

The error norm of Clenshaw–Curtis and related quadrature formulae

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Abstract We consider interpolatory quadrature formulae relative to the Legendre weight function $w(t) = 1$ on the interval $[-1, 1]$. On certain spaces of analytic functions the error term of these formulae is a continuous linear functional. We obtain new estimates for the norm of the error functional when the latter does not keep a constant sign at the monomials. Subsequently, the derived estimates are applied into the case of the Clenshaw–Curtis formula, the Basu formula and the Fejér formula of the first kind.

Keywords Error norm · Interpolatory quadrature formulae · Clenshaw–Curtis quadrature formula · Basu quadrature formula · Fejér quadrature formula of the first kind

Mathematics Subject Classification 65D32

1 Introduction

We consider the interpolatory quadrature formula relative to the Legendre weight function $w(t) = 1$ on the interval $[-1, 1]$

$$\int_{-1}^1 f(t)dt = \sum_{v=1}^N w_v f(\tau_v) + R_N(f), \quad (1.1)$$

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where the nodes $\tau_\nu = \tau_\nu^{(N)}$, ordered decreasingly, are all in the interval $[-1, 1]$, and the weights $w_\nu = w_\nu^{(N)}$ are real numbers. Formula (1.1) has degree of exactness d at least $N - 1$, i.e., $R_N(f) = 0$ for all $f \in \mathbb{P}_{N-1}$.

For functions $f \in C^{d+1}[-1, 1]$, the error term of formula (1.1) can be estimated by

$$|R_N(f)| \leq c_d \max_{-1 \leq t \leq 1} |f^{(d+1)}(t)|, \quad c_d = \int_{-1}^1 |K_d(t)| dt, \tag{1.2}$$

where K_d is the d th Peano kernel (cf. [3, Section 4.3]). Estimate (1.2), although frequently quoted, is of limited use. For one reason, higher order derivatives of a function are not readily available, and, even if they are, the resulting error bound cannot be applied to functions of lower-order continuity. Furthermore, estimates like (1.2) do not lend themselves for comparing quadrature formulae with different degrees of exactness.

A more practical estimate can be obtained by means of a Hilbert space method proposed by Hämmerlin in [5]. If f is a single-valued holomorphic function in the disc $C_r = \{z \in \mathbb{C} : |z| < r\}$, $r > 1$, then it can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in C_r.$$

Define

$$|f|_r = \sup \left\{ |a_k| r^k : k \in \mathbb{N}_0 \text{ and } R_N(t^k) \neq 0 \right\}, \tag{1.3}$$

which is a seminorm in the space

$$X_r = \{f : f \text{ holomorphic in } C_r \text{ and } |f|_r < \infty\}. \tag{1.4}$$

Then it can be shown that the error term R_N of formula (1.1) is a continuous linear functional in $(X_r, |\cdot|_r)$, and its norm is given by

$$\|R_N\| = \sum_{k=0}^{\infty} \frac{|R_N(t^k)|}{r^k}. \tag{1.5}$$

If, in addition, for an $\varepsilon \in \{-1, 1\}$,

$$\varepsilon R_N(t^k) \geq 0, \quad k \geq 0, \tag{1.6_i}$$

or

$$\varepsilon (-1)^k R_N(t^k) \geq 0, \quad k \geq 0, \tag{1.6_{ii}}$$

then, letting

$$\pi_N(t) = \prod_{\nu=1}^N (t - \tau_\nu),$$

we can derive the representations

$$\|R_N\| = r \left| \frac{1}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt \right|, \tag{1.7}$$

or

$$\|R_N\| = r \left| \frac{1}{\pi_N(-r)} \int_{-1}^1 \frac{\pi_N(t)}{r+t} dt \right|, \tag{1.7ii}$$

respectively (cf. [11, Section 2]). The error norm can lead to estimates for the error functional itself. If $f \in X_R$, then

$$|R_N(f)| \leq \|R_N\| |f|_r, \quad 1 < r \leq R,$$

which, optimized as a function of r , gives

$$|R_N(f)| \leq \inf_{1 < r \leq R} (\|R_N\| |f|_r). \tag{1.8}$$

Furthermore, if $|f|_r$ is estimated by $\max_{|z|=r} |f(z)|$, which exists at least for $r < R$ (cf. [11, Equation (2.9)]), we get

$$\begin{aligned} |R_N(f)| &\leq \|R_N\| \max_{|z|=r} |f(z)|, \quad 1 < r < R, \\ |R_N(f)| &\leq \inf_{1 < r < R} \left(\|R_N\| \max_{|z|=r} |f(z)| \right). \end{aligned} \tag{1.9}$$

The latter can also be derived by a contour integration technique on circular contours (cf. [4]).

Representations (1.7i) and (1.7ii) have been successfully applied for computing the norm of the error functional for many well-known quadrature formulae, among them the Gauss, Gauss–Lobatto, Gauss–Radau and Gauss–Kronrod quadrature formulae for various weight functions as well as the Fejér formula of the second kind also known as Filippi rule.

However, how do we proceed if the error term of formula (1.1) does not satisfy one of conditions (1.6i) and (1.6ii), in particular, if $R_N(t^k)$ does not keep a constant sign for all $k \geq 0$? This is the case with a number of well-known formulae, such as the Clenshaw–Curtis formula, the Basu formula and the Fejér formula of the first kind also known as Pólya rule. In the present paper, we show that if $R_N(t^k)$ changes sign, only once, at some specific $k = k_N$ (cf. (3.2) and (3.16) below), then $\|R_N\|$ can still be effectively estimated by means of (1.5). We do this in Sect. 3. Our approach is new and general, and in that sense it could likely be applied to interpolatory quadrature formulae whose error term at the monomials changes sign more than once, although this case would require a more delicate handling. In Sect. 4, we apply the estimates derived in Sect. 3 to the Clenshaw–Curtis formula, the Basu formula and the Fejér formula of the first kind. A detailed numerical example, depicting the quality of our bounds, is given in Sect. 5. Our exposition begins in Sect. 2, with some integral formulas for the Chebyshev polynomials of the first and second kind useful in our development.

2 Integral formulas for Chebyshev polynomials

The formulas presented in this section, besides been important in their own right, are also useful in deriving the error norm of Clenshaw–Curtis and Basu formulae.

Throughout this and all subsequent sections, by $[\cdot]$ we denote the integer part of a real number, while the notation \sum' means that the last term in the sum must be halved when n is odd.

Let T_n and U_n be the n th degree Chebyshev polynomials of the first and second kind, respectively, expressed by

$$T_n(\cos \theta) = \cos n\theta, \tag{2.1}$$

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}. \tag{2.2}$$

Both satisfy the three-term recurrence relation

$$p_{k+1}(t) = 2tp_k(t) - p_{k-1}(t), \quad k = 1, 2, \dots, \tag{2.3}$$

where

$$\begin{aligned} p_0(t) &= 1, \quad p_1(t) = t \text{ if } p_n = T_n, \\ p_0(t) &= 1, \quad p_1(t) = 2t \text{ if } p_n = U_n. \end{aligned} \tag{2.4}$$

Proposition 2.1 *Let $r \in \mathbb{R}$ with $|r| > 1$.*

(i) *We have*

$$\begin{aligned} \int_{-1}^1 \frac{(t^2 - 1)T_n(t)}{r - t} dt &= (r^2 - 1)T_n(r) \ln\left(\frac{r+1}{r-1}\right) - 4(r^2 - 1) \sum'_{k=1}^{[(n+1)/2]} \frac{T_{n-2k+1}(r)}{2k-1} \\ &+ \begin{cases} \frac{2r}{n^2-1}, & n \text{ even,} \\ \frac{2}{n^2-4}, & n \text{ odd.} \end{cases} \end{aligned} \tag{2.5}$$

Moreover,

$$\int_{-1}^1 \frac{(t^2 - 1)T_n(t)}{r + t} dt = (-1)^n \int_{-1}^1 \frac{(t^2 - 1)T_n(t)}{r - t} dt. \tag{2.6}$$

(ii) *We have*

$$\begin{aligned} \int_{-1}^1 \frac{(t^2 - 1)U_n(t)}{r - t} dt &= (r^2 - 1)U_n(r) \ln\left(\frac{r+1}{r-1}\right) - 4(r^2 - 1) \sum'_{k=1}^{[(n+1)/2]} \frac{U_{n-2k+1}(r)}{2k-1} \\ &- \begin{cases} \frac{2r}{n+1}, & n \text{ even,} \\ \frac{2(n+1)}{n(n+2)}, & n \text{ odd.} \end{cases} \end{aligned} \tag{2.7}$$

Moreover,

$$\int_{-1}^1 \frac{(t^2 - 1)U_n(t)}{r + t} dt = (-1)^n \int_{-1}^1 \frac{(t^2 - 1)U_n(t)}{r - t} dt. \tag{2.8}$$

Proof (i) Writing

$$\int_{-1}^1 \frac{(t^2 - 1)T_n(t)}{r - t} dt = \int_{-1}^1 \frac{(t^2 - r^2 + r^2 - 1)T_n(t)}{r - t} dt,$$

splitting the integral on the right-hand side in two, and using

$$tT_n(t) = \frac{1}{2}T_{n+1}(t) + \frac{1}{2}T_{n-1}(t)$$

(cf. (2.3) and (2.4)), we get

$$\begin{aligned} \int_{-1}^1 \frac{(t^2 - 1)T_n(t)}{r - t} dt &= (r^2 - 1) \int_{-1}^1 \frac{T_n(t)}{r - t} dt \\ &\quad - \frac{1}{2} \int_{-1}^1 T_{n+1}(t) dt - r \int_{-1}^1 T_n(t) dt - \frac{1}{2} \int_{-1}^1 T_{n-1}(t) dt. \end{aligned} \tag{2.9}$$

Now,

$$\int_{-1}^1 \frac{T_n(t)}{r - t} dt = T_n(r) \ln \left(\frac{r + 1}{r - 1} \right) - 4 \sum_{k=1}^{[(n+1)/2]} \frac{T_{n-2k+1}(r)}{2k - 1}$$

(cf. [10, Proposition 2.2, Equation (2.8)]), and

$$\int_{-1}^1 T_m(t) dt = \begin{cases} -\frac{2}{m^2-1}, & m \text{ even,} \\ 0, & m \text{ odd} \end{cases}$$

(cf. [8, Equation (2.43)]), which, inserted into (2.9), give (2.5).

To prove (2.6), all we have to do is to set $-t$ for t into the integral on the left-hand side and take into account that $T_n(-t) = (-1)^n T_n(t)$ (cf. (2.1)).

(ii) The proof of (2.7) and (2.8) is similar to that of (2.5) and (2.6), respectively, except that here we use

$$\int_{-1}^1 \frac{U_n(t)}{r - t} dt = U_n(r) \ln \left(\frac{r + 1}{r - 1} \right) - 4 \sum_{k=1}^{[(n+1)/2]} \frac{U_{n-2k+1}(r)}{2k - 1}$$

(cf. [10, Proposition 2.2, Equation (2.9)]),

$$\int_{-1}^1 U_m(t) dt = \begin{cases} \frac{2}{m+1}, & m \text{ even,} \\ 0, & m \text{ odd} \end{cases} \tag{2.10}$$

(cf. [8, Equation (2.46)]), and $U_n(-t) = (-1)^n U_n(t)$ (cf. (2.2)). □

3 Estimates for the error norm of interpolatory quadrature formulae

In the present section, we assume that the error term of formula (1.1) does not satisfy one of conditions (1.6_i) and (1.6_{ii}), but instead $R_N(t^k)$ changes sign, once, at some specific $k = k_N$. Then, based on (1.5), we obtain estimates for the error norm, which are both efficient and easy to apply.

Theorem 3.1 *Consider the quadrature formula (1.1) satisfying*

$$\sum_{v=1}^N |w_v| \leq M \tag{3.1}$$

and

$$R_N(t^k) \begin{cases} \geq 0, & 0 \leq k \leq k_N, \\ \leq 0, & k > k_N, \end{cases} \tag{3.2}$$

where $M > 0$ and $k_N = k_N^{(N)}$ are constants. Then

$$\begin{aligned} \|R_N\| &\leq \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt + \frac{2M}{r^{k_N}(r-1)} \\ &\quad - 2r \left\{ \ln \left(\frac{r+1}{r-1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k-1)r^{2k-1}} \right\}. \end{aligned} \tag{3.3}$$

Proof From (1.5), we have, in view of (3.2),

$$\begin{aligned} \|R_N\| &= \sum_{k=0}^{k_N} \frac{R_N(t^k)}{r^k} - \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k} \\ &= \sum_{k=0}^{\infty} \frac{R_N(t^k)}{r^k} - 2 \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k}. \end{aligned} \tag{3.4}$$

The first part on the right-hand side of the last line in (3.4), using the continuity of R_N on $(C[-1, 1], \|\cdot\|_\infty)$, and proceeding as in the proof of Theorem 2.1(a) in [11], computes to

$$\sum_{k=0}^{\infty} \frac{R_N(t^k)}{r^k} = \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt. \tag{3.5}$$

The second part, on the other hand, again by the continuity of R_N on $(C[-1, 1], \|\cdot\|_\infty)$, can be written as

$$- 2 \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k} = -2R_N \left(\sum_{k=k_N+1}^{\infty} \left(\frac{t}{r} \right)^k \right) = -\frac{2}{r^{k_N}} R_N \left(\frac{t^{k_N+1}}{r-t} \right), \tag{3.6}$$

and, setting $f(t) = t^{k_N+1}/(r - t)$ in formula (1.1), (3.6) takes the form

$$-2 \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k} = \frac{2}{r^{k_N}} \left(\sum_{\nu=1}^N w_{\nu} \frac{\tau_{\nu}^{k_N+1}}{r - \tau_{\nu}} - \int_{-1}^1 \frac{t^{k_N+1}}{r - t} dt \right). \tag{3.7}$$

Given that

$$t^{k_N+1} = (r - t) \left(-t^{k_N} - r t^{k_N-1} - \dots - r^{k_N} \right) + r^{k_N+1},$$

the integral on the right-hand side of (3.7) computes to

$$\begin{aligned} \int_{-1}^1 \frac{t^{k_N+1}}{r - t} dt &= - \sum_{k=0}^{k_N} r^{k_N-k} \int_{-1}^1 t^k dt + r^{k_N+1} \int_{-1}^1 \frac{dt}{r - t} \\ &= r^{k_N+1} \left\{ \ln \left(\frac{r + 1}{r - 1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k - 1)r^{2k-1}} \right\}. \end{aligned} \tag{3.8}$$

Moreover, by means of $|\tau_{\nu}| \leq 1, \nu = 1, 2, \dots, N$, and (3.1),

$$\sum_{\nu=1}^N w_{\nu} \frac{\tau_{\nu}^{k_N+1}}{r - \tau_{\nu}} \leq \sum_{\nu=1}^N |w_{\nu}| \frac{|\tau_{\nu}|^{k_N+1}}{r - \tau_{\nu}} \leq \frac{M}{r - 1}, \tag{3.9}$$

which inserted, together with (3.8), into (3.7), gives

$$-2 \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k} \leq \frac{2M}{r^{k_N}(r - 1)} - 2r \left\{ \ln \left(\frac{r + 1}{r - 1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k - 1)r^{2k-1}} \right\}. \tag{3.10}$$

Now, (3.4), together with (3.5) and (3.10), yields (3.3). □

An immediate consequence is the following

Corollary 3.2 (a) Consider the quadrature formula (1.1) having all weights nonnegative and satisfying condition (3.2). Then

$$\begin{aligned} \|R_N\| &\leq \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r - t} dt + \frac{4}{r^{k_N}(r - 1)} \\ &\quad - 2r \left\{ \ln \left(\frac{r + 1}{r - 1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k - 1)r^{2k-1}} \right\}. \end{aligned} \tag{3.11}$$

(b) Consider the quadrature formula (1.1) being symmetric, having all weights nonnegative and satisfying condition (3.2). Then

$$\|R_N\| \leq \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt + \frac{4}{r^{k_N}(r^2-1)} - 2r \left\{ \ln \left(\frac{r+1}{r-1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k-1)r^{2k-1}} \right\}. \tag{3.12}$$

Proof (a) If all weights of formula (1.1) are nonnegative, then

$$\sum_{\nu=1}^N |w_\nu| = \sum_{\nu=1}^N w_\nu = \int_{-1}^1 dt = 2,$$

hence, $M = 2$ (cf. (3.1)), which, in view of (3.3), implies (3.11).

(b) Formula (1.1) is symmetric if

$$\tau_{N-\nu+1} = -\tau_\nu, \quad w_{N-\nu+1} = w_\nu, \quad \nu = 1, 2, \dots, N, \tag{3.13}$$

and as, for N odd, $\tau_{(N+1)/2} = 0$, we have

$$\sum_{\nu=1}^N w_\nu \frac{\tau_\nu^{k_N+1}}{r-\tau_\nu} = \sum_{\nu=1}^{\lfloor N/2 \rfloor} w_\nu \frac{\tau_\nu^{k_N+1}}{r-\tau_\nu} + \sum_{\nu=1}^{\lfloor N/2 \rfloor} w_\nu \frac{(-\tau_\nu)^{k_N+1}}{r+\tau_\nu}. \tag{3.14}$$

Furthermore, by symmetry,

$$R_N(t^{2l-1}) = 0, \quad l \geq 1,$$

consequently, k_N in (3.2) is even, and (3.14), by virtue of $|\tau_\nu| \leq 1, \nu = 1, 2, \dots, N$, takes the form

$$\begin{aligned} \sum_{\nu=1}^N w_\nu \frac{\tau_\nu^{k_N+1}}{r-\tau_\nu} &= 2 \sum_{\nu=1}^{\lfloor N/2 \rfloor} w_\nu \frac{\tau_\nu^{k_N+2}}{r^2-\tau_\nu^2} \\ &\leq 2 \sum_{\nu=1}^{\lfloor N/2 \rfloor} w_\nu \frac{1}{r^2-1} \leq \frac{1}{r^2-1} \sum_{\nu=1}^N w_\nu = \frac{2}{r^2-1}. \end{aligned} \tag{3.15}$$

Now, following the proof of Theorem 3.1, and replacing (3.9) by (3.15), we obtain (3.12). □

A case similar to that of Theorem 3.1 is presented in the following

Theorem 3.3 Consider the quadrature formula (1.1) satisfying condition (3.1) and

$$R_N(t^k) \begin{cases} \leq 0, & 0 \leq k \leq k_N, \\ \geq 0, & k > k_N, \end{cases} \tag{3.16}$$

where $k_N = k_N^{(N)}$ is a constant. Then

$$\begin{aligned} \|R_N\| \leq & 2r \left[\ln \left(\frac{r+1}{r-1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k-1)r^{2k-1}} \right] + \frac{2M}{r^{k_N}(r-1)} \\ & - \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt. \end{aligned} \tag{3.17}$$

Proof From (1.5), we have, in view of (3.16),

$$\begin{aligned} \|R_N\| &= - \sum_{k=0}^{k_N} \frac{R_N(t^k)}{r^k} + \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k} \\ &= - \sum_{k=0}^{\infty} \frac{R_N(t^k)}{r^k} + 2 \sum_{k=k_N+1}^{\infty} \frac{R_N(t^k)}{r^k}. \end{aligned}$$

Then, proceeding as in the proof of Theorem 3.1, we obtain (3.17). □

Also, in a like manner as in Corollary 3.2, we derive the following consequence of Theorem 3.3.

Corollary 3.4 (a) Consider the quadrature formula (1.1) having all weights nonnegative and satisfying condition (3.16). Then

$$\begin{aligned} \|R_N\| \leq & 2r \left[\ln \left(\frac{r+1}{r-1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k-1)r^{2k-1}} \right] \\ & + \frac{4}{r^{k_N}(r-1)} - \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt. \end{aligned} \tag{3.18}$$

(b) Consider the quadrature formula (1.1) being symmetric, having all weights nonnegative and satisfying condition (3.16). Then

$$\begin{aligned} \|R_N\| \leq & 2r \left[\ln \left(\frac{r+1}{r-1} \right) - 2 \sum_{k=1}^{\lfloor k_N/2 \rfloor + 1} \frac{1}{(2k-1)r^{2k-1}} \right] \\ & + \frac{4}{r^{k_N}(r^2-1)} - \frac{r}{\pi_N(r)} \int_{-1}^1 \frac{\pi_N(t)}{r-t} dt. \end{aligned} \tag{3.19}$$

4 Estimates for the error norm of the Clenshaw–Curtis formula, Basu formula and Fejér formula of the first kind

We treat each case separately.

4.1 The Clenshaw–Curtis formula

This is the quadrature formula

$$\int_{-1}^1 f(t)dt = w_0^{*(2)} f(1) + \sum_{\nu=1}^n w_{\nu}^{*(2)} f(\tau_{\nu}^{(2)}) + w_{n+1}^{*(2)} f(-1) + R_n^{*(2)}(f), \tag{4.1}$$

with

$$\tau_{\nu}^{(2)} = \cos \theta_{\nu}^{(2)}, \quad \theta_{\nu}^{(2)} = \frac{\nu}{n+1} \pi, \quad \nu = 1, 2, \dots, n, \tag{4.2}$$

the zeros of the n th degree Chebyshev polynomial of the second kind U_n (cf. (2.2)), and

$$w_{\nu}^{*(2)} = \frac{2}{n+1} \left(1 - 2 \sum_{k=1}^{[(n+1)/2]} \frac{\cos 2k\theta_{\nu}^{(2)}}{4k^2 - 1} \right), \quad \nu = 1, 2, \dots, n, \tag{4.3}$$

$$w_0^{*(2)} = w_{n+1}^{*(2)} = \begin{cases} \frac{1}{(n+1)^2}, & n \text{ even,} \\ \frac{1}{n(n+2)}, & n \text{ odd.} \end{cases}$$

The weights $w_{\nu}^{*(2)}$, $\nu = 1, 2, \dots, n$, are all positive, and formula (4.1) has precise degree of exactness $d = n + 1$ if n is even and $d = n + 2$ if n is odd, i.e., $d = 2[(n + 1)/2] + 1$ (cf. [9]).

In order to obtain an estimate for $\|R_n^{*(2)}\|$, we need to get an assessment for the sign of $R_n^{*(2)}(t^k)$, $k \geq 0$. Our findings are summarized in the following

Lemma 4.1 *The error term of the Clenshaw–Curtis quadrature formula (4.1), when $n \geq 2$, satisfies*

$$R_n^{*(2)}(t^k) \begin{cases} \geq 0, & 0 \leq k \leq k_n^{*(2)}, \\ \leq 0, & k > k_n^{*(2)}, \end{cases} \tag{4.4}$$

where $k_n^{*(2)} > 2[(n + 1)/2] + 2$ is a constant.

For $n = 1$, there holds

$$R_1^{*(2)}(t^k) \leq 0, \quad k \geq 0. \tag{4.5}$$

Proof First of all,

$$R_n^{*(2)}(t^k) = 0, \quad k = 0, 1, \dots, \begin{cases} n + 1, & n \text{ even,} \\ n + 2, & n \text{ odd.} \end{cases} \tag{4.6}$$

Moreover, formula (4.1) is symmetric (cf. (3.13)), hence,

$$R_n^{*(2)}(t^{2l-1}) = 0, \quad l \geq 1. \tag{4.7}$$

Thus, we have to look at the sign of $R_n^{*(2)}(t^{2l})$. Let $n(\text{even}) \geq 2$. Then, by (4.6) and (4.1),

$$R_n^{*(2)}(t^{n+2}) = R_n^{*(2)}\left((t^2 - 1)\frac{1}{2^n}U_n(t)\right) = \frac{1}{2^n} \int_{-1}^1 (t^2 - 1)U_n(t)dt,$$

from which, in view of

$$(t^2 - 1)U_n(t) = \frac{1}{4}U_{n+2}(t) - \frac{1}{2}U_n(t) + \frac{1}{4}U_{n-2}(t)$$

(cf. (2.3) and (2.4) or [8, Equation (2.42)]), we get, by means of (2.10),

$$\begin{aligned} R_n^{*(2)}(t^{n+2}) &= \frac{1}{2^n} \left(\frac{1}{4} \int_{-1}^1 U_{n+2}(t)dt - \frac{1}{2} \int_{-1}^1 U_n(t)dt + \frac{1}{4} \int_{-1}^1 U_{n-2}(t)dt \right) \\ &= \frac{1}{2^{n-2}(n-1)(n+1)(n+3)}. \end{aligned}$$

In a like manner, we obtain, for $n(\text{odd}) \geq 3$,

$$R_n^{*(2)}(t^{n+3}) = \frac{n+1}{2^{n-2}(n-2)n(n+2)(n+4)}.$$

Hence, all together

$$R_n^{*(2)}(t^{2[(n+1)/2]+2}) > 0, \quad n \geq 2. \tag{4.8}$$

Furthermore, setting $f(t) = t^{2l}$ in (4.1), we have

$$R_n^{*(2)}(t^{2l}) = \frac{2}{2l+1} - w_0^{*(2)} - \sum_{\nu=1}^n w_\nu^{*(2)}(\tau_\nu^{(2)})^{2l} - w_{n+1}^{*(2)}, \tag{4.9}$$

and, as $\frac{2}{2l+1}$ is decreasing with l while $w_0^{*(2)} = w_{n+1}^{*(2)} > 0$ are independent of l ,

$$R_n^{*(2)}(t^{2l}) < 0, \quad l > k_n^{*(2)}/2, \tag{4.10}$$

for some constant $k_n^{*(2)} > 2[(n+1)/2] + 2$. Combining (4.6)–(4.8) and (4.10), we obtain (4.4).

For $n = 1$, the Clenshaw–Curtis formula (4.1) is Simpson’s rule on the interval $[-1, 1]$,

$$\int_{-1}^1 f(t)dt = \frac{1}{3}f(1) + \frac{4}{3}f(0) + \frac{1}{3}f(-1) + R_1^{*(2)}(f),$$

with

$$R_1^{*(2)}(f) = -\frac{1}{90}f^{(4)}(\xi), \quad -1 < \xi < 1. \tag{4.11}$$

From (4.11), there follows that

$$R_1^{*(2)}(t^{2l}) \leq 0, \quad l \geq 2,$$

which, combined with (4.6) and (4.7), yields (4.5). □

From (4.9), in view of (4.2) and (4.3), we can compute the precise values of $k_n^{*(2)}$. It suffices to find the highest value of $l = l^{*(2)}$ such that $R_n^{*(2)}(t^{2l^{*(2)}}) > 0$; then $k_n^{*(2)} = 2l^{*(2)}$. The values of $k_n^{*(2)}$, $2 \leq n \leq 40$, are given in Table 1.

Based on the previous lemma, we can derive estimates for $\|R_n^{*(2)}\|$.

Theorem 4.2 Consider the Clenshaw–Curtis quadrature formula (4.1). For $n \geq 2$, we have

$$\begin{aligned} \|R_n^{*(2)}\| \leq & 4r \sum_{k=1}^{\lfloor k_n^{*(2)}/2 \rfloor + 1} \frac{1}{(2k-1)r^{2k-1}} + \frac{4}{r^{k_n^{*(2)}}(r^2-1)} - r \ln \left(\frac{r+1}{r-1} \right) \\ & - \frac{4r}{U_n(r)} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{U_{n-2k+1}(r)}{2k-1} - \begin{cases} \frac{2r^2}{(n+1)(r^2-1)U_n(r)}, & n \text{ even,} \\ \frac{2(n+1)r}{n(n+2)(r^2-1)U_n(r)}, & n \text{ odd.} \end{cases} \end{aligned} \tag{4.12}$$

On the other hand, for $n = 1$, we have

$$\|R_1^{*(2)}\| = \frac{2(3r^2-2)}{3(r^2-1)} - r \ln \left(\frac{r+1}{r-1} \right). \tag{4.13}$$

Proof The Clenshaw–Curtis formula (4.1) is the case of the quadrature formula (1.1) with $N = n + 2$, $\tau_1 = 1$, $\tau_{\nu+1} = \tau_\nu^{(2)}$, $\nu = 1, 2, \dots, n$, and $\tau_{n+2} = -1$.

Table 1 Values of $k_n^{*(2)}$, $2 \leq n \leq 40$

n	$k_n^{*(2)}$	n	$k_n^{*(2)}$	n	$k_n^{*(2)}$	n	$k_n^{*(2)}$
2	6	12	150	22	472	32	974
3	12	13	172	23	514	33	1034
4	22	14	200	24	558	34	1096
5	30	15	226	25	604	35	1160
6	42	16	258	26	652	36	1226
7	54	17	288	27	700	37	1292
8	72	18	322	28	752	38	1362
9	86	19	356	29	804	39	1432
10	108	20	394	30	860	40	1506
11	126	21	432	31	916		

For $n \geq 2$, in view of (4.4), we get, from Eq. (3.12) in Corollary 3.2(b),

$$\|R_n^{*(2)}\| \leq \frac{r}{(r^2 - 1)U_n(r)} \int_{-1}^1 \frac{(t^2 - 1)U_n(t)}{r - t} dt + \frac{4}{r^{k_n^{*(2)}}(r^2 - 1)} - 2r \left\{ \ln \left(\frac{r + 1}{r - 1} \right) - 2 \sum_{k=1}^{\lfloor k_n^{*(2)}/2 \rfloor + 1} \frac{1}{(2k - 1)r^{2k-1}} \right\},$$

and, inserting (2.7), we obtain (4.12).

On the other hand, for $n = 1$, in view of (4.5) (cf. (1.6_i) with $\varepsilon = -1$), we have, from (1.7_i),

$$\|R_1^{*(2)}\| = -\frac{r}{(r^2 - 1)U_1(r)} \int_{-1}^1 \frac{(t^2 - 1)U_1(t)}{r - t} dt,$$

and, again by (2.7), we get (4.13). □

Remark 4.1 The value of $U_m(r)$ in (4.12) can be computed by either the three-term recurrence relation (2.3) and (2.4) or directly as

$$U_m(r) = \frac{1 - \tau^{2m+2}}{2\tau^{m+1}\sqrt{r^2 - 1}}, \quad m \geq 0,$$

where $\tau = r - \sqrt{r^2 - 1}$ (cf. [8, Equation (1.52)]).

4.2 The Basu formula

This is the quadrature formula

$$\int_{-1}^1 f(t) dt = w_0^{*(1)} f(1) + \sum_{\nu=1}^n w_\nu^{*(1)} f(\tau_\nu^{(1)}) + w_{n+1}^{*(1)} f(-1) + R_n^{*(1)}(f), \quad (4.14)$$

with
$$\tau_\nu^{(1)} = \cos \theta_\nu^{(1)}, \quad \theta_\nu^{(1)} = \frac{2\nu - 1}{2n} \pi, \quad \nu = 1, 2, \dots, n, \quad (4.15)$$

the zeros of the n th degree Chebyshev polynomial of the first kind T_n (cf. (2.1)), and

$$w_\nu^{*(1)} = \begin{cases} \frac{2}{n} \left(1 - 2 \sum_{k=1}^{n/2} \frac{\cos 2k\theta_\nu^{(1)}}{4k^2 - 1} + \frac{(-1)^{\nu-1} \cot \theta_\nu^{(1)}}{n^2 - 1} \right), & n \text{ even,} \\ \frac{2}{n} \left(1 - 2 \sum_{k=1}^{(n-1)/2} \frac{\cos 2k\theta_\nu^{(1)}}{4k^2 - 1} + \frac{(-1)^{\nu-1} \csc \theta_\nu^{(1)}}{n^2 - 4} \right), & n \text{ odd,} \end{cases} \quad \nu = 1, 2, \dots, n, \quad (4.16)$$

$$w_0^{*(1)} = w_{n+1}^{*(1)} = \begin{cases} -\frac{1}{n^2 - 1}, & n \text{ even,} \\ -\frac{1}{n^2 - 4}, & n \text{ odd.} \end{cases}$$

The weights $w_v^{*(1)}$, $v = 1, 2, \dots, n$, are all positive, and formula (4.14) has precise degree of exactness $d = n + 1$ if n is even and $d = n + 2$ if n is odd, i.e., $d = 2[(n + 1)/2] + 1$ (cf. [9]).

As in the case of the Clenshaw–Curtis formula, we need an assessment for the sign of $R_n^{*(1)}(t^k)$, $k \geq 0$.

Lemma 4.3 *The error term of the Basu quadrature formula (4.14), when $n \geq 4$, satisfies*

$$R_n^{*(1)}(t^k) \begin{cases} \leq 0, & 0 \leq k \leq k_n^{*(1)}, \\ \geq 0, & k > k_n^{*(1)}, \end{cases} \tag{4.17}$$

where $k_n^{*(1)} > 2[(n + 1)/2] + 2$ is a constant.

For $n = 1, 2$ or 3 , there holds

$$R_1^{*(1)}(t^k) \leq 0, \quad k \geq 0, \tag{4.18}$$

$$R_2^{*(1)}(t^k) \geq 0, \quad k \geq 0, \tag{4.19}$$

$$R_3^{*(1)}(t^k) \geq 0, \quad k \geq 0. \tag{4.20}$$

Proof First of all,

$$R_n^{*(1)}(t^k) = 0, \quad k = 0, 1, \dots, \begin{cases} n + 1, & n \text{ even,} \\ n + 2, & n \text{ odd,} \end{cases} \tag{4.21}$$

and, as formula (4.14) is symmetric (cf. (3.13)),

$$R_n^{*(1)}(t^{2l-1}) = 0, \quad l \geq 1. \tag{4.22}$$

Moreover, proceeding exactly as in the case of the Clenshaw–Curtis formula, we find, for n (even) ≥ 4 ,

$$R_n^{*(1)}(t^{n+2}) = -\frac{3}{2^{n-3}(n^2 - 1)(n^2 - 9)},$$

and, for n (odd) ≥ 5 ,

$$R_n^{*(1)}(t^{n+3}) = -\frac{3}{2^{n-3}(n^2 - 4)(n^2 - 16)},$$

i.e.,

$$R_n^{*(1)}(t^{2[(n+1)/2]+2}) < 0, \quad n \geq 4. \tag{4.23}$$

Furthermore, from (4.14),

$$R_n^{*(1)}(t^{2l}) = \frac{2}{2l + 1} - w_0^{*(1)} - \sum_{v=1}^n w_v^{*(1)}(\tau_v^{(1)})^{2l} - w_{n+1}^{*(1)}, \tag{4.24}$$

and, as $w_0^{*(1)} = w_{n+1}^{*(1)} < 0$ while $|\tau_\nu^{(1)}| < 1$, $\nu = 1, 2, \dots, n$, and therefore $\sum_{\nu=1}^n w_\nu^{*(1)} (\tau_\nu^{(1)})^{2l}$ is decreasing with l ,

$$R_n^{*(1)}(t^{2l}) > 0, \quad l > k_n^{*(1)}/2, \tag{4.25}$$

for some constant $k_n^{*(1)} > 2[(n + 1)/2] + 2$. Now, combining (4.21)–(4.23) and (4.25), we obtain (4.17).

For $n = 1$, the Basu formula (4.14) is Simpson’s rule, hence, (4.18) follows as in Lemma 4.1.

For $n = 2$, formula (4.14) has the form

$$\int_{-1}^1 f(t)dt = -\frac{1}{3}f(1) + \frac{4}{3}f\left(\frac{\sqrt{2}}{2}\right) + \frac{4}{3}f\left(-\frac{\sqrt{2}}{2}\right) - \frac{1}{3}f(-1) + R_2^{*(1)}(f),$$

and, after a simple computation,

$$R_2^{*(1)}(t^{2l}) = 2 \left\{ \frac{1}{2l + 1} + \frac{1}{3} \left(1 - \frac{1}{2^{l-2}} \right) \right\} > 0, \quad l \geq 2,$$

which, combined with (4.21) and (4.22), yields (4.19).

Similarly, for $n = 3$, formula (4.14) is

$$\begin{aligned} \int_{-1}^1 f(t)dt = & -\frac{1}{5}f(1) + \frac{32}{45}f\left(\frac{\sqrt{3}}{2}\right) + \frac{44}{45}f(0) \\ & + \frac{32}{45}f\left(-\frac{\sqrt{3}}{2}\right) - \frac{1}{5}f(-1) + R_3^{*(1)}(f), \end{aligned}$$

hence,

$$R_3^{*(1)}(t^{2l}) = 2 \left\{ \frac{1}{2l + 1} + \frac{1}{5} \left(1 - 2 \left(\frac{3}{4} \right)^{l-2} \right) \right\} > 0, \quad l \geq 3,$$

and, as in the previous case, gives (4.20). □

From (4.24), in view of (4.15) and (4.16), we can compute, in a like manner as in the Clenshaw–Curtis formula, the constants $k_n^{*(1)}$; their values, for $4 \leq n \leq 40$, are given in Table 2.

We are now in a position to derive estimates for $\|R_n^{*(1)}\|$.

Theorem 4.4 Consider the Basu quadrature formula (4.14). For $n \geq 4$, we have

$$\begin{aligned} \|R_n^{*(1)}\| \leq & r \ln \left(\frac{r + 1}{r - 1} \right) - 4r \sum_{k=1}^{\lfloor k_n^{*(1)}/2 \rfloor + 1} \frac{1}{(2k - 1)r^{2k-1}} + \frac{4}{r^{k_n^{*(1)}}(r^2 - 1)} \\ & + \frac{4r}{T_n(r)} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{T_{n-2k+1}(r)}{2k - 1} - \begin{cases} \frac{2r^2}{(n^2-1)(r^2-1)T_n(r)}, & n \text{ even,} \\ \frac{2r}{(n^2-4)(r^2-1)T_n(r)}, & n \text{ odd.} \end{cases} \end{aligned} \tag{4.26}$$

Table 2 Values of $k_n^{*(1)}$, $4 \leq n \leq 40$

n	$k_n^{*(1)}$	n	$k_n^{*(1)}$	n	$k_n^{*(1)}$	n	$k_n^{*(1)}$
4	8	14	120	24	356	34	714
5	12	15	136	25	384	35	754
6	22	16	158	26	416	36	800
7	28	17	176	27	448	37	844
8	38	18	200	28	484	38	892
9	48	19	220	29	518	39	938
10	60	20	246	30	556	40	988
11	72	21	270	31	592		
12	88	22	298	32	632		
13	102	23	324	33	670		

On the other hand, we have, for $n = 1$,

$$\|R_1^{*(1)}\| = \frac{2(3r^2 - 2)}{3(r^2 - 1)} - r \ln \left(\frac{r + 1}{r - 1} \right), \tag{4.27}$$

for $n = 2$,

$$\|R_2^{*(1)}\| = r \ln \left(\frac{r + 1}{r - 1} \right) - \frac{2r^2(6r^2 - 7)}{3(r^2 - 1)(2r^2 - 1)}, \tag{4.28}$$

and, for $n = 3$,

$$\|R_3^{*(1)}\| = r \ln \left(\frac{r + 1}{r - 1} \right) - \frac{2r(60r^4 - 85r^2 + 22)}{15(r^2 - 1)(4r^3 - 3r)}. \tag{4.29}$$

Proof The Basu formula (4.14) is the case of the quadrature formula (1.1) with $N = n + 2$, $\tau_1 = 1$, $\tau_{v+1} = \tau_v^{(1)}$, $v = 1, 2, \dots, n$, and $\tau_{n+2} = -1$.

For $n \geq 4$, in view of (4.17), and in spite of $w_0^{*(1)}$ and $w_{n+1}^{*(1)}$ being negative, a minor modification of Corollary 3.4(b), together with (2.5), yields (4.26).

On the other hand, the case $n = 1$ was treated in Theorem 4.2; while, for $n = 2$ or 3, in view of (4.19) and (4.20) (cf. (1.6_i) with $\varepsilon = 1$), we have, from (1.7_i),

$$\|R_n^{*(1)}\| = \frac{r}{(r^2 - 1)T_n(r)} \int_{-1}^1 \frac{(t^2 - 1)T_n(t)}{r - t} dt,$$

which, by (2.5), gives (4.28) and (4.29). □

Remark 4.2 As for $U_m(r)$ (cf. Remark 4.1), the value of $T_m(r)$ in (4.26) can be computed by either the three-term recurrence relation (2.3) and (2.4) or directly as

$$T_m(r) = \frac{1 + \tau^{2m}}{2\tau^m}, \quad m \geq 0$$

(cf. [8, Equation (1.49)]).

4.3 The Fejér formula of the first kind

This is the quadrature formula

$$\int_{-1}^1 f(t)dt = \sum_{\nu=1}^n w_{\nu}^{(1)} f(\tau_{\nu}^{(1)}) + R_n^{(1)}(f), \tag{4.30}$$

with the $\tau_{\nu}^{(1)}$ given by (4.15), and

$$w_{\nu}^{(1)} = \frac{2}{n} \left(1 - 2 \sum_{k=1}^{[n/2]} \frac{\cos 2k\theta_{\nu}^{(1)}}{4k^2 - 1} \right), \quad \nu = 1, 2, \dots, n. \tag{4.31}$$

The weights $w_{\nu}^{(1)}$, $\nu = 1, 2, \dots, n$, are all positive, and formula (4.30) has precise degree of exactness $d = n - 1$ if n is even and $d = n$ if n is odd, i.e., $d = 2[(n + 1)/2] - 1$ (cf. [9]).

As in the previous two cases, we begin our analysis by examining the sign of $R_n^{(1)}(t^k)$, $k \geq 0$.

Lemma 4.5 *The error term of the Fejér quadrature formula of the first kind (4.30), when $n \geq 2$, satisfies*

$$R_n^{(1)}(t^k) \begin{cases} \leq 0, & 0 \leq k \leq k_n^{(1)}, \\ \geq 0, & k > k_n^{(1)}, \end{cases} \tag{4.32}$$

where $k_n^{(1)} > 2[(n + 1)/2]$ is a constant.

For $n = 1$, there holds

$$R_1^{(1)}(t^k) \geq 0, \quad k \geq 0. \tag{4.33}$$

Proof First of all,

$$R_n^{(1)}(t^k) = 0, \quad k = 0, 1, \dots, \begin{cases} n - 1, & n \text{ even,} \\ n, & n \text{ odd,} \end{cases} \tag{4.34}$$

and, as formula (4.30) is symmetric (cf. (3.13)),

$$R_n^{(1)}(t^{2l-1}) = 0, \quad l \geq 1. \tag{4.35}$$

Also, proceeding as in the case of the Clenshaw–Curtis formula, we find, for $n(\text{even}) \geq 2$,

$$R_n^{(1)}(t^n) = -\frac{1}{2^{n-2}(n^2 - 1)},$$

and, for $n(\text{odd}) \geq 3$,

$$R_n^{(1)}(t^{n+1}) = -\frac{1}{2^{n-2}(n^2 - 4)},$$

i.e.,
$$R_n^{(1)}(t^{2[(n+1)/2]}) < 0, \quad n \geq 2. \tag{4.36}$$

Furthermore, from (4.30),

$$R_n^{(1)}(t^{2l}) = \frac{2}{2l+1} - \sum_{\nu=1}^n w_\nu^{(1)}(\tau_\nu^{(1)})^{2l} = \frac{1}{2l+1} \left\{ 2 - \sum_{\nu=1}^n w_\nu^{(1)}(2l+1)(\tau_\nu^{(1)})^{2l} \right\}, \tag{4.37}$$

and, as $|\tau_\nu^{(1)}| < 1, \nu = 1, 2, \dots, n$, hence,

$$\lim_{l \rightarrow \infty} (2l+1)(\tau_\nu^{(1)})^{2l} = 0, \quad \nu = 1, 2, \dots, n,$$

we get
$$R_n^{(1)}(t^{2l}) > 0, \quad l > k_n^{(1)}/2, \tag{4.38}$$

for some constant $k_n^{(1)} > 2[(n+1)/2]$. Now, combining (4.34)–(4.36) and (4.38), we obtain (4.32).

For $n = 1$, the Fejér formula of the first kind (4.30) is the 1-point Gauss formula for the Legendre weight function $w(t) = 1$ on $[-1, 1]$,

$$\int_{-1}^1 f(t)dt = 2f(0) + R_1^{(1)}(f),$$

with

$$R_1^{(1)}(f) = \frac{1}{3}f''(\xi), \quad -1 < \xi < 1,$$

consequently,

$$R_1^{(1)}(t^{2l}) \geq 0, \quad l \geq 1,$$

which, combined with (4.34) and (4.35), yields (4.33). □

The constants $k_n^{(1)}$ are computed from (4.37), in view of (4.15) and (4.31); their values, for $2 \leq n \leq 40$, are given in Table 3.

We can now derive the estimates for $\|R_n^{(1)}\|$.

Theorem 4.6 *Consider the Fejér quadrature formula of the first kind (4.30). For $n \geq 2$, we have*

$$\begin{aligned} \|R_n^{(1)}\| \leq & r \ln \left(\frac{r+1}{r-1} \right) - 4r \sum_{k=1}^{[k_n^{(1)}/2]+1} \frac{1}{(2k-1)r^{2k-1}} + \frac{4}{r^{k_n^{(1)}}(r^2-1)} \\ & + \frac{4r}{T_n(r)} \sum_{k=1}^{[(n+1)/2]'} \frac{T_{n-2k+1}(r)}{2k-1}. \end{aligned} \tag{4.39}$$

Table 3 Values of $k_n^{(1)}$, $2 \leq n \leq 40$

n	$k_n^{(1)}$	n	$k_n^{(1)}$	n	$k_n^{(1)}$	n	$k_n^{(1)}$
2	4	12	218	22	736	32	1562
3	12	13	256	23	806	33	1660
4	22	14	298	24	878	34	1762
5	36	15	342	25	952	35	1868
6	54	16	390	26	1030	36	1976
7	72	17	438	27	1110	37	2088
8	96	18	492	28	1194	38	2202
9	122	19	548	29	1282	39	2320
10	150	20	608	30	1372	40	2440
11	182	21	670	31	1464		

On the other hand, for $n = 1$, we have

$$\|R_1^{(1)}\| = r \ln \left(\frac{r + 1}{r - 1} \right) - 2. \tag{4.40}$$

Proof The Fejér formula of the first kind (4.30) is the case of the quadrature formula (1.1) with $N = n$ and $\tau_\nu = \tau_\nu^{(1)}$, $\nu = 1, 2, \dots, n$.

For $n \geq 2$, in view of (4.32), Corollary 3.4(b), together with [10, Proposition 2.2, Equation (2.8)], yields (4.39).

On the other hand, for $n = 1$, in view of (4.33) (cf. (1.6_i) with $\varepsilon = 1$), we have, from (1.7_i),

$$\|R_1^{(1)}\| = \frac{r}{T_1(r)} \int_{-1}^1 \frac{T_1(t)}{r - t} dt,$$

which, again by [10, Proposition 2.2, Equation (2.8)], gives (4.40). □

5 A numerical example

We choose the same example as in [10, Section 4], for the reader to be able to make comparisons.

We want to approximate the integral

$$\int_{-1}^1 e^{\omega t} dt = \frac{e^\omega - e^{-\omega}}{\omega}, \quad \omega > 0, \tag{5.1}$$

by using either the Clenshaw–Curtis formula (4.1) or the Fejér formula of the first kind (4.30).

The function $f(z) = e^{\omega z} = \sum_{k=0}^{\infty} \frac{\omega^k z^k}{k!}$ is entire and, in view of (1.3) and (4.7), we have

$$|f|_r^{*(2)} = \begin{cases} \frac{\omega^{2((n+1)/2+1)} r^{2((n+1)/2+1)}}{(2((n+1)/2+1))!}, & 1 < r \leq \frac{\sqrt{(2[(n+1)/2]+3)(2[(n+1)/2]+4)}}{\omega}, \\ \frac{\omega^{2((n+1)/2+k+1)} r^{2((n+1)/2+k+1)}}{(2((n+1)/2+k+1))!}, & \frac{\sqrt{(2[(n+1)/2]+2k+1)(2[(n+1)/2]+2k+2)}}{\omega} < r \leq \frac{\sqrt{(2[(n+1)/2]+2k+3)(2[(n+1)/2]+2k+4)}}{\omega}, \end{cases} \quad k = 1, 2, \dots \tag{5.2}$$

The above formula holds as it stands if $\sqrt{(2[(n+1)/2]+3)(2[(n+1)/2]+4)} > \omega$; otherwise, the formula for $|f|_r^{*(2)}$ starts at the branch of (5.2) for which $\sqrt{(2[(n+1)/2]+2k+3)(2[(n+1)/2]+2k+4)} > \omega$. Similarly,

$$|f|_r^{(1)} = \begin{cases} \frac{\omega^{2[(n+1)/2]} r^{2[(n+1)/2]}}{(2[(n+1)/2])!}, & 1 < r \leq \frac{\sqrt{(2[(n+1)/2]+1)(2[(n+1)/2]+2)}}{\omega}, \\ \frac{\omega^{2((n+1)/2+k)} r^{2((n+1)/2+k)}}{(2((n+1)/2+k))!}, & \frac{\sqrt{(2[(n+1)/2]+2k-1)(2[(n+1)/2]+2k)}}{\omega} < r \leq \frac{\sqrt{(2[(n+1)/2]+2k+1)(2[(n+1)/2]+2k+2)}}{\omega}, \end{cases} \quad k = 1, 2, \dots,$$

under restrictions similar to those in (5.2). Hence, in both cases, $f \in X_{\infty}$ (cf. (1.4)). Moreover,

$$\max_{|z|=r} |f(z)| = e^{\omega r}.$$

Therefore, by (1.8) and (1.9),

$$|R_n^{*(2)}(f)| \leq \inf_{1 < r < \infty} \left(\|R_n^{*(2)}\| |f|_r^{*(2)} \right), \tag{5.3}$$

$$|R_n^{(1)}(f)| \leq \inf_{1 < r < \infty} \left(\|R_n^{(1)}\| |f|_r^{(1)} \right), \tag{5.4}$$

and

$$|R_n^{*(2)}(f)| \leq \inf_{1 < r < \infty} \left(\|R_n^{*(2)}\| e^{\omega r} \right), \tag{5.5}$$

$$|R_n^{(1)}(f)| \leq \inf_{1 < r < \infty} \left(\|R_n^{(1)}\| e^{\omega r} \right), \tag{5.6}$$

with $\|R_n^{*(2)}\|$ and $\|R_n^{(1)}\|$ estimated or given by (4.12) and (4.13) and (4.39) and (4.40), respectively.

It is now worthwhile to see how (5.3) and (5.5) compare with already existing error bounds.

First of all, estimates for $\|R_n^{*(2)}\|$ using Hämmerlin’s method (cf. Sect. 1) have been obtained by Akrivis in his Ph.D. thesis; he did that by either Approximation

Theory techniques or Chebyshev polynomials’ expansions (cf. [1, Sections 1.4–1.5 and 1.7, Equations (1.7.9), (1.4.35) and (1.5.58)–(1.5.59)]). From the three estimates he derived, we choose the one that gives the best results in our case; setting $m = [(n + 1)/2] + 2$ in Equation (1.7.9) in [1] and using Equation (1.5.58), we have

$$\|R_n^{*(2)}\| \leq 2 \left\{ d_m \left(1 - \frac{2m(2m - 5)}{(2m - 3)(2m + 1)} \tau^2 \right) + \frac{4(m + 1)^2}{4(m + 1)^2 - 1} \frac{\tau^3}{\sqrt{r^2 - 1}} \right\} \frac{r\tau^{2m-2}}{\sqrt{r^2 - 1}} + \frac{|R_m^{*(2)}(t^{2m-2})| - |R_m^L(t^{2m-2})|}{r^{2m-2}}, \tag{5.7}$$

where

$$d_m = \frac{2^{4m-4} m(m - 1)^3 ((m - 2)!)^4}{(2m - 1)((2m - 2)!)^2}, \tag{5.8}$$

$\tau = r - \sqrt{r^2 - 1}$ and R_m^L is the error term of the m -point Gauss-Lobatto quadrature formula for the Legendre weight function $w(t) = 1$ on the interval $[-1, 1]$ (with the end nodes ± 1 included in the m points). Then, we apply (5.3) and (5.5) with $\|R_n^{*(2)}\|$ estimated by (5.7) and (5.8).

Also, a few years before Akrivis, a number of error bounds have been derived for functions analytic on circular or elliptic contours.

If f is analytic on the disc C_r (cf. Sect. 1), Jayarajan obtained bounds for the Chebyshev-Fourier coefficients of f , leading, in our case, to the estimate

$$|R_n^{*(2)}(f)| \leq \inf_{1 < r < \infty} \left\{ \frac{64(n + 1)}{(n - 2)n(n + 2)(n + 4)} \frac{r\tau^{n+3}}{\sqrt{r^2 - 1}} e^{\omega r} \right\}, \quad n \text{ odd} \tag{5.9}$$

(cf. [6, Equation (26)]).

If, on the other hand, f is analytic on the ellipse $\mathcal{E}_\rho = \{z \in \mathbb{C} : z = \frac{1}{2}(u + u^{-1}), u = \rho e^{i\theta}, 0 \leq \theta \leq 2\pi\}$ with foci at $z = \pm 1$ and sum of semiaxes ρ , $\rho > 1$, error bounds for $R_n^{*(2)}$ have been given by Chawla (cf. [2]), Kambo (cf. [7]) and Riess and Johnson (cf. [12]). We choose the latter, which is based on Chebyshev polynomials’ expansions, and, together with the estimate of Kambo, gives the best results and it is also very easy to apply. It has the form

$$|R_n^{*(2)}(f)| \leq \inf_{1 < \rho < \infty} \left(2 \left\{ \frac{16(n + 1)}{(n - 2)n(n + 2)(n + 4)} + \frac{1}{\rho^2} + O\left(\frac{1}{\rho^{n+1}}\right) \right\} \frac{1}{\rho^{n+3}} \max_{z \in \mathcal{E}_\rho} |f(z)| \right), \quad n \text{ odd}, \tag{5.10}$$

and, as in our case,

$$\max_{z \in \mathcal{E}_\rho} |f(z)| = e^{\frac{1}{2}\omega(\rho + \rho^{-1})}, \tag{5.11}$$

Table 4 Error bounds and actual error in approximating the integral (5.1) using formula (4.1)

ω	n	r_{opt}	Bound (5.3)	r_{opt}	Bound (5.5)	Error
0.5	5	18.973	7.777(−11)	16.063	5.597(−10)	7.777(−11)
	10	26.981	1.556(−18)	24.030	1.363(−17)	1.556(−18)
1.0	5	9.486	2.060(−8)	8.123	1.502(−7)	2.059(−8)
	10	13.490	6.491(−15)	12.060	5.711(−14)	6.490(−15)
2.0	5	4.743	6.037(−6)	4.226	4.587(−5)	6.009(−6)
	10	6.745	2.859(−11)	6.119	2.559(−10)	2.856(−11)
	15	9.746	1.346(−18)	9.068	1.453(−17)	1.346(−18)
4.0	5	2.372	2.636(−3)	2.359	2.135(−2)	2.469(−3)
	10	3.373	1.585(−7)	3.229	1.489(−6)	1.551(−7)
	15	4.873	4.223(−13)	4.634	4.665(−12)	4.199(−13)
	20	5.873	7.937(−18)	5.611	5.592(−17)	7.911(−18)
8.0	5	1.576	3.689(0)	1.550	3.257(+1)	2.746(0)
	10	1.937	2.178(−3)	1.904	2.205(−2)	1.850(−3)
	15	2.529	2.383(−7)	2.505	2.684(−6)	2.174(−7)
	20	2.937	6.231(−11)	2.963	7.711(−10)	5.902(−11)

we get

$$|R_n^{*(2)}(f)| \leq \inf_{1 < \rho < \infty} \left(2 \left\{ \frac{16(n+1)}{(n-2)n(n+2)(n+4)} + \frac{1}{\rho^2} + O\left(\frac{1}{\rho^{n+1}}\right) \right\} \frac{1}{\rho^{n+3}} e^{\frac{1}{2}\omega(\rho+\rho^{-1})} \right), \quad n \text{ odd} \tag{5.12}$$

(cf. [12, Equation (11)]).

Our results are summarized in Tables 4, 5 and 6 for formula (4.1) and in Table 7 for formula (4.30); in particular, in Table 4, we present the results based on estimates (5.3) and (5.5) with $\|R_n^{*(2)}\|$ estimated by (4.12); in Table 5, we give the corresponding estimates but with $\|R_n^{*(2)}\|$ estimated by (5.7) and (5.8); finally, in Table 6, we present the results based on estimates (5.9) and (5.12). (Numbers in parentheses indicate decimal exponents.) All computations were performed on a SUN Ultra 5 computer in quad precision (machine precision $1.93 \cdot 10^{-34}$). The value of r and ρ , at which the infimum in each of bounds (5.3)–(5.6), (5.9) and (5.12) was attained, is given in the column headed r_{opt} and ρ_{opt} , respectively, which is placed immediately before the column of the corresponding bound. As n and r increase, $\|R_n^{*(2)}\|$ and $\|R_n^{(1)}\|$ decrease and close to machine precision they can even take a negative value. This actually happens, for $\|R_n^{*(2)}\|$ when $\omega = 0.5$ and $n \geq 13$, $\omega = 1.0$ and $n \geq 15$, and $\omega = 2.0$ and $n \geq 19$; and for $\|R_n^{(1)}\|$ when $\omega = 0.5$ and $n \geq 15$, and $\omega = 1.0$ and $n \geq 17$. The reason is that, in all these cases, the infimums in bounds (5.3)–(5.6) are attained at rather high values of $r > 15$.

Bounds (5.3) and (5.4) provide an excellent estimate of the actual error, and they are always better than bounds (5.5) and (5.6), respectively. This is to be expected, as

Table 5 Error bounds and actual error in approximating the integral (5.1) using formula (4.1)

ω	n	r_{opt}	Bound (5.3)	r_{opt}	Bound (5.5)	Error
0.5	5	18.973	8.606(−11)	16.546	6.403(−10)	7.777(−11)
	10	26.982	3.548(−18)	26.312	3.364(−17)	1.556(−18)
1.0	5	9.487	2.920(−8)	8.742	2.289(−7)	2.059(−8)
	10	13.491	3.939(−14)	13.689	3.747(−13)	6.490(−15)
2.0	5	4.744	1.533(−5)	4.724	1.242(−4)	6.009(−6)
	10	6.752	5.866(−10)	6.994	5.536(−9)	2.856(−11)
	15	9.747	2.885(−17)	9.981	3.250(−16)	1.346(−18)
4.0	5	2.872	1.394(−2)	2.575	1.144(−1)	2.469(−3)
	10	3.873	1.026(−5)	3.582	9.949(−5)	1.551(−7)
	15	5.373	3.141(−11)	5.052	3.581(−10)	4.199(−13)
	20	6.373	2.285(−15)	6.045	2.844(−14)	7.911(−18)
8.0	5	1.607	2.867(+1)	1.542	2.535(+2)	2.746(0)
	10	1.937	2.789(−1)	1.946	2.834(0)	1.850(−3)
	15	2.687	4.536(−5)	2.626	5.351(−4)	2.174(−7)
	20	3.187	5.002(−8)	3.101	6.381(−7)	5.902(−11)

Table 6 Error bounds and actual error in approximating the integral (5.1) using formula (4.1)

ω	n	r_{opt}	Bound (5.9)	ρ_{opt}	Bound (5.12)	Error
0.5	5	16.039	1.112(−9)	32.107	5.605(−10)	7.777(−11)
1.0	5	8.078	2.932(−7)	16.206	1.510(−7)	2.059(−8)
2.0	5	4.152	8.423(−5)	8.366	4.678(−5)	6.009(−6)
	15	9.061	2.885(−17)	18.870	2.468(−17)	1.346(−18)
4.0	5	2.284	3.349(−2)	4.543	2.268(−2)	2.469(−3)
	15	4.621	9.086(−12)	9.820	1.661(−11)	4.199(−13)
8.0	5	1.471	4.002(+1)	2.666	3.642(+1)	2.746(0)
	15	2.482	4.858(−6)	5.146	2.472(−5)	2.174(−7)

(5.5) and (5.6) can be derived from (5.3) and (5.4) if $|f|_r^{*(2)}$ and $|f|_r^{(1)}$ are estimated by $\max_{|z|=r} |f(z)|$ (cf. Sect. 1).

Furthermore, bounds (5.3) and (5.5) with $\|R_n^{*(2)}\|$ estimated by (4.12) are better than the corresponding bounds with $\|R_n^{*(2)}\|$ estimated by (5.7) and (5.8), particularly as ω and n increase. Estimate (5.7) was derived by approximating $\|R_n^{*(2)}\|$ by $\|R_m^L\|$ and then adding a correction factor, while (4.12) was obtained by a tailored made process, which apparently explains the difference between the two estimates.

Similarly, bounds (5.3) and (5.5) are better than (5.9) and (5.12), particularly as ω gets large, and this in spite of the fact that (5.12) is based on elliptical contours, which have the advantage of shrinking around the interval $[-1, 1]$ as $\rho \rightarrow 1$. However, (5.10) is reasonably good for large ρ (when the ellipse looks more and more like a circle), and this cannot happen when ω is large, as then $\max_{z \in \mathcal{E}_\rho} |f(z)|$ becomes exceedingly high (cf. (5.11)), which is apparently the reason for (5.3) and (5.5) outperforming (5.12).

Table 7 Error bounds and actual error in approximating the integral (5.1) using formula (4.30)

ω	n	r_{opt}	Bound (5.4)	r_{opt}	Bound (5.6)	Error
0.5	5	14.966	1.309(−7)	12.080	8.206(−7)	1.309(−7)
	10	22.978	1.069(−14)	20.036	8.563(−14)	1.069(−14)
1.0	5	7.483	8.710(−6)	6.155	5.571(−5)	8.705(−6)
	10	11.489	1.117(−11)	10.071	9.006(−11)	1.117(−11)
	15	17.492	2.682(−20)	16.038	2.711(−19)	2.681(−20)
2.0	5	3.741	6.525(−4)	3.277	4.437(−3)	6.471(−4)
	10	5.744	1.242(−8)	5.140	1.025(−7)	1.240(−8)
	15	8.746	1.842(−15)	8.074	1.879(−14)	1.841(−15)
	20	10.744	4.341(−21)	10.058	4.925(−20)	4.341(−21)
4.0	5	1.871	8.036(−2)	1.926	5.829(−1)	7.096(−2)
	10	2.872	1.800(−5)	2.763	1.575(−4)	1.741(−5)
	15	4.373	1.464(−10)	4.146	1.535(−9)	1.453(−10)
	20	5.373	5.337(−15)	5.118	6.175(−14)	5.316(−15)
8.0	5	1.393	3.465(+1)	1.346	2.883(+2)	2.402(+1)
	10	1.687	7.215(−2)	1.698	6.868(−1)	5.807(−2)
	15	2.367	2.190(−5)	2.275	2.351(−4)	1.965(−5)
	20	2.687	1.096(−8)	2.727	1.297(−7)	1.025(−8)

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