R_{m,l}(s)ds + h\int_0^{c_i} R_{m,n}(s)ds$. So by (4.4) and (4.5), we have

$$
\begin{aligned}
& e_h(t_n) + h\sum_{j=1}^{m} a_{ij}\varepsilon_{n,j} + h^{m+1}R_{m,n}(c_i) \\
&\quad = h\sum_{l=0}^{n-1} e_h(t_l) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{l,j} + hc_i e_h(t_n) + h^2\sum_{j=1}^{m} b_{ij}\varepsilon_{n,j} + h^{m+1}\tilde{R}_{m,n}(c_i).
\end{aligned}
\tag{4.6}
$$

By the standard technique used by Brunner (see [2]), rewriting (4.6) with n replaced by $n-1$ and with $i=m$ and subtract it from (4.6), we find

$$
\begin{aligned}
& e_h(t_n) - e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{ij}\varepsilon_{n,j} - h\sum_{j=1}^{m} a_{mj}\varepsilon_{n-1,j} \\
&\quad + h^{m+1}R_{m,n}(c_i) - h^{m+1}R_{m,n-1}(c_m) \\
&\quad = he_h(t_{n-1}) + h^2\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{n-1,j} + hc_i e_h(t_n) - hc_m e_h(t_{n-1}) \\
&\quad + h^2\sum_{j=1}^{m} b_{ij}\varepsilon_{n,j} - h^2\sum_{j=1}^{m} b_{mj}\varepsilon_{n-1,j} + h^{m+1}\tilde{R}_{m,n}(c_i) - h^{m+1}\tilde{R}_{m,n-1}(c_m).
\end{aligned}
$$

This can be written in the more concise form

$$
\begin{aligned}
& (e_h(t_n) - e_h(t_{n-1}))\, e + hA\varepsilon_n - hee_m^T A\varepsilon_{n-1} \\
&\quad = he_h(t_{n-1})e + h^2 e\gamma^T\varepsilon_{n-1} + hCee_h(t_n) - hc_m ee_h(t_{n-1}) \\
&\qquad + h^2 B\varepsilon_n - h^2 ee_m^T B\varepsilon_{n-1} + h^{m+1}R_{m,n},
\end{aligned}
\tag{4.7}
$$

with obvious meaning of $R_{m,n}$, and with $C := diag(c_1, \ldots, c_m)$, $\varepsilon_n := (\varepsilon_{n,1}, \ldots, \varepsilon_{n,m})^T$, $B := (b_{ij})_{m\times m}$, $\gamma := (\gamma_1(1), \ldots, \gamma_m(1))^T$.

Since e_h is continuous in I, and hence at the mesh points, by (4.3) and $e_h(0) = 0$, we also have the relation (see [2, (1.1.27)])

$$
\begin{aligned}
e_h(t_n) &= e_h(t_{n-1}+h) = e_h(t_{n-1}) + h\sum_{j=1}^{m} b_j \varepsilon_{n-1,j} + h^{m+1} R_{m,n-1}(1) \\
&= h\sum_{l=0}^{n-1}\sum_{j=1}^{m} b_j \varepsilon_{l,j} + h^{m+1}\sum_{l=0}^{n-1} R_{m,l}(1) = h\sum_{l=0}^{n-1} b^T\varepsilon_l + h^{m+1}\sum_{l=0}^{n-1} R_{m,l}(1),
\end{aligned}
\tag{4.8}
$$

where $b_j := \int_0^1 L_j(s)ds$ and $b^T := (b_1, \ldots, b_m)$.

Substituting (4.8) into (4.7), we have

$$
\begin{aligned}
&eb^T\varepsilon_{n-1} + h^m e R_{m,n-1}(1) + A\varepsilon_n - ee_m^T A\varepsilon_{n-1} \\
&\quad = (1-c_m)e\left[h\sum_{l=0}^{n-2} b^T\varepsilon_l + h^m\sum_{l=0}^{n-2} hR_{m,l}(1)\right] + he\gamma^T\varepsilon_{n-1} \\
&\qquad + Ce\left[h\sum_{l=0}^{n-1} b^T\varepsilon_l + h^m\sum_{l=0}^{n-1} hR_{m,l}(1)\right] + hB\varepsilon_n - hee_m^T B\varepsilon_{n-1} + h^m R_{m,n}.
\end{aligned}
$$

This equation can be written in the form

$$
\begin{aligned}
(A-hB)\,\varepsilon_n &= \left(ee_m^T A - eb^T + he\gamma^T - hee_m^T B\right)\varepsilon_{n-1} + h(1-c_m)e\sum_{l=0}^{n-2} b^T\varepsilon_l \\
&\quad + hCe\sum_{l=0}^{n-1} b^T\varepsilon_l + h^m\tilde{R}_{m,n},
\end{aligned}
\tag{4.9}
$$

or

$$
\varepsilon_n = \left(A^{-1}\left(ee_m^T A - eb^T\right) + O(h)\right)\varepsilon_{n-1} + hD\sum_{l=0}^{n-1}\varepsilon_l + h^m\bar{R}_{m,n}, \tag{4.10}
$$

with obvious meaning of $\tilde{R}_{m,n}$, $\bar{R}_{m,n}$ and D.

Since

$$
ee_m^T A - eb^T = \begin{pmatrix} a_{m1}-b_1 & a_{m2}-b_2 & \cdots & a_{mm}-b_m \\ a_{m1}-b_1 & a_{m2}-b_2 & \cdots & a_{mm}-b_m \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}-b_1 & a_{m2}-b_2 & \cdots & a_{mm}-b_m \end{pmatrix},
$$

the rank of the matrix $ee_m^T A - eb^T$ is one, implying that the rank of the matrix $A^{-1}(ee_m^T A - eb^T)$ is also one. This means that this matrix has exactly one nonzero eigenvalue. Setting $A^{-1} := (v_{ij})_{m\times m}$, we have

$$A^{-1}\left(ee_m^T A - eb^T\right) = \begin{pmatrix} (a_{m1}-b_1)\sum_{j=1}^m v_{1j} & (a_{m2}-b_2)\sum_{j=1}^m v_{1j} & \cdots & (a_{mm}-b_m)\sum_{j=1}^m v_{1j} \\ (a_{m1}-b_1)\sum_{j=1}^m v_{2j} & (a_{m2}-b_2)\sum_{j=1}^m v_{2j} & \cdots & (a_{mm}-b_m)\sum_{j=1}^m v_{2j} \\ \cdots & \cdots & \cdots & \cdots \\ (a_{m1}-b_1)\sum_{j=1}^m v_{mj} & (a_{m2}-b_2)\sum_{j=1}^m v_{mj} & \cdots & (a_{mm}-b_m)\sum_{j=1}^m v_{mj} \end{pmatrix},$$

and the nonzero eigenvalue is

$$\lambda\left(A^{-1}(ee_m^T A - eb^T)\right) = \sum_{i=1}^m (a_{mi}-b_i)\sum_{j=1}^m v_{ij} = 1 - b^T A^{-1} e.$$

By Proposition 3.8 and Theorem 3.10 of [5], we know that the stability function $R(z) = P(z)/Q(z)$ of the collocation method has the value $R(\infty) = 1 - b^T A^{-1} e$, where $Q(z)$ and $P(z)$ are the polynomials

$$Q(z) = M^{(m)}(0) + M^{(m-1)}(0)z + \cdots + M(0)z^m,$$
$$P(z) = M^{(m)}(1) + M^{(m-1)}(1)z + \cdots + M(1)z^m,$$

with

$$M(z) = \frac{1}{m!}\prod_{i=1}^m (z - c_i).$$

Therefore,

$$R(\infty) = \frac{M(1)}{M(0)} = (-1)^m \prod_{i=1}^m \frac{1-c_i}{c_i};$$

that is, the only nonzero eigenvalue of $A^{-1}(ee_m^T A - eb^T)$ is

$$1 - b^T A^{-1} e = (-1)^m \prod_{i=1}^m \frac{1-c_i}{c_i} = \rho_m.$$

Therefore $A^{-1}\left(ee_m^T A - eb^T\right)$ is diagonalizable and there exists a nonsingular matrix T such that

$$T^{-1}A^{-1}\left(ee_m^TA - eb^T\right)T =: F = diag(\rho_m, \underbrace{0, \dots, 0}_{m-1}).$$

Multiplying (4.10) by T^{-1} and setting $Z_n := T^{-1}\varepsilon_n$, we obtain

$$Z_n = (F + O(h))Z_{n-1} + hT^{-1}DT\sum_{l=0}^{n-1} Z_l + h^mT^{-1}\bar{R}_{m,n}. \tag{4.11}$$

We consider the following three cases:

Case I $-1 < \rho_m < 1$

Using standard techniques of error estimation for collocation solutions of VIEs (see [2,5,7]), we know that there exists a constant C_1, such that

$$\|\varepsilon_n\|_1 \le C_1h^m. \tag{4.12}$$

It follows from (4.6) that there exist constants C_2 and C_3 such that

$$|e_h(t_n)| \le hC_2\sum_{l=0}^{n-1}|e_h(t_l)| + C_3h^{m+1}, \tag{4.13}$$

and hence by the discrete Gronwall inequality (see [2]), there exists a constant C_4, such that

$$|e_h(t_n)| \le C_4h^{m+1} \quad (n = 1, \dots, N). \tag{4.14}$$

Case II $\rho_m = -1$

Rewriting (4.11) with n replaced by $n-1$ and subtract it from (4.11), we have

$$\begin{aligned} Z_n - Z_{n-1} = (F + O(h))(Z_{n-1} - Z_{n-2}) + hT^{-1}DTZ_{n-1} \\ + h^mT^{-1}(\bar{R}_{m,n} - \bar{R}_{m,n-1}). \end{aligned} \tag{4.15}$$

Therefore,

$$\begin{pmatrix} Z_n \\ Z_{n-1} \end{pmatrix} = \begin{pmatrix} I_m + F + O(h) & -F + O(h) \\ I_m & 0 \end{pmatrix}\begin{pmatrix} Z_{n-1} \\ Z_{n-2} \end{pmatrix} + \begin{pmatrix} h^mT^{-1}\left(\bar{R}_{m,n} - \bar{R}_{m,n-1}\right) \\ 0 \end{pmatrix}.$$

Since $\bar{R}_{m,n} - \bar{R}_{m,n-1} = O(h)$ for $u \in C^{m+2}$, we define $r_{m,n} := \frac{\bar{R}_{m,n} - \bar{R}_{m,n-1}}{h}$ and set

$$X_n := \begin{pmatrix} Z_n \\ Z_{n-1} \end{pmatrix}, G := \begin{pmatrix} I_m + F & -F \\ I_m & 0 \end{pmatrix}, \bar{r}_{m,n} := \begin{pmatrix} T^{-1}r_{m,n} \\ 0 \end{pmatrix}.$$

We may then write

$$X_n = GX_{n-1} + O(h)X_{n-1} + h^{m+1}\bar{r}_{m,n}. \tag{4.16}$$

The eigenvalues of the matrix G are $\underbrace{1, 1, \ldots, 1}_{m}; -1, \underbrace{0, \ldots, 0}_{m-1}$. The eigenvalue 1 of multiplicity m has m linearly independent eigenvectors, while to the eigenvalue 0 of multiplicity $m-1$ there correspond $m-1$ linearly independent eigenvectors. Therefore, G is diagonalizable, and there exists a nonsingular matrix P such that

$$P^{-1}GP =: \Lambda = diag(\underbrace{1, \ldots, 1}_{m}, -1, \underbrace{0, \ldots, 0}_{m-1}).$$

Defining $Y_n := P^{-1}X_n$ we obtain

$$Y_n = (\Lambda + O(h))Y_{n-1} + h^{m+1}P^{-1}\bar{r}_{m,n}. \tag{4.17}$$

Similar to [7] we can assert that there exist constants C_5, C_6 so that

$$\|Y_n\| \leq (1 + C_5h)\|Y_{n-1}\| + C_6h^{m+1}.$$

An induction argument then leads to

$$\|Y_n\| \leq (1 + C_5h)^n\|Y_0\| + \frac{(1 + C_5h)^n - 1}{C_5h}C_6h^{m+1},$$

and we can then show that there exists a constant C_7 such that,

$$\|\varepsilon_n\|_1 \leq C_7h^m. \tag{4.18}$$

By (4.6), and as in Case I, there exists hence a constant C_8 so that

$$|e_h(t_n)| \leq C_8h^{m+1}. \tag{4.19}$$

Case III $\rho_m = 1$

Here, the eigenvalues of G defined in Case II are $\underbrace{1, 1, \ldots, 1}_{m}; 1, \underbrace{0, \ldots, 0}_{m-1}$, where now the eigenvalue 1 of multiplicity $m + 1$ also has m linearly independent eigenvectors. This means that G is not diagonalizable, but there exists a nonsingular matrix Q, such that

$$Q^{-1}GQ = \begin{pmatrix} 1 & 1 & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}.$$

Defining $\bar{\Lambda} := Q^{-1}GQ$, $\bar{Y}_n := Q^{-1}X_n$ and recalling (4.16) we obtain

$$\bar{Y}_n = (\bar{\Lambda} + O(h))\bar{Y}_{n-1} + h^{m+1}Q^{-1}\bar{r}_{m,n}. \tag{4.20}$$

An induction argument yields

$$\bar{Y}_n = (\bar{\Lambda} + O(h))^n \bar{Y}_0 + h^{m+1}\sum_{l=0}^{n-1}(\bar{\Lambda} + O(h))^l Q^{-1}\bar{r}_{m,n-l},$$

and thus there exist constants C_9, C_{10} such that

$$\|\bar{Y}_n\| \le C_9\|\bar{\Lambda}^n\|\|\bar{Y}_0\| + h^{m+1}C_{10}\sum_{l=0}^{n-1}\|\bar{\Lambda}^l\|.$$

It is easily to check that

$$\bar{\Lambda}^n = \begin{pmatrix} 1 & n & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}.$$

Therefore, there exists a constant C_{11} such that

$$\|\varepsilon_n\|_1 \le C_{11}h^{m-1}, \tag{4.21}$$

and an argument analogous to the one employed in the analysis of Case I shows that there exists a constant C_{12} such that

$$|e_h(t_n)| \le C_{12}h^m. \tag{4.22}$$

Obviously, the collocation solution u_h is divergent if $|\rho_m| > 1$. The proof is completed by recalling (2.3), (4.1) and (4.3).

In the following, we prove the results for general, non-constant kernels. Now, by (4.4), (1.1)–(2.2) and using (4.3), we obtain

$$\begin{aligned}
e_h(t_{n,i}) &= e_h(t_n) + h\sum_{j=1}^{m} a_{ij}\varepsilon_{n,j} + h^{m+1}R_{m,n}(c_i)\\
&= \int_0^{t_{n,i}} K(t_{n,i},s)e_h(s)ds\\
&= h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i},t_l+sh)e_h(t_l+sh)ds + h\int_0^{c_i} K(t_{n,i},t_n+sh)e_h(t_n+sh)ds\\
&= h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i},t_l+sh)\left[e_h(t_l)+h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{l,j}\right]ds\\
&\quad + h\int_0^{c_i} K(t_{n,i},t_n+sh)\left[e_h(t_n)+h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{n,j}\right]ds\\
&\quad + h^{m+1}\sum_{l=0}^{n-1} h\int_0^1 K(t_{n,i},t_l+sh)R_{m,l}(s)ds\\
&\quad + h^{m+2}\int_0^{c_i} K(t_{n,i},t_n+sh)R_{m,n}(s)ds\\
&= h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i},t_l+sh)ds\, e_h(t_l) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^1 K(t_{n,i},t_l+sh)\beta_j(s)ds\,\varepsilon_{l,j}\\
&\quad + h\int_0^{c_i} K(t_{n,i},t_n+sh)ds\, e_h(t_n) + h^2\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i},t_n+sh)\beta_j(s)ds\,\varepsilon_{n,j}\\
&\quad + h^{m+1}\sum_{l=0}^{n-1} h\int_0^1 K(t_{n,i},t_l+sh)R_{m,l}(s)ds + h^{m+2}\int_0^{c_i} K(t_{n,i},t_n+sh)R_{m,n}(s)ds.
\end{aligned} \tag{4.23}$$

By the standard technique used by Brunner (see [2]), rewriting (4.23) with n replaced by $n-1$ and with $i=m$ and subtract it from (4.23), we find

$$\begin{aligned}
&e_h(t_n) - e_h(t_{n-1}) + h\sum_{j=1}^{m} a_{ij}\varepsilon_{n,j} - h\sum_{j=1}^{m} a_{mj}\varepsilon_{n-1,j}\\
&= h\int_0^1 K(t_{n,i},t_{n-1}+sh)ds\, e_h(t_{n-1})\\
&\quad + h^2(c_i+1-c_m)\sum_{l=0}^{n-2}\int_0^1 K_1'(\xi_{n,i},t_l+sh)ds\, e_h(t_l)\\
&\quad + h^2\sum_{j=1}^{m}\int_0^1 K(t_{n,i},t_{n-1}+sh)\beta_j(s)ds\,\varepsilon_{n-1,j}
\end{aligned}$$

$$+h^3(c_i+1-c_m)\sum_{l=0}^{n-2}\sum_{j=1}^{m}\int_0^1 K_1'(\xi_{n,i},t_l+sh)\beta_j(s)ds\varepsilon_{l,j}$$
$$+h\int_0^{c_i}K(t_{n,i},t_n+sh)dse_h(t_n)-h\int_0^{c_m}K(t_{n-1,m},t_{n-1}+sh)dse_h(t_{n-1})$$
$$+h^2\sum_{j=1}^{m}\int_0^{c_i}K(t_{n,i},t_n+sh)\beta_j(s)ds\varepsilon_{n,j}$$
$$-h^2\sum_{j=1}^{m}\int_0^{c_m}K(t_{n-1,m},t_{n-1}+sh)\beta_j(s)ds\varepsilon_{n-1,j}$$
$$+h^{m+1}\tilde{R}_{m,n}(c_i),$$

where $\tilde{R}_{m,n}(c_i):=-R_{m,n}(c_i)+R_{m,n-1}(c_m)+h\int_0^1 K(t_{n,i},t_{n-1}+sh)R_{m,n-1}(s)ds+\sum_{l=0}^{n-2}h\int_0^1[K(t_{n,i},t_l+sh)-K(t_{n-1,m},t_l+sh)]R_{m,l}(s)ds+h\int_0^{c_i}K(t_{n,i},t_n+sh)R_{m,n}(s)ds-h\int_0^{c_m}K(t_{n-1,m},t_{n-1}+sh)R_{m,n-1}(s)ds$, $\xi_{n,i}\in(t_{n-1,m},t_{n,i})$.

This can be written in the more concise form

$$
\begin{aligned}
&(e_h(t_n)-e_h(t_{n-1}))\,e+hA\varepsilon_n-hee_m^TA\varepsilon_{n-1}\\
&\quad=hC_n^{(n-1)}ee_h(t_{n-1})+h^2\,(C+(1-c_m)I_m)\sum_{l=0}^{n-2}\bar{C}_n^{(l)}ee_h(t_l)+h^2B_n^{(n-1)}\varepsilon_{n-1}\\
&\qquad+h^3\,(C+(1-c_m)I_m)\sum_{l=0}^{n-2}\bar{B}_n^l\varepsilon_l+hC_nee_h(t_n)-hee_m^TC_{n-1}ee_h(t_{n-1})\\
&\qquad+h^2B_n\varepsilon_n-h^2ee_m^TB_{n-1}\varepsilon_{n-1}+h^{m+1}R_{m,n}^{(1)},
\end{aligned}
\tag{4.24}
$$

with obvious meaning of $R_{m,n}^{(1)}$, and with $\bar{C}_n^{(l)}:=diag(\int_0^1 K_1'(\xi_{n,i},t_l+sh)ds)$ $(0\le l\le N-1)$ and $\bar{B}_n^{(l)}:=(\int_0^1 K_1'(\xi_{n,i},t_l+sh)\beta_j(s)ds)$ $(0\le l\le N-1)$.

Substituting (4.8) into (4.24), we have

$$
\begin{aligned}
&eb^T\varepsilon_{n-1}+A\varepsilon_n-ee_m^TA\varepsilon_{n-1}\\
&\quad=hC_n^{n-1}eb^T\sum_{l=0}^{n-2}\varepsilon_l+h^2\,(C+(1-c_m)I_m)\sum_{l=0}^{n-2}\bar{C}_n^{(l)}e\sum_{k=0}^{l-1}\varepsilon_k+hB_n^{(n-1)}\varepsilon_{n-1}\\
&\qquad+h^2\,(C+(1-c_m)I_m)\sum_{l=0}^{n-2}\bar{B}_n^l\varepsilon_l+hC_neb^T\sum_{l=0}^{n-1}\varepsilon_l-hee_m^TC_{n-1}eeb^T\sum_{l=0}^{n-2}\varepsilon_l\\
&\qquad+hB_n\varepsilon_n-hee_m^TB_{n-1}\varepsilon_{n-1}+h^m\bar{\tilde{R}}_{m,n},
\end{aligned}
$$

with obvious meaning of $\bar{\bar{R}}_{m,n}$. This equation can be written in the form

$$
\begin{aligned}
&(A - hB_n)\,\varepsilon_n \\
&= \left(ee_m^T A - eb^T + hB_n^{(n-1)} - hee_m^T B_{n-1}\right)\varepsilon_{n-1} + hC_n^{n-1} eb^T \sum_{l=0}^{n-2} \varepsilon_l \\
&\quad + h^2\,(C + (1 - c_m)I_m) \sum_{l=0}^{n-2} \bar{C}_n^{(l)} e \sum_{k=0}^{l-1} \varepsilon_k + h^2\,(C + (1 - c_m)I_m) \sum_{l=0}^{n-2} \bar{B}_n^l \varepsilon_l \\
&\quad + hC_n eb^T \sum_{l=0}^{n-1} \varepsilon_l - hee_m^T C_{n-1} eeb^T \sum_{l=0}^{n-2} \varepsilon_l + h^m \bar{\bar{R}}_{m,n},
\end{aligned}
\tag{4.25}
$$

or

$$
\varepsilon_n = \left(A^{-1}\left(ee_m^T A - eb^T\right) + O(h)\right)\varepsilon_{n-1} + h\tilde{D}_n \sum_{l=0}^{n-1} \varepsilon_l + h^m \bar{\bar{R}}_{m,n}, \tag{4.26}
$$

with obvious meaning of $\bar{\bar{R}}_{m,n}$ and $\tilde{D}_n$.

Comparison (4.10) of the case $K(t,s) \equiv 1$ with (4.26), and similar to the proof of the case $K(t,s) \equiv 1$, we can obtain now (4.11) becomes

$$
Z_n = (F + O(h))Z_{n-1} + hT^{-1}\tilde{D}_n T \sum_{l=0}^{n-1} Z_l + h^m T^{-1} \bar{\bar{R}}_{m,n}. \tag{4.27}
$$

We also consider the following three cases:

Case I $-1 < \rho_m < 1$

By the same technique of the case $K(t,s) \equiv 1$, we can prove that there exists a constant $\tilde{C}_4$, such that

$$
|e_h(t_n)| \le \tilde{C}_4 h^{m+1} \quad (n = 1, \ldots, N). \tag{4.28}
$$

Case II $\rho_m = -1$

Rewriting (4.27) with n replaced by $n-1$ and subtract it from (4.27), we find

$$
\begin{aligned}
Z_n - Z_{n-1} = {} & (F + O(h))(Z_{n-1} - Z_{n-2}) + hT^{-1}\tilde{D}_n T Z_{n-1} \\
& + hT^{-1}\left(\tilde{D}_n - \tilde{D}_{n-1}\right) T \sum_{l=0}^{n-2} Z_l \\
& + h^m T^{-1}(\bar{\bar{R}}_{m,n} - \bar{\bar{R}}_{m,n-1}).
\end{aligned}
\tag{4.29}
$$

Notice that $\tilde{D}_n - \tilde{D}_{n-1} = O(h)$, therefore,

$$\begin{pmatrix} Z_n \\ Z_{n-1} \end{pmatrix} = \begin{pmatrix} I_m + F + O(h) & -F + O(h) \\ I_m & 0 \end{pmatrix} \begin{pmatrix} Z_{n-1} \\ Z_{n-2} \end{pmatrix} + \sum_{l=1}^{n-2} \begin{pmatrix} O(h^2) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_l \\ Z_{l-1} \end{pmatrix} + \begin{pmatrix} h^m T^{-1} \left(\bar{\bar{R}}_{m,n} - \bar{\bar{R}}_{m,n-1} \right) \\ 0 \end{pmatrix}.$$

Now (4.17) becomes

$$Y_n = (\Lambda + O(h))Y_{n-1} + O(h^2) \sum_{l=1}^{n-2} Y_l + O(h^{m+1}). \tag{4.30}$$

Similar to the case of $K(t,s) \equiv 1$, we can assert that there exist constants $\tilde{C}_5, \tilde{C}_5', \tilde{C}_6$ so that

$$\|Y_n\| \leq (1 + \tilde{C}_5 h)\|Y_{n-1}\| + \tilde{C}_5' h^2 \sum_{l=1}^{n-2} \|Y_l\| + \tilde{C}_6 h^{m+1}.$$

An induction argument then leads to

$$\|Y_n\| \leq (1 + \tilde{C}_5 h)^n \|Y_0\| + \tilde{C}_5' h^2 \frac{(1 + \tilde{C}_5 h)^n - 1}{\tilde{C}_5 h} \sum_{l=1}^{n-2} \|Y_l\| + \tilde{C}_6 \frac{(1 + \tilde{C}_5 h)^n - 1}{\tilde{C}_5 h} h^{m+1}.$$

Therefore, by the discrete Gronwall inequality (see [2]), we can get that there exists a constant $\tilde{C}_7$ such that,

$$\|\varepsilon_n\|_1 \leq \tilde{C}_7 h^m, \tag{4.31}$$

and similar to the case of $K(t,s) \equiv 1$, we can then show that there exists a constant $\tilde{C}_8$ so that

$$|e_h(t_n)| \leq \tilde{C}_8 h^{m+1}. \tag{4.32}$$

Case III $\rho_m = 1$

Using the technique of [7], we write the collocation approximation u_h and the exact solution in the form

$$u_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) u_h(t_{n,j}) + h^m \frac{u_h^{(m)}(\eta_n)}{m!} \prod_{i=1}^{m} (s - c_i), \tag{4.33}$$

and

$$u(t_n+sh)=\sum_{j=1}^{m}L_j(s)u(t_{n,j})+h^m\frac{u^{(m)}(\eta_n')}{m!}\prod_{i=1}^{m}(s-c_i), \tag{4.34}$$

where $\eta_n, \eta_n' \in (t_n, t_{n+1})$.

So (4.34)–(4.33) yields

$$e_h((t_n+sh)=\sum_{j=1}^{m}L_j(s)e_h(t_{n,j})+h^m\hat{R}_n(s), \tag{4.35}$$

where $\hat{R}_n(s):=\frac{u^{(m)}(\eta_n')-u_h^{(m)}(\eta_n)}{m!}\prod_{i=1}^{m}(s-c_i)$.

Now, by (1.1)–(2.2) and using (4.35), we obtain

$$\begin{aligned}e_h(t_{n,i})&=h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i},t_l+sh)e_h(t_l+sh)ds\\&\quad+h\int_0^{c_i}K(t_{n,i},t_n+sh)e_h(t_n+sh)ds\\&=h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i},t_l+sh)\sum_{j=1}^{m}L_j(s)dse_h(t_{l,j})\\&\quad+h\int_0^{c_i}K(t_{n,i},t_n+sh)\sum_{j=1}^{m}L_j(s)dse_h(t_{n,j})\\&\quad+h^m\bar{\hat{R}}_n(s),\end{aligned} \tag{4.36}$$

with obvious meanings of $\bar{\hat{R}}_n(s)$.

Rewriting (4.36) with n replaced by $n-1$ and $i=m$ and subtract it from (4.36), we can get

$$\begin{aligned}&e_h(t_{n,i})-e_h(t_{n-1,m})\\&=h\int_0^1 K(t_{n,i},t_{n-1}+sh)\sum_{j=1}^{m}L_j(s)dse_h(t_{n-1,j})\\&\quad+h^2(c_i+1-c_m)\sum_{l=0}^{n-2}\int_0^1 K_1'(\xi_{n,i},t_l+sh)\sum_{j=1}^{m}L_j(s)dse_h(t_{l,j})\\&\quad+h\int_0^{c_i}K(t_{n,i},t_n+sh)\sum_{j=1}^{m}L_j(s)dse_h(t_{n,j})\end{aligned}$$

$$
\begin{aligned}
&- h \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh) \sum_{j=1}^{m} L_j(s) ds e_h(t_{n-1,j}) \\
&+ h^m \bar{\hat{R}}_n(s) - h^m \bar{\hat{R}}_{n-1}(s). \qquad (4.37)
\end{aligned}
$$

Denoting $E_n := (e_h(t_{n,1}), \ldots, e_h(t_{n,m}))^T$ and noticing that $\bar{\hat{R}}_n(s) - \bar{\hat{R}}_{n-1}(s) = O(h)$, we can rewrite (4.37) as the more concise form

$$
E_n - ee_m^T E_{n-1} = O(h) E_{n-1} + O(h^2) \sum_{l=0}^{n-2} E_l + O(h) E_n + O(h^{m+1}). \qquad (4.38)
$$

Since

$$
ee_m^T = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 \end{pmatrix},
$$

the rank of ee_m^T is 1, and the unique nonzero eigenvalue is 1, so ee_m^T is diagonalizable and there exists a nonsingular matrix $\tilde{T}$, such that

$$
\tilde{T}^{-1} ee_m^T \tilde{T} =: \tilde{F} = diag(1, \underbrace{0, \ldots, 0}_{m-1}).
$$

Denote $\tilde{E}_n = \tilde{T}^{-1} E_n$. Then (4.38) becomes

$$
\tilde{E}_n = (\tilde{F} + O(h)) \tilde{E}_{n-1} + O(h^2) \sum_{l=0}^{n-2} \tilde{E}_l + O(h^{m+1}).
$$

Therefore, there exist constants $\bar{C}_5$, $\bar{C}'_5$ and $\bar{C}_6$, such that

$$
\|\tilde{E}_n\| \le (1 + \bar{C}_5 h) \|\tilde{E}_{n-1}\| + \bar{C}'_5 h^2 \sum_{l=0}^{n-2} \|\tilde{E}_l\| + \bar{C}_6 h^{m+1}. \qquad (4.39)
$$

Similar to Case II, we can get that there exists a constant $\bar{C}_7$ such that

$$
\|E_n\| \le \bar{C}_7 h^m.
$$

By (4.35), we can then get there exists a constant $\bar{C}_8$ such that

$$
|e_h(t_n + sh)| \le \bar{C}_8 h^m.
$$

Obviously, the collocation solution u_h is divergent also if $|\rho_m| > 1$. The proof is completed by recalling (2.3), (4.1) and (4.3).

5 Proof of Theorems 3.1

On $\sigma_n^{(\mu)} := (t_n^{(\mu)}, t_{n+1}^{(\mu)}]$, the derivative u_h' of the collocation solution has the local Lagrange representation,

$$u_h'(t_n^{(\mu)} + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}^{(\mu)}, \quad s \in (0, 1], \tag{5.1}$$

where $U_{n,i}^{(\mu)} := u_h'(t_{n,i}^{(\mu)})$. Upon integration of (5.1) we obtain

$$u_h(t_n^{(\mu)} + sh) = u_h(t_n^{(\mu)}) + h \sum_{j=1}^{m} \beta_j(s) U_{n,j}^{(\mu)}, \quad s \in [0, 1]. \tag{5.2}$$

Since on each subinterval $\sigma_n^{(\mu)}$ the exact solution of the delay VIE (1.2) is in C^{m+2}, we may write

$$u'(t_n^{(\mu)} + sh) = \sum_{j=1}^{m} L_j(s) u'(t_{n,j}^{(\mu)}) + h^m R_{m,n}^{(1,\mu)}(s), \quad s \in (0, 1], \tag{5.3}$$

where the Peano remainder term is given by

$$R_{m,n}^{(1,\mu)}(v) := \int_0^1 K_m(v, z) u^{(m+1)}(t_n^{(\mu)} + zh) dz.$$

Integration of (5.3) leads to

$$u(t_n^{(\mu)} + sh_n) = u(t_n^{(\mu)}) + h \sum_{j=1}^{m} \beta_j(s) u'(t_{n,j}^{(\mu)}) + h^{m+1} R_{m,n}^{(\mu)}(s), \quad s \in [0, 1], \tag{5.4}$$

with $R_{m,n}^{(\mu)}(s) := \int_0^s R_{m,n}^{(1,\mu)}(v) dv$.

For ease of notation we will again assume that $K(t, s) \equiv 1$, and we can extend to the proof to the non-constant kernel by the same technique as the proof of Theorem 2.1.

By (5.2) and (5.4), the collocation error $e_h := u - u_h$ on $\bar{\sigma_n}^{(\mu)} := [t_n^{(\mu)}, t_{n+1}^{(\mu)}]$ can be written as

$$e_h(t_n^{(\mu)} + sh) = e_h(t_n^{(\mu)}) + h \sum_{j=1}^{m} \beta_j(s) \varepsilon_{n,j}^{(\mu)} + h^{m+1} R_{m,n}^{(\mu)}(s), \tag{5.5}$$

where $\varepsilon_{n,i}^{(\mu)} = u'(t_{n,i}^{(\mu)}) - u_h'(t_{n,i}^{(\mu)})$. Particularly,

$$e_h(t_{n,i}^{(\mu)}) = e_h(t_n^{(\mu)}) + h\sum_{j=1}^{m} a_{ij}\varepsilon_{n,j}^{(\mu)} + h^{m+1}R_{m,n}^{(\mu)}(c_i). \tag{5.6}$$

By (1.2)–(3.3) and using (5.5), we have

$$\begin{aligned}
e_h(t_{n,i}^{(\mu)}) &= \int_{t_{n,i}^{(\mu-1)}}^{t_{n,i}^{(\mu)}} e_h(s)ds = h\int_{c_i}^{1} e_h(t_n^{(\mu-1)}+sh)ds + h\sum_{l=n+1}^{N-1}\int_0^1 e_h(t_l^{(\mu-1)}+sh)ds \\
&\quad + h\sum_{l=0}^{n-1}\int_0^1 e_h(t_l^{(\mu)}+sh)ds + h\int_0^{c_i} e_h(t_n^{(\mu)}+sh)ds \\
&= h\int_{c_i}^{1}\left[e_h(t_n^{(\mu-1)}) + h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{n,j}^{(\mu-1)}\right]ds \\
&\quad + h\sum_{l=n+1}^{N-1}\int_0^1\left[e_h(t_l^{(\mu-1)}) + h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{l,j}^{(\mu-1)}\right]ds \\
&\quad + h\sum_{l=0}^{n-1}\int_0^1\left[e_h(t_l^{(\mu)}) + h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{l,j}^{(\mu)}\right]ds \\
&\quad + h\int_0^{c_i}\left[e_h(t_n^{(\mu)}) + h\sum_{j=1}^{m}\beta_j(s)\varepsilon_{n,j}^{(\mu)}\right]ds \\
&\quad + h^{m+1}\sum_{l=n+1}^{N-1} h\int_0^1 R_{m,l}^{(\mu-1)}(s)ds + h^{m+2}\int_{c_i}^{1} R_{m,n}^{(\mu-1)}(s)ds \\
&\quad + h^{m+1}\sum_{l=0}^{n-1} h\int_0^1 R_{m,l}^{(\mu)}(s)ds + h^{m+2}\int_0^{c_i} R_{m,n}^{(\mu)}(s)ds \\
&= h(1-c_i)e_h(t_n^{(\mu-1)}) + h^2\sum_{j=1}^{m}(\gamma_j(1)-b_{ij})\varepsilon_{n,j}^{(\mu-1)} + h\sum_{l=n+1}^{N-1} e_h(t_l^{(\mu-1)}) \\
&\quad + h^2\sum_{l=n+1}^{N-1}\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{l,j}^{(\mu-1)} + h\sum_{l=0}^{n-1} e_h(t_l^{(\mu)}) \\
&\quad + h^2\sum_{l=0}^{n-1}\sum_{j=1}^{m}\gamma_j(1)\varepsilon_{l,j}^{(\mu)} + hc_i e_h(t_n^{(\mu)}) \\
&\quad + h^2\sum_{j=1}^{m} b_{ij}\varepsilon_{n,j}^{(\mu)} + h^{m+1}\tilde{R}_{m,n}^{(\mu)}(c_i),
\end{aligned} \tag{5.7}$$

where $\tilde{R}_{m,n}^{(\mu)}(c_i) = \sum_{l=n+1}^{N-1} h\int_0^1 R_{m,l}^{(\mu-1)}(s)ds + h\int_{c_i}^1 R_{m,n}^{(\mu-1)}(s)ds + \sum_{l=0}^{n-1} h\int_0^1 R_{m,l}^{(\mu)}(s)ds + h\int_0^{c_i} R_{m,n}^{(\mu)}(s)ds$. Thus, by (5.5) and (5.7), we have

$$\begin{aligned} &e_h(t_n^{(\mu)}) + h\sum_{j=1}^m a_{ij}\varepsilon_{n,j}^{(\mu)} + h^{m+1}R_{m,n}^{(\mu)}(c_i) \\ &\quad = h(1-c_i)e_h(t_n^{(\mu-1)}) + h^2\sum_{j=1}^m(\gamma_j(1)-b_{ij})\varepsilon_{n,j}^{(\mu-1)} + h\sum_{l=n+1}^{N-1} e_h(t_l^{(\mu-1)}) \\ &\quad\quad + h^2\sum_{l=n+1}^{N-1}\sum_{j=1}^m \gamma_j(1)\varepsilon_{l,j}^{(\mu-1)} + h\sum_{l=0}^{n-1} e_h(t_l^{(\mu)}) + h^2\sum_{l=0}^{n-1}\sum_{j=1}^m \gamma_j(1)\varepsilon_{l,j}^{(\mu)} \\ &\quad\quad + hc_ie_h(t_n^{(\mu)}) + h^2\sum_{j=1}^m b_{ij}\varepsilon_{n,j}^{(\mu)} + h^{m+1}\tilde{R}_{m,n}^{(\mu)}(c_i). \end{aligned} \tag{5.8}$$

Rewriting (5.8) with n replaced by $n-1$ and with $i=m$ and subtract it from (5.8), we can obtain

$$\begin{aligned} &e_h(t_n^{(\mu)}) - e_h(t_{n-1}^{(\mu)}) + h\sum_{j=1}^m a_{ij}\varepsilon_{n,j}^{(\mu)} - h\sum_{j=1}^m a_{mj}\varepsilon_{n-1,j}^{(\mu)} \\ &\quad + h^{m+1}R_{m,n}^{(\mu)}(c_i) - h^{m+1}R_{m,n-1}^{(\mu)}(c_m) \\ &= h(1-c_i)e_h(t_n^{(\mu-1)}) - h(1-c_m)e_h(t_{n-1}^{(\mu-1)}) + h^2\sum_{j=1}^m(\gamma_j(1)-b_{ij})\varepsilon_{n,j}^{(\mu-1)} \\ &\quad - h^2\sum_{j=1}^m(\gamma_j(1)-b_{mj})\varepsilon_{n-1,j}^{(\mu-1)} - he_h(t_n^{(\mu-1)}) - h^2\sum_{j=1}^m\gamma_j(1)\varepsilon_{n,j}^{(\mu-1)} \\ &\quad + he_h(t_{n-1}^{(\mu)}) + h^2\sum_{j=1}^m\gamma_j(1)\varepsilon_{n-1,j}^{(\mu)} + hc_ie_h(t_n^{(\mu)}) - hc_me_h(t_{n-1}^{(\mu)}) \\ &\quad + h^2\sum_{j=1}^m b_{ij}\varepsilon_{n,j}^{(\mu)} - h^2\sum_{j=1}^m b_{mj}\varepsilon_{n-1,j}^{(\mu)} + h^{m+1}\tilde{R}_{m,n}^{(\mu)}(c_i) - h^{m+1}\tilde{R}_{m,n-1}^{(\mu)}(c_m). \end{aligned} \tag{5.9}$$

By (5.5), the continuity of e_h on I, and $e_h(0)=0$, we find by induction

$$\begin{aligned} e_h(t_n^{(\mu)}) &= e_h(t_{n-1}^{(\mu)}+h) = e_h(t_{n-1}^{(\mu)}) + h\sum_{j=1}^m b_j\varepsilon_{n-1,j}^{(\mu)} + h^{m+1}R_{m,n-1}^{(\mu)}(1) \\ &= h\sum_{\nu=1}^{\mu}\sum_{l=0}^{N-1} b^T\varepsilon_l^{(\nu-1)} + h^{m+1}\sum_{\nu=1}^{\mu}\sum_{l=0}^{N-1} R_{m,l}^{(\nu-1)}(1) + h\sum_{l=0}^{n-1} b^T\varepsilon_l^{(\mu)} \\ &\quad + h^{m+1}\sum_{l=0}^{n-1} R_{m,l}^{(\mu)}(1). \end{aligned} \tag{5.10}$$

For $\mu = 0$, it follows from (5.9) and (5.10) that we can write the error equation in the form

$$\begin{aligned} &e\left(b^T\varepsilon_{n-1}^{(0)} + h^m R_{m,n-1}^{(0)}(1)\right) + A\varepsilon_n^{(0)} - ee_m^T A\varepsilon_{n-1}^{(0)} \\ &\quad = (1-c_m)e\left(h\sum_{l=0}^{n-2} b^T\varepsilon_l^{(0)} + h^{m+1}\sum_{l=0}^{n-2} R_{m,l}^{(0)}(1)\right) + he\gamma^T\varepsilon_{n-1}^{(0)} \\ &\qquad + Ce\left(h\sum_{l=0}^{n-1} b^T\varepsilon_l^{(0)} + h^m\sum_{l=0}^{n-1} R_{m,l}^{(0)}(1)\right) + hB\varepsilon_n^{(0)} - hee_m^T B\varepsilon_{n-1}^{(0)} + h^m R_{m,n}^{(0)}, \end{aligned}$$

or

$$\begin{aligned} (A - hB)\,\varepsilon_n^{(0)} &= \left(ee_m^T A - eb^T + he\gamma^T - hee_m^T B\right)\varepsilon_{n-1}^{(0)} \\ &\quad + (1-c_m)eh\sum_{l=0}^{n-2} b^T\varepsilon_l^{(0)} + Ceh\sum_{l=0}^{n-1} b^T\varepsilon_l^{(0)} + h^m\tilde{R}_{m,n}^{(0)}, \end{aligned}$$

with obvious meaning of $\tilde{R}_{m,n}^{(0)}$.

The proof of Theorem 2.1 reveals that $\varepsilon_n^{(0)}$ converges if, and only if

$$-1 \le \rho_m = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \le 1,$$

and that there exist constant $C_1^{(0)}$, such that

$$\|\varepsilon_n^{(0)}\|_1 \le C_1^{(0)} \begin{cases} h^m, & \text{if } -1 \le \rho_m < 1; \\ h^{m-1}, & \text{if } \rho_m = 1. \end{cases} \tag{5.11}$$

Equation (5.8) implies that there exist constants $C_2^{(0)}$ and $C_3^{(0)}$ such that

$$|e_h(t_n^{(0)})| \le hC_2^{(0)}\sum_{l=0}^{n-1}|e_h(t_l^{(0)})| + C_3^{(0)} \begin{cases} h^{m+1}, & \text{if } -1 \le \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1, \end{cases} \tag{5.12}$$

and thus the discrete Gronwall inequality (see [2]) guarantees the existence of a constant $C_4^{(0)}$ for which

$$|e_h(t_n^{(0)})| \le C_4^{(0)} \begin{cases} h^{m+1}, & \text{if } -1 \le \rho_m < 1, \\ h^m, & \text{if } \rho_m = 1, \end{cases} \tag{5.13}$$

holds. Assume that for $\nu = 1, \dots, \mu - 1$, $\varepsilon_n^{(\nu)}$ converges if, and only if

$$-1 \le \rho_m = (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \le 1,$$

and that there exist constants $C_1^{(\nu)}$ and $C_4^{(\nu)}$ such that

$$\|\varepsilon_n^{(\nu)}\|_1 \le C_1^{(\nu)} \begin{cases} h^m, & \text{if } -1 \le \rho_m < 1, \\ h^{m-1}, & \text{if } \rho_m = 1, \end{cases} \tag{5.14}$$

and

$$|e_h(t_n^{(\nu)})| \le C_4^{(\nu)} \begin{cases} h^{m+1}, & \text{if } -1 \le \rho_m < 1; \\ h^m, & \text{if } \rho_m = 1. \end{cases} \tag{5.15}$$

By (5.9) and (5.10), we have

$$\begin{aligned} &eb^T \varepsilon_{n-1}^{(\mu)} + A\varepsilon_n^{(\mu)} - ee_m^T A\varepsilon_{n-1}^{(\mu)} \\ &\quad = (I_m - C)ee_h(t_n^{(\mu-1)}) - (1 - c_m)ee_h(t_{n-1}^{(\mu-1)}) + h(e\gamma^T - B)\varepsilon_n^{(\mu-1)} \\ &\quad\quad - he(\gamma^T - e_m^T B)\varepsilon_{n-1}^{(\mu-1)} - ee_h(t_n^{(\mu-1)}) - he\gamma^T \varepsilon_n^{(\mu-1)} \\ &\quad\quad + (1 - c_m)e\left(h \sum_{\nu=1}^{\mu} \sum_{l=0}^{N-1} b^T \varepsilon_l^{(\nu-1)} + h \sum_{l=0}^{n-2} b^T \varepsilon_l^{(\mu)} \right) + he\gamma^T \varepsilon_{n-1}^{(\mu)} \\ &\quad\quad + Ce\left(h \sum_{\nu=1}^{\mu} \sum_{l=0}^{N-1} b^T \varepsilon_l^{(\nu-1)} + h \sum_{l=0}^{n-1} b^T \varepsilon_l^{(\mu)} \right) + hB\varepsilon_n^{(\mu)} \\ &\quad\quad - hee_m^T B\varepsilon_{n-1}^{(\mu)} + h^m R_{m,n}^{(\mu)}, \end{aligned} \tag{5.16}$$

with obvious meaning of $R_{m,n}^{(\mu)}$.

In the remaining part of the proof we consider the following three cases.

Case I $-1 < \rho_m < 1$

By assumption (5.14) and (5.15), we obtain from (5.16)

$$\begin{aligned} (A - hB)\, \varepsilon_n^{(\mu)} &= \left(ee_m^T A - eb^T + he\gamma^T - hee_m^T B + hCeb^T \right) \varepsilon_{n-1}^{(\mu)} \\ &\quad + h\,((1 - c_m)e + Ce) \sum_{l=0}^{n-2} b^T \varepsilon_l^{(\mu)} + O(h^m). \end{aligned}$$

Proceeding as in the proof of Theorem 2.1 we see that there exists a constant $C_1^{(\mu)}$ such that

$$\|\varepsilon_n^{(\mu)}\|_1 \le C_1^{(\mu)} h^m, \tag{5.17}$$

and hence, by (5.8) and the discrete Gronwall lemma (see [2]), there exist constants $C_2^{(\mu)}$, $C_3^{(\mu)}$ and $C_4^{(\mu)}$ for which

$$|e_h(t_n^{(\mu)})| \le hC_2^{(\mu)} \sum_{l=0}^{n-1} |e_h(t_l^{(\mu)})| + C_3^{(\mu)} h^{m+1}$$

and

$$|e_h(t_n^{(\mu)})| \le C_4^{(\mu)} h^{m+1}$$

are true.

Case II $\rho_m = -1$

Rewriting (5.16) with n replaced by $n-1$ and with $i = m$ and subtract it from (5.16), and by (5.10), we can obtain

$$\begin{aligned}
&eb^T\varepsilon_{n-1}^{(\mu)} - eb^T\varepsilon_{n-2}^{(\mu)} + A\varepsilon_n^{(\mu)} - A\varepsilon_{n-1}^{(\mu)} - ee_m^T A\varepsilon_{n-1}^{(\mu)} + ee_m^T A\varepsilon_{n-2}^{(\mu)} \\
&\quad = (I_m - C)e\left(e_h(t_n^{(\mu-1)}) - e_h(t_{n-1}^{(\mu-1)})\right) - (1-c_m)e\left(e_h(t_{n-1}^{(\mu-1)}) - e_h(t_{n-2}^{(\mu-1)})\right) \\
&\qquad + h(e\gamma^T - B)\left(\varepsilon_n^{(\mu-1)} - \varepsilon_{n-1}^{(\mu-1)}\right) - he(\gamma^T - e_m^T B)\left(\varepsilon_{n-1}^{(\mu-1)} - \varepsilon_{n-2}^{(\mu-1)}\right) \\
&\qquad - e\left(e_h(t_n^{(\mu-1)}) - e_h(t_{n-1}^{(\mu-1)})\right) - he\gamma^T\left(\varepsilon_n^{(\mu-1)} - \varepsilon_{n-1}^{(\mu-1)}\right) \\
&\qquad + h(1-c_m)eb^T\varepsilon_{n-2}^{(\mu)} + he\gamma^T\left(\varepsilon_{n-1}^{(\mu)} - \varepsilon_{n-2}^{(\mu)}\right) + hCeb^T\varepsilon_{n-1}^{(\mu)} \\
&\qquad + hB\left(\varepsilon_n^{(\mu)} - \varepsilon_{n-1}^{(\mu)}\right) - hee_m^T B\left(\varepsilon_{n-1}^{(\mu)} - \varepsilon_{n-2}^{(\mu)}\right) + h^m\left(R_{m,n}^{(\mu)} - R_{m,n-1}^{(\mu)}\right) \\
&\quad = -Ce\left(e_h(t_n^{(\mu-1)}) - e_h(t_{n-1}^{(\mu-1)})\right) - (1-c_m)e\left(e_h(t_{n-1}^{(\mu-1)}) - e_h(t_{n-2}^{(\mu-1)})\right) \\
&\qquad - hB\left(\varepsilon_n^{(\mu-1)} - \varepsilon_{n-1}^{(\mu-1)}\right) - he(\gamma^T - e_m^T B)\left(\varepsilon_{n-1}^{(\mu-1)} - \varepsilon_{n-2}^{(\mu-1)}\right) \\
&\qquad + h(1-c_m)eb^T\varepsilon_{n-2}^{(\mu)} + he\left(\gamma^T - e_m^T B\right)\left(\varepsilon_{n-1}^{(\mu)} - \varepsilon_{n-2}^{(\mu)}\right) \\
&\qquad + hCeb^T\varepsilon_{n-1}^{(\mu)} + hB\left(\varepsilon_n^{(\mu)} - \varepsilon_{n-1}^{(\mu)}\right) + h^m\left(R_{m,n}^{(\mu)} - R_{m,n-1}^{(\mu)}\right) \\
&\quad = -hCeb^T\varepsilon_{n-1}^{(\mu-1)} - h(1-c_m)eb^T\varepsilon_{n-2}^{(\mu-1)} - hB\left(\varepsilon_n^{(\mu-1)} - \varepsilon_{n-1}^{(\mu-1)}\right) \\
&\qquad - he(\gamma^T - e_m^T B)\left(\varepsilon_{n-1}^{(\mu-1)} - \varepsilon_{n-2}^{(\mu-1)}\right) + h(1-c_m)eb^T\varepsilon_{n-2}^{(\mu)} \\
&\qquad + he\left(\gamma^T - e_m^T B\right)\left(\varepsilon_{n-1}^{(\mu)} - \varepsilon_{n-2}^{(\mu)}\right) + hCeb^T\varepsilon_{n-1}^{(\mu)} \\
&\qquad + hB\left(\varepsilon_n^{(\mu)} - \varepsilon_{n-1}^{(\mu)}\right) + h^m\bar{R}_{m,n}^{(\mu)},
\end{aligned} \tag{5.18}$$

with obvious meaning of $\bar{R}^{(\mu)}_{m,n}$. Therefore,

$$\begin{pmatrix} A-hB & 0 \\ 0 & I_m \end{pmatrix}\begin{pmatrix} \varepsilon_n^{(\mu)} \\ \varepsilon_{n-1}^{(\mu)} \end{pmatrix}$$
$$= \begin{pmatrix} A+e\left(e_m^TA-b^T\right) & e\left(b^T-e_m^TA\right) \\ +h\left(e\gamma^T+Ceb^T-B-ee_m^TB\right) & +he\left((1-c_m)b^T-\gamma^T+e_m^TB\right) \\ I_m & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{n-1}^{(\mu)} \\ \varepsilon_{n-2}^{(\mu)} \end{pmatrix}$$
$$+\begin{pmatrix} -hCeb^T\varepsilon_{n-1}^{(\mu-1)}-h(1-c_m)eb^T\varepsilon_{n-2}^{(\mu-1)}-hB\left(\varepsilon_n^{(\mu-1)}-\varepsilon_{n-1}^{(\mu-1)}\right) \\ -he(\gamma^T-e_m^TB)\left(\varepsilon_{n-1}^{(\mu-1)}-\varepsilon_{n-2}^{(\mu-1)}\right)+h^m\bar{R}^{(\mu)}_{m,n} \\ 0 \end{pmatrix}.$$

Obviously, the inverse of the coefficient matrix is $\begin{pmatrix} (A-hB)^{-1} & 0 \\ 0 & I_m \end{pmatrix}$, so that by assumption (5.14) we obtain

$$\begin{pmatrix} \varepsilon_n^{(\mu)} \\ \varepsilon_{n-1}^{(\mu)} \end{pmatrix} = \begin{pmatrix} I_m+A^{-1}e(e_m^TA-b^T)+O(h) & -A^{-1}e(e_m^TA-b^T)+O(h) \\ I_m & 0 \end{pmatrix}\begin{pmatrix} \varepsilon_{n-1}^{(\mu)} \\ \varepsilon_{n-2}^{(\mu)} \end{pmatrix}$$
$$+\begin{pmatrix} h^m\tilde{\bar{R}}^{(\mu)}_{m,n} \\ 0 \end{pmatrix}, \tag{5.19}$$

where the meaning of $\tilde{\bar{R}}^{(\mu)}_{m,n}$ is clear. Since the eigenvalues of the matrix on the right-hand side are $\underbrace{1,1,\ldots,1}_{m};-1,\underbrace{0,\ldots,0}_{m-1}$, we may use an argument similar to the one in Case II for Theorem 2.1 to establish the existence of constants $C_5^{(\mu)}$ and $C_6^{(\mu)}$ such that

$$\|\varepsilon_n^{(\mu)}\|\le C_5^{(\mu)}h^m,\quad |e_h(t_n^{(\mu)})|\le C_6^{(\mu)}h^{m+1}. \tag{5.20}$$

Case III $\rho_m=1$

Using the technique of [7], on $\sigma_n^{(\mu)}:=(t_n^{(\mu)},t_{n+1}^{(\mu)}]$, we write the collocation approximation u_h and the exact solution in the form

$$u_h(t_n^{(\mu)}+sh)=\sum_{j=1}^m L_j(s)u_h(t_{n,j}^{(\mu)})+h^m\frac{u_h^{(m)}(\eta_n^{(\mu)})}{m!}\prod_{i=1}^m(s-c_i), \tag{5.21}$$

and

$$u((t_n^{(\mu)}+sh)=\sum_{j=1}^m L_j(s)u(t_{n,j}^{(\mu)})+h^m\frac{u^{(m)}((\eta_n^{(\mu)})')}{m!}\prod_{i=1}^m(s-c_i), \tag{5.22}$$

where $\eta_n^{(\mu)}, (\eta_n^{(\mu)})' \in (t_n^{(\mu)}, t_{n+1}^{(\mu)})$.

So (5.22)–(5.21) yields

$$e_h((t_n^{(\mu)} + sh) = \sum_{j=1}^{m} L_j(s) e_h(t_{n,j}^{(\mu)}) + h^m \hat{R}_n^{(\mu)}(s), \tag{5.23}$$

where $\hat{R}_n^{(\mu)}(s) := \frac{u^{(m)}((\eta_n^{(\mu)})') - u_h^{(m)}(\eta_n^{(\mu)})}{m!} \prod_{i=1}^{m}(s - c_i)$. Therefore, the proof is completed by resorting to the finial argument in the proof of Theorem 2.1 and the mathematical induction.

6 Numerical examples

In this section, we present two examples to illustrate the foregoing convergence results. For the numerical solution of (1.1) and (1.2), we choose $m = 1, m = 2$ and $m = 3$. For $m = 1$ we use $c_1 = \frac{1}{3}, 0.49, 0.5, 0.8, 1$ respectively, and $\rho_m = -2, -\frac{51}{49}, -1, -\frac{1}{4}, 0$ respectively. For $m = 2$ we use the Gauss collocation parameters, $c_1 = \frac{3-\sqrt{3}}{6}, c_2 = \frac{3+\sqrt{3}}{6}$; the Radau IIA collocation parameters, $c_1 = \frac{1}{3}, c_2 = 1$; and three sets of arbitrary collocation parameters, $c_1 = \frac{1}{2}, c_2 = 1; c_1 = \frac{1}{3}, c_2 = \frac{2}{3}; c_1 = \frac{1}{6}, c_2 = 0.82$ respectively, and $\rho_m = 1, 0, 0, 1, \frac{45}{41}$ respectively. For $m = 3$ we use the Gauss collocation parameters, $c_1 = \frac{5-\sqrt{15}}{10}, c_2 = \frac{1}{2}, c_3 = \frac{5+\sqrt{15}}{10}$; the Radau IIA collocation parameters, $c_1 = \frac{4-\sqrt{6}}{10}, c_2 = \frac{4+\sqrt{6}}{10}, c_3 = 1$; and three sets of arbitrary collocation parameters, $c_1 = \frac{1}{2}, c_2 = \frac{2}{3}, c_3 = 1; c_1 = \frac{1}{3}, c_2 = \frac{1}{2}, c_3 = \frac{2}{3}; c_1 = \frac{1}{4}, c_2 = \frac{1}{2}, c_3 = 0.7$ respectively, and $\rho_m = -1, 0, 0, -1, \frac{9}{7}$ respectively. In Tables 1, 2, 3, 4, 5 and 6 we list the absolute errors for the five collocation parameters and for $m = 1, m = 2$ or $m = 3$.

Example 6.1 In (1.1) let $K(t, s) = e^{t-s}$ and with $g(t)$ such that the exact solution is $u(t) = e^{-t}$.

Table 1 The absolute errors for Example 6.1 with $m = 1$ at $t = 1$

N	$c_1 = \frac{1}{3}$ ($\rho_m = -2$)	$c_1 = 0.49$ ($\rho_m = -\frac{51}{49}$)	$c_1 = 0.5$ ($\rho_m = -1$)	$c_1 = 0.8$ ($\rho_m = -\frac{1}{4}$)	$c_1 = 1$ ($\rho_m = 0$)
2^4	1.6769e+01	−1.1245e−03	−1.3797e−03	−1.1654e−03	−3.8162e−04
2^5	2.7879e+05	−1.5970e−04	−3.4871e−04	−2.9908e−04	−9.5274e−05
2^6	3.0145e+14	1.3160e−04	−8.7586e−05	−7.5737e−05	−2.3810e−05
2^7	1.3950e+33	7.4499e−04	−2.1943e−05	−1.9055e−05	−5.9521e−06
2^8	1.1887e+71	3.2355e−02	−5.4913e−06	−4.7789e−06	−1.4880e−06
2^9	3.4440e+147	2.2707e+02	−1.3735e−06	−1.1966e−06	−3.7200e−07
Ratio	–	–	3.9980e+00	3.9937e+00	4.0000e+00

Table 2 The absolute errors for Example 6.1 with $m = 2$ at $t = 1$

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{2}, 1)$ ($\rho_m = 0$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, 0.82)$ ($\rho_m = \frac{45}{41}$)
2^2	8.7567e−04	−1.5895e−04	1.0229e−05	1.6246e−03	7.2562e−04
2^3	2.2484e−04	−2.0608e−05	5.8522e−07	4.1289e−04	2.4094e−04
2^4	5.6609e−05	−2.6137e−06	3.4833e−08	1.0368e−04	1.0263e−04
2^5	1.4178e−05	−3.2878e−07	2.1220e−09	2.5949e−05	7.9295e−05
2^6	3.5461e−06	−4.1218e−08	1.3090e−10	6.4890e−06	2.3295e−04
2^7	8.8663e−07	−5.1596e−09	8.1272e−12	1.6224e−06	1.2191e−02
Ratio	3.9995e+00	7.9886e+00	1.6106e+01	3.9996e+00	–

Table 3 The absolute errors for Example 6.1 with $m = 3$ at $t = 1$

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{2}, \frac{2}{3}, 1)$ ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$ ($\rho_m = -1$)	$(\frac{1}{4}, \frac{1}{2}, 0.7)$ ($\rho_m = \frac{9}{7}$)
2^2	1.6331e−06	−3.5692e−07	2.8600e−06	6.2657e−08	1.0811e−05
2^3	1.0388e−07	−1.1580e−08	1.8879e−07	−6.6111e−09	2.5177e−06
2^4	6.5215e−09	−3.6737e−10	1.2104e−08	−5.8375e−10	1.3325e−06
2^5	4.0806e−10	−1.1556e−11	7.6582e−10	−3.9167e−11	4.7302e−06
2^6	2.5513e−11	−3.6238e−13	4.8151e−11	−2.4919e−12	9.1983e−04
2^7	1.5855e−12	−1.1047e−14	3.0186e−12	−1.7691e−13	5.5577e+02
Ratio	1.6091e+01	3.2803e+01	1.5951e+01	1.4086e+01	–

Table 4 The absolute errors for Example 6.2 with $m = 1$ at $t = 2$

N	$c_1 = \frac{1}{3}$ ($\rho_m = -2$)	$c_1 = 0.49$ ($\rho_m = -\frac{51}{49}$)	$c_1 = 0.5$ ($\rho_m = -1$)	$c_1 = 0.8$ ($\rho_m = -\frac{1}{4}$)	$c_1 = 1$ ($\rho_m = 0$)
2^4	−2.6353e+05	−4.4259e−03	−4.0549e−03	−5.3670e−03	−8.0719e−03
2^5	−2.7239e+14	−1.3943e−03	−1.0451e−03	−1.3628e−03	−2.0292e−03
2^6	−1.2319e+33	−1.2031e−03	−2.6516e−04	−3.4339e−04	−5.0897e−04
2^7	−1.0377e+71	−3.4330e−02	−6.6776e−05	−8.6191e−05	−1.2747e−04
2^8	−2.9890e+147	−2.3627e+02	−1.6754e−05	−2.1591e−05	−3.1898e−05
Ratio	–	–	3.9857e+00	3.9920e+00	3.9962e+00

Example 6.2 Consider (1.2) with $K(t, s) = e^{t-s}$, $\tau = 1$ and $\phi(t) = 1$, and with $g(t)$ such that the exact solution is $u(t) = \cos t$ for $t \geq 0$.

From Tables 1, 2, 3, 4, 5 and 6, we can see that the numerical results are consistent with our theoretical analysis.

Table 5 The absolute errors for Example 6.2 with $m = 2$ at $t = 2$

N	Gauss ($\rho_m = 1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{2}, 1)$ ($\rho_m = 0$)	$(\frac{1}{3}, \frac{2}{3})$ ($\rho_m = 1$)	$(\frac{1}{6}, 0.82)$ ($\rho_m = \frac{45}{41}$)
2^2	−2.1203e−03	4.2557e−04	−4.2732e−05	−4.6661e−03	−1.4824e−03
2^3	−5.3314e−04	5.5962e−05	−2.6318e−06	−1.1686e−03	−5.3870e−04
2^4	−1.3350e−04	7.1778e−06	−1.6357e−07	−2.9229e−04	−3.0280e−04
2^5	−3.3388e−05	9.0894e−07	−1.0200e−08	−7.3082e−05	−5.1290e−04
2^6	−8.3478e−06	1.1436e−07	−6.3685e−10	−1.8271e−05	−1.3128e−02
2^7	−2.0870e−06	1.4342e−08	−3.9786e−11	−4.5679e−06	−1.2168e+02
Ratio	3.9999e+00	7.9740e+00	1.6007e+01	4.0000e+00	–

Table 6 The absolute errors for Example 6.2 with $m = 3$ at $t = 2$

N	Gauss ($\rho_m = -1$)	Radau IIA ($\rho_m = 0$)	$(\frac{1}{2}, \frac{2}{3}, 1)$ ($\rho_m = 0$)	$(\frac{1}{3}, \frac{1}{2}, \frac{2}{3})$ ($\rho_m = -1$)	$(\frac{1}{4}, \frac{1}{2}, 0.7)$ ($\rho_m = \frac{9}{7}$)
2^2	6.3829e−06	−7.3679e−07	1.1411e−05	−3.8353e−07	3.9902e−05
2^3	3.9579e−07	−2.3465e−08	7.3205e−07	−3.6064e−08	1.8447e−05
2^4	2.4685e−08	−7.4033e−10	4.6325e−08	−2.4297e−09	6.0003e−05
2^5	1.5420e−09	−2.3252e−11	2.9129e−09	−1.5455e−10	1.1480e−02
2^6	9.6366e−11	−7.2664e−13	1.8260e−10	−9.6959e−12	6.9302e+03
Ratio	1.6001e+01	3.1999e+01	1.5952e+01	1.5940e+01	–

In practical applications one will rarely use collocation space $S_m^{(0)}(I_h)$ with $m > 3$, since $m = 3$ yields the global convergence order $p = m + 1 = 4$ and very small absolute errors already for moderately small stepsizes.

7 Concluding remark

As we mentioned in Sect. 1, the main purpose of this paper was to close a gap in previous convergence analyses of continuous piecewise polynomial collocation solutions for second-kind Volterra integral equations. While such globally continuous collocation approximations may occasionally be desirable (for example in VFIEs with non-vanishing delays), their accuracy is in general inferior to the one obtained by using discontinuous piecewise polynomials (at essentially the same computational cost).

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