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Spectral method for solving high order nonlinear boundary value problems via operational matrices

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Abstract A spectral method based on operational matrices of Bernstein polynomials using collocation method is elaborated and employed for solving nonlinear ordinary and partial differential equations with multi-point boundary conditions. First, properties of Bernstein polynomial, operational matrices of integration, differentiation and product are introduced and then utilized to reduce the given differential equation to the solution of a system of algebraic equations. This new approach provides a significant computational advantage by converting the given original problem to an equivalent integro-differential equation which implies all boundary condition. Approximate solution is achieved by expanding the desired function in terms of a Bernstein basis and employing operational matrices. Unknown coefficients are determined by collocation. The method is compared with modified Adomian decomposition method, Birkhoff-type interpolation method, reproducing kernel Hilbert space method, fixed point method, finite-difference Keller-box method, multilevel augmentation method and shooting method. Illustrative examples are included to demonstrate the high precision, validity and good performance of the new scheme even for solving nonlinear singular differential equations.

Keywords Nonlinear differential equations · Multi-point boundary value problem · Bernstein basis · Operational matrix · Collocation spectral method

Mathematics Subject Classification 34B15 · 34K28 · 35G25 · 65N35 · 78M22

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1 Introduction

In this paper, we aim to exhibit a collocation spectral method for numerically solving the nonlinear boundary value problems (BVPs) using operational matrices of Bernstein polynomials to make a comparison between the proposed method and the some other existing methods. Therefore, we utilize all of the boundary values to derive an equivalent integro-differential equation before applying the scheme to calculate the solution components. Unknown functions in the equivalent integro-differential equation are approximated in terms of Bernstein polynomials. By applying the operational matrices and collocation method, a system of algebraic equations is obtained which is solved by $Mathematica^{TM}$.

As mentioned in [10], for the sake of generality, kth-order nonlinear differential equation is considered as

$$\frac{d^k u(x)}{dx^k} = Nu(x) + g(x), \qquad k \geq 2 \tag{1.1}$$

subject to k boundary conditions

$$u^{(p_0)}(x_0) = \alpha_0 \,, \quad u^{(p_1)}(x_1) = \alpha_1 \,, \ldots, u^{(p_{k-1})}(x_{k-1}) = \alpha_{k-1} \,, \tag{1.2}$$

where $x_0,\ x_1,\ \ldots,\ x_{k-1}$ are not all equal, $0 = p_0 \leq p_1 \leq \cdots \leq p_{k-1} \leq k-1$, $p_j \leq j$ for $j = 1, 2, \ldots, k-1$, $p_i \neq p_j$ if $x_i = x_j$. The field of x in Eq. (1.1) is $\min\{x_0, x_1, \ldots, x_{k-1}\} \leq x \leq \max\{x_0, x_1, \ldots, x_{k-1}\}$. If there are l different points in $x_0,\ x_1,\ \ldots,\ x_{k-1}$ then we treat an l-point BVP. By $p_0 = 0$ we presume that the boundary value problem is in the Dirichlet form. Nu is an analytic nonlinear operator, and $g(x)$ is the system input, which is a given continuous function. We consider the nonlinear BVPs subject to multi-point boundary conditions for at most $k = 8$ and suppose that their solutions exist uniquely [2]. Presented method in this paper can be generalized similarly for appropriate higher order differential equations and partial differential equations [28,29].

We continue in Sect. 2 by presenting the properties of Bernstein polynomials. In Sect. 3, we describe function approximation by using the Bernstein polynomial basis and an upper bound of the approximation error is deduced. We introduce shortly operational matrices of integration, differentiation, dual and product of Bernstein polynomial in Sect. 4. Numerical scheme for the solution of (1.1)–(1.2) is elaborated in Sect. 5. In Sect. 6, we report our numerical findings and make comparisons between the presented method and some other methods. Section 7 consists of a brief conclusion.

2 Properties of Bernstein polynomials

The Bernstein polynomials of mth degree are defined on the interval $[a, b]$ as [27–29]

$$B_{i,m}(x) = \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m}, \quad 0 \leq i \leq m$$

where

$$\binom{m}{i} = \frac{m!}{i!(m-i)!}.$$

These Bernstein polynomials form a basis on [a,b]. In Bernstein polynomials of mth degree, there are $m+1$ mth degree polynomials which this causes to increase the efficiency and performance of the new method in comparison with methods which use of other polynomials. For convenience, we set $B_{i,m}(x) = 0$, if $i < 0$ or $i > m$. A recursive definition can also be used to generate the Bernstein polynomials over $[a, b]$ such that the ith mth-degree Bernstein polynomials can be written

$$B_{i,m}(x) = \frac{(b-x)}{b-a} B_{i,m-1}(x) + \frac{x-a}{b-a} B_{i-1,m-1}(x).$$

It can easily be shown that the Bernstein polynomials are positive, linear independent and the sum of all the Bernstein polynomials is unity for all real $x \in [a, b]$, i.e., $\sum_{i=0}^{m} B_{i,m}(x) = 1$ (it is said that the Bernstein polynomials have the partition unity property). It is easy to show that any given polynomial of degree m can be expanded in terms of these basis functions. It is well known [4,12] that the Bernstein–Vandermonde matrix A is a strictly totally positive matrix when the points satisfy $0 < t_0 < t_1 < \ldots < t_m < 1$ which $A = [a_{i+1,j+1}]$ and $a_{i+1,j+1} = B_{j,m}(t_i)$ for $i, j = 0, \cdots, m$. It is noteworthy that matrix A is used in this method for solving the system of algebraic equations by collocation technique.

3 Approximation of functions

Suppose that $H = L^2[a, b]$ where $a, b \in \mathbb{R}$, let $\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\} \subset H$ is the set of Bernstein polynomials of mth degree and

$$Y = Span\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\}$$

and f is an arbitrary element in H. Since Y is a finite dimensional vector space, f has a unique best approximation out of Y, say $y_0 \in Y$, that is

$$\exists y_0 \in Y;\ \forall y \in Y\ ||f - y_0||_2 \leq ||f - y||_2,$$

where $||f||_2 = \sqrt{\langle f, f \rangle}$ and $\langle f, g \rangle = \int_a^b f_{(t)} g_{(t)} dt$.

In [27], it is shown that the unique coefficient vector $c^T = [c_0, c_1, \ldots, c_m]$ exists such as

$$f \simeq y_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T \phi,$$

where $\phi^T = [B_{0,m}, B_{1,m}, \dots, B_{m,m}]$ and c^T can be obtained

$$c^T = \left(\int_a^b f(x)\phi(x)^T dx\right) Q^{-1},$$

which Q is said dual matrix of ϕ and is defined as

$$Q = \langle \phi \ , \ \phi \rangle = \int_a^b \phi(x)\phi(x)^T dx.$$

Theorem 1 *Suppose that H be a Hilbert space and Y be a closed subspace of H such that $dim \quad Y < \infty$ and $\{y_1, y_2, \dots, y_n\}$ is any basis for Y. Let x be an arbitrary element in H and y_0 be the unique best approximation to x out of Y. Then*

$$||x - y_0||_2^2 = \frac{G(x, y_1, y_2, \dots, y_n)}{G(y_1, y_2, \dots, y_n)},$$

where

$$G(x, y_1, y_2, \dots, y_n) = \begin{vmatrix} \langle x, x \rangle & \langle x, y_1 \rangle & \cdots & \langle x, y_n \rangle \\ \langle y_1, x \rangle & \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ \vdots & \vdots & & \vdots \\ \langle y_n, x \rangle & \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle \end{vmatrix}.$$

Proof [18].
The exact value of approximation error is presented by the Theorem 1 and in the following lemma we present an upper bound of approximation error.

Lemma 1 *Suppose that the function $g : [a, b] \to \mathbb{R}$ be $m + 1$ times continuously differentiable, $g \in C^{m+1}[a, b]$, and $Y = Span\{B_{0,m}, B_{1,m}, \dots, B_{m,m}\}$. If $c^T\phi$ be the best approximation g out of Y then the mean error bound is presented as follows:*

$$\left\|g - c^T\phi\right\|_2 \leq \frac{M(b-a)^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}},$$

where $M = \max_{x\in[a,b]} |g^{(m+1)}(x)|$.

Proof [27].
Lemma 1 shows that the method of approximation converges to f when $m \to \infty$.

4 Operational matrices of Bernstein polynomial

Operational matrices of the integration P, differentiation D, dual Q and product $\widehat{C}$ are introduced as following which the details of obtaining of these matrices are given in [27]

$$\int_0^x \phi(t)dt \simeq P\phi(x), \qquad \frac{d}{dx}\phi(x) \simeq D\phi(x),$$
$$Q = \int_a^b \phi(x)\phi^T(x)dx, \quad c^T\phi(x)\phi(x)^T \simeq \phi(x)^T\widehat{C}.$$

5 Solution of problem

In this section, we convert the main problem to an equivalent integro-differential equation which implies all conditions of main problem. Then, the unknown coefficients are identified by a numerical scheme which will be described in the following.

It should be noticed that:

1. Without loss of generality, we can assume that the given differential equation is defined on interval [0, 1], otherwise we transfer the given interval from $[a, b]$ to [0, 1] (e.g. see Example 3).
2. Since there are many different cases of differential equation with multi-point boundary conditions, therefore we perform our method on the following forth order differential equation (similar to Example 2). This method can be generalized analogously to other second, third, fourth and higher order differential equations.

Consider the following differential equation (similar to Example 2)

$$u^{(4)}(x) = Nu(x) + g(x), \tag{5.1}$$
$$u(0) = A, \quad u''(x_1) = B, \quad u'''(x_2) = E, \quad u(1) = F, \quad 0 \le x_2 < x_1 \le 1 \tag{5.2}$$

which A, B, F, E are scalar values and Nu is an analytic nonlinear operator, and $g(x)$ is the system input, which is a known continuous function. Using integration, we have

$$u^{(3)}(x) = u^{(3)}(0) + \int_0^x u^{(4)}(s)ds, \tag{5.3}$$
$$u^{(2)}(x) = u^{(2)}(0) + xu^{(3)}(0) + \int_0^x \int_0^r u^{(4)}(s)dsdr, \tag{5.4}$$
$$u'(x) = u'(0) + xu^{(2)}(0) + \frac{x^2}{2}u^{(3)}(0) + \int_0^x \int_0^z \int_0^r u^{(4)}(s)dsdrdz,$$
$$u(x) = A + xu'(0) + \frac{x^2}{2}u^{(2)}(0) + \frac{x^3}{6}u^{(3)}(0) + \int_0^x \int_0^n \int_0^z \int_0^r u^{(4)}(s)dsdrdzdn, \tag{5.5}$$

which $u'(0)$, $u^{(2)}(0)$ and $u^{(3)}(0)$ are unknown and are determined by (5.2). From (5.2), (5.3), (5.4) and (5.5), we achieve

$$u^{(3)}(0) = E - \int_0^{x_2} u^{(4)}(s)ds,$$
$$u^{(2)}(0) = B - \left[x_1u^{(3)}(0) + \int_0^{x_1} \int_0^r u^{(4)}(s)dsdr\right],$$

and

$$u'(0) = F - \left[A + \frac{u^{(2)}(0)}{2} + \frac{u^{(3)}(0)}{6} + \int_0^1 \int_0^n \int_0^z \int_0^r u^{(4)}(s) ds dr dz dn \right].$$

Now, an equivalent integro-differential equation which implies all conditions of main problem is obtained by substituting achieved values $u'(0)$, $u^{(2)}(0)$ and $u^{(3)}(0)$ in (5.5), i.e.

$$\begin{aligned} u(x) = A + x \left(F - \left[A + \frac{u^{(2)}(0)}{2} + \frac{u^{(3)}(0)}{6} + \int_0^1 \int_0^n \int_0^z \int_0^r u^{(4)}(s) ds dr dz dn \right] \right) \\ + \frac{x^2}{2} \left(B - \left[x_1 u^{(3)}(0) + \int_0^{x_1} \int_0^r u^{(4)}(s) ds dr \right] \right) \\ + \frac{x^3}{6} \left(E - \int_0^{x_2} u^{(4)}(s) ds \right) + \int_0^x \int_0^n \int_0^z \int_0^r u^{(4)}(s) ds dr dz dn. \end{aligned} \quad (5.6)$$

If we expand the function $u^{(4)}(x)$ in terms of Bernstein polynomial as

$$u^{(4)}(x) \simeq c^T \phi(x) \quad (5.7)$$

and utilize the operational matrix, then we will have

$$\begin{aligned} u^{(3)}(0) &= E - c^T \int_0^{x_2} \phi(s) ds = y_4, \\ u^{(2)}(0) &= B - \left[x_1 y_4 + c^T p \int_0^{x_1} \phi(s) ds \right] = y_3, \\ u'(0) &= F - \left[A + \frac{y_3}{2} + \frac{y_4}{6} + c^T p^3 \int_0^1 \phi(s) ds \right] = y_2. \end{aligned}$$

Replacing them in (5.6) gives

$$u(x) = A + x y_2 + \frac{x^2}{2} y_3 + \frac{x^3}{6} y_4 + c^T p^4 \phi(x) = y_1(x). \quad (5.8)$$

Since the approximate solution (5.8) must satisfy in (5.1), we define the error remainder function $y(x)$ as

$$y(x) = u^{(4)}(x) - N u(x) - g(x)$$

and substitute the (5.7) and (5.8) in $y(x)$

$$y(x) = c^T \phi(x) - N y_1(x) - g(x)$$

and then specify the unknown coefficient c using collocation.

This method can readily generalized to appropriate two-dimensional partial differential equations (see Example 4) and higher order ordinary differential equations by multi-points boundary conditions.

6 Illustrative examples

To demonstrate the validity, application and efficiency of the described method, the obtained results for several examples are presented in this section. In all examples the package of Mathematica (8.0) has been used to solve the test problems.

Example 1 This example is adapted from [21] and studied by modified Adomian decomposition method [10] and also in [5,6]. Consider the two-point BVP for the second-order nonlinear differential equation with an exponential nonlinearity

$$\begin{aligned} &u''(x) = e^{u(x)}, \quad 0 \le x \le 1, \\ &u(0) = 0, \quad u(1) = 0, \end{aligned}$$

which has the exact solution

$$u^*(x) = 2\ln\left(k \sec \frac{k(2x-1)}{4}\right) - \ln(2),$$

where k satisfies $k\sec\left(\frac{k}{4}\right) = \sqrt{2}$, hence, with 16 significant figures $k = 1.336055694906108$. As mentioned in [21], the problem is mildly nonlinear and is easy to solve. In fact, if an algorithm does not work well on this problem, the algorithm should be suspect. Using proposed method, we present the error values $u(x)$ in some points and $||u_m - u^*||_\infty$ for $m = 3,\ 6,\ 9$ in Table 1 to highlight the rapid rate of convergence and plot the error function for $m = 9$ in Fig. 1.

In contrast to the new approach, we note that the maximum approximation error of modified Adomian decomposition method related to $E_3(x)$ in [10] is a multiple of 10^{-3} whereas the maximum approximation error of the presented method is a multiple of 10^{-8} when both methods yield a polynomial of degree 6. Also, the minimal approximation errors of recursion scheme (the multilevel augmentation method) in [5] are a multiple of 10^{-4} and 10^{-8} for norms $||.||_1$ and $||.||_0$, respectively, and the maximum approximation error of the presented method is a multiple of 10^{-10} with $m = 9$ by $||.||_\infty$. Furthermore, the maximum approximation absolute errors of the presented methods in [21] and [6] are a multiple and 10^{-11} and of 10^{-8}, respectively.

We see that the value of m (degree of Bernstein polynomial) can affect the rate of convergence of method. Thus we can use this phenomenon to design more efficient schemes by increasing the value of m. Our method does not depend on any parameter while the method in [10] includes a parameter and the scheme in [5] contains three undetermined coefficients.

Example 2 Consider the four-point BVP for the fourth-order nonlinear differential equation with a product nonlinearity [10,22]

Table 1 Error values of $u(x)$ of method in Example 1

x	m = 3	m = 6	m = 9
0	−0.000513999	5.62176×10^{-9}	-4.65181×10^{-10}
0.1	0.000105718	-2.29883×10^{-8}	-2.05659×10^{-10}
0.2	0.00019385	-6.15119×10^{-9}	1.16237×10^{-10}
0.3	0.0000515635	1.74553×10^{-8}	-1.77296×10^{-11}
0.4	−0.000113844	-1.61038×10^{-8}	-2.07494×10^{-11}
0.5	−0.000181124	-4.45384×10^{-8}	1.60203×10^{-10}
0.6	−0.000110612	-2.16532×10^{-8}	-1.60264×10^{-11}
0.7	0.0000572202	-5.84318×10^{-11}	-1.27651×10^{-11}
0.8	0.000200315	-4.15635×10^{-8}	1.39026×10^{-10}
0.9	0.000110567	-6.79627×10^{-8}	-1.2632×10^{-10}
1	−0.000513999	-5.62176×10^{-9}	-4.65181×10^{-10}
$\|\|u_m - u^*\|\|_\infty$	0.000513999	7.21702×10^{-8}	4.65181×10^{-10}

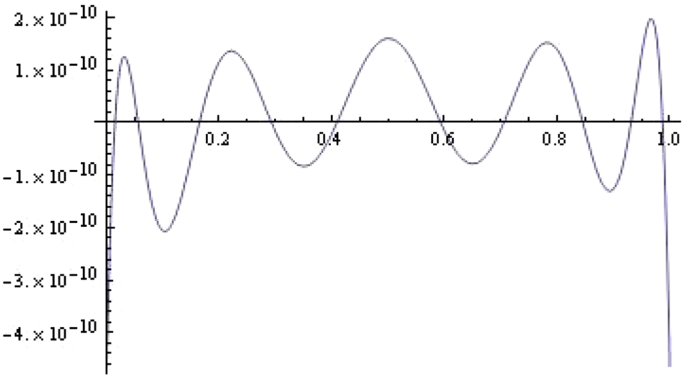

Fig. 1 Error function in Example 1

$$
\begin{aligned}
&u^{(4)}(x) + u(x)u'(x) - 4x^7 - 24 = 0, \quad 0 \le x \le 1 \\
&u(0) = 0, \quad u'''(0.25) = 6, \quad u''(0.5) = 3, \quad u(1) = 1,
\end{aligned}
$$

with the exact solution $u(x) = x^4$. In Fig. 2, we plot the error of $u(x)$ for $m = 6$ and display the error values $u(t)$ for $m = 3,\ 6$ in some points in Table 2.

In contrast to the new method, we note that, in [10] maximum approximation error of modified Adomian decomposition method related to $E_1(x)$ and $E_2(x)$ (which yields polynomials of degree 25 and 39, respectively,) are a multiple of 10^{-4} and 10^{-7}, respectively, but maximum approximation error of the presented method for $m = 6$ (which yields a polynomial of degree 6) is a multiple of 10^{-16} which it shows the high preciseness and good performance of method. We also note that the convergence of the new method does not depend on any parameter while the scheme in [22] contains three undetermined coefficients and the scheme in [10] includes an undetermined parameter.

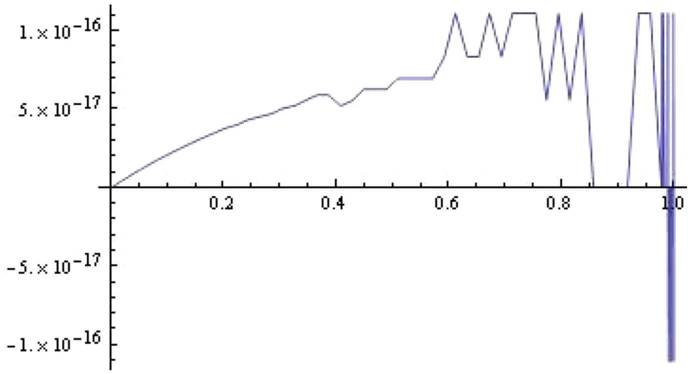

Fig. 2 Error figure of $u(x)$ for $m = 6$ in Example 2

Table 2 Error values of $u(x)$ for $m = 3,\ 6$ in Example 2

x	m = 3	m = 6
0	−0.0142852	-5.42101×10^{-19}
0.1	0.00332894	2.06676×10^{-17}
0.2	0.00582865	3.70797×10^{-17}
0.3	0.00161403	5.0307×10^{-17}
0.4	−0.00331487	5.55112×10^{-17}
0.5	−0.005358	6.93889×10^{-17}
0.6	−0.00331529	8.32667×10^{-17}
0.7	0.00161329	8.32667×10^{-17}
0.8	0.0058278	5.55112×10^{-17}
0.9	0.0033283	1.11022×10^{-17}
1	−0.0142852	0

Example 3 Consider the BVP for the fourth-order nonlinear differential equation with an exponential nonlinearity [10,23]

$$u^{(4)}(x) = -6\,e^{-4u(x)}, \quad 0 \le x \le 4 - e,$$
$$u(0) = 1, \quad u''(0) = -\frac{1}{e^2}, \quad u(4-e) = \ln(4), \quad u''(4-e) = -\frac{1}{16},$$

which has the exact solution $u(x) = \ln(e + x)$.

We first let $x = (4 - e)t$ to transfer the differential equation from the interval $[0, (4 - e)]$ to $[0, 1]$, so the original differential equation is altered to

$$v^{(4)}(t) = -6(4-e)^4\, e^{-4v(t)}, \quad 0 \le t \le 1$$
$$v(0) = 1, \quad v''(0) = -\frac{(4-e)^2}{e^2}, \quad v(1) = \ln(4), \quad v''(1) = -\frac{(4-e)^2}{16},$$

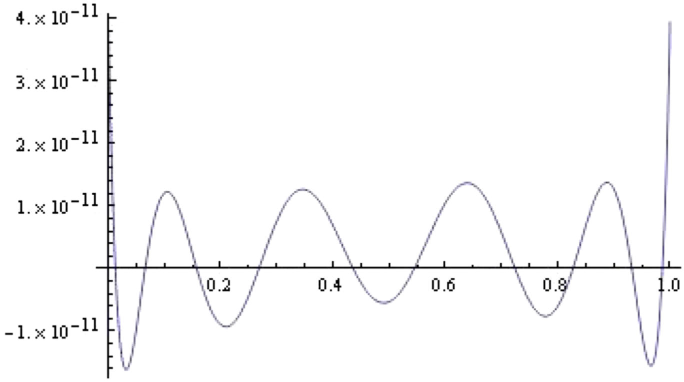

Fig. 3 Error figure of $v(t)$ for $m=9$ in Example 3

Table 3 Error values of $v(t)$ for $m=3,\ 6,\ 9$ in Example 3

x	m = 3	m = 6	m =9
0	0.0000797358	5.28958×10^{-8}	3.94209×10^{-11}
0.1	−0.0000205312	-1.83897×10^{-8}	1.20577×10^{-11}
0.2	−0.0000286405	2.20312×10^{-9}	-8.9484×10^{-12}
0.3	-1.35209×10^{-6}	-2.42924×10^{-8}	7.60036×10^{-12}
0.4	0.0000253848	-4.07791×10^{-8}	6.89115×10^{-12}
0.5	0.0000331912	-2.64039×10^{-8}	-5.29665×10^{-12}
0.6	0.000018626	-1.48458×10^{-8}	1.04627×10^{-11}
0.7	-8.97284×10^{-6}	-3.01546×10^{-8}	5.7121×10^{-12}
0.8	−0.0000292688	-3.95987×10^{-8}	-6.01341×10^{-12}
0.9	−0.000012409	4.75189×10^{-10}	1.27183×10^{-11}
1	0.0000797358	-5.28958×10^{-8}	3.94209×10^{-11}

with the exact solution $v(t)=\ln((4-e)t+e)$ which $v(t)=u((4-e)t)$. In Fig. 3, we plot the error of $v(t)$ for $m=9$ and exhibit the error values $v(t)$ for $m=3,\ 6,\ 9$ in some points in Table 3.

In contrast with the new scheme, we notice that in [10] maximum approximation error of modified Adomian decomposition method related to $E_2(x)$, $E_3(x)$ and $E_4(x)$ (which yields polynomials of degree $8,\ 12,\ 16$, respectively,) are a multiple of 10^{-3}, 10^{-3} and 10^{-4}, but maximum approximation error of the presented method for $m=6$ and $m=9$ (which yields polynomials of degree 6 and 9) are a multiple of 10^{-8} and 10^{-11}. Numerical results in Table 3 demonstrate a remarkable accuracy for such low orders of approximation by the presented method because the error function quickly approaches zero by increasing the value of m. We also notice that the scheme in [23] contains two undetermined coefficients while our method does not depend on any parameter.

Example 4 Consider the nonlinear second-order homogeneous partial differential equation [10,26]

$$\theta_{xx}(x,t)+\frac{\varepsilon}{1+\varepsilon\theta(x,t)}(\theta_x(x,t))^2-K^2\frac{\theta(x,t)}{1+\varepsilon\theta(x,t)}-\frac{\theta_t(x,t)}{1+\varepsilon\theta(x,t)}=0,$$

where K depends on the physical properties and design parameters, and $\theta(x,t)$ has the domain of definition $x\in[0,1]$, $t\in[0,\infty)$ and subject to a mixed set of homogeneous Neumann and inhomogeneous Dirichlet boundary conditions which includes a sinusoidally varying boundary value,

$$\theta_x(0,t)=0,\quad \theta(1,t)=1+S\cos(\beta t).$$

The physical variables and parameters are $\theta, x, t, \varepsilon, K, S$ and β, which represent the dimensionless temperature, distance, time, thermal conductivity parameter, fin parameter, amplitude of oscillation and frequency of oscillation, respectively. The interested reader can refer to [25,26] for further details in regard to the derivation and design limitations of this engineering model.

Exact solution $\theta^*(x,t)$ of this problem is unknown. In order to investigate the approximate solution $\theta_{n,m}(x,t)$ and examine the convergence of method to the exact solution, we consider the error remainder function

$$\begin{aligned}E(\theta_{n,m}(x,t))=&\frac{\partial^2}{\partial x^2}\theta_{n,m}(x,t)+\frac{\varepsilon}{1+\varepsilon\theta_{n,m}(x,t)}\left(\frac{\partial}{\partial x}\theta_{n,m}(x,t)\right)^2\\&-K^2\frac{\theta_{n,m}(x,t)}{1+\varepsilon\theta_{n,m}(x,t)}-\frac{\frac{\partial}{\partial t}\theta_{n,m}(x,t)}{1+\varepsilon\theta_{n,m}(x,t)},\end{aligned}$$

which $\theta_{n,m}(x,t)=\phi_n^T(x)C\phi_m(t)$ where ϕ_m and C are Bernstein polynomials of order m and a $(n+1)\times(m+1)$ matrix, respectively.

For $S=0.1,\ \beta=1,\ K=0.5,\ \varepsilon=0.2$, we plot the error surfaces $E\theta_{n,m}(x,t)$ for $0\le x\le 1$, $0\le t\le 4\pi$ and $n=m=3$ in Fig. 4.

In contrast to the new method, we note that in [10] maximum approximation error of modified Adomian decomposition method is a multiple of 10^{-2}, while maximum approximation error of the presented method for $m=n=3$ is a multiple of 10^{-4} which shows the high accuracy of method. In Figs. 5, 6 and 7, we plot the the error surfaces $E(\theta_{3,3}(x,t))$ on $0\le x\le 1,\ 0\le t\le 4\pi$ for $S=0.1,\ \beta=1,\ K=0.5$ and different values $\varepsilon=0.3, 0, -0.3$, respectively.

Example 5 Consider the nonlinear third-order differential equation [9,16]

$$u'''(x)+(2+u(x))(1-u'(x))^2+(u'''(x))^3=0,\quad 0<x<1$$

with boundary conditions

$$u(0)=0,\quad u''(0)=0,\quad -2u'(1)-(u''(1))^2=0.$$

The existence and uniqueness of the solution for this problem are shown in [8] for $x\in[0,1]$. Applying the presented method with $m=3,6,9$, we plot in Fig. 8 the error remainder function

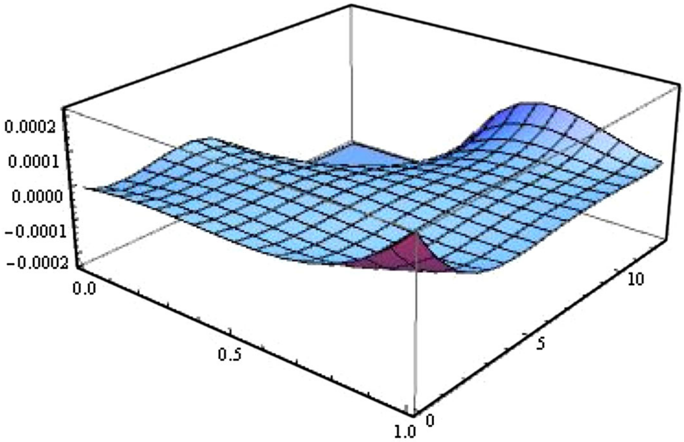

Fig. 4 Error remainder function $E(\theta_{3,3}(x,t))$ for $\varepsilon = 0.2$ in Example 4

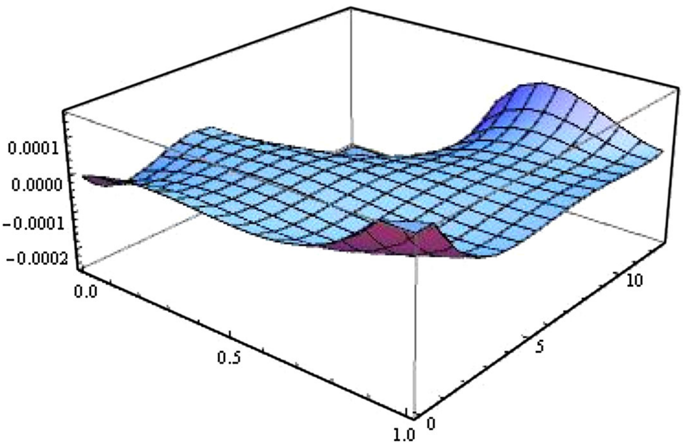

Fig. 5 Error remainder function $E(\theta_{3,3}(x,t))$ for $\varepsilon = 0.3$ in Example 4

$$E_m(x) = u_m'''(x) + (2 + u_m(x))(1 - u_m'(x))^2 + (u_m'''(x))^3$$

for $m = 9$ on [0,1] and list some values of $E_m(x)$ in Table 4. In contrast to the new scheme, we notice that in [9] the maximum approximation error of the Adomian decomposition method is a multiple of 10^{-2}, whereas the maximum approximation error of the presented method for $m = 6$ and $m = 9$ (which yields polynomials of degree 6 and 9) are a multiple of 10^{-4} and 10^{-6}.

Example 6 Consider the nonlinear differential equation for a cantilever nano-electro mechanical system (NEMS) [11,20]

$$u^{(4)}(x) + \frac{\alpha_K}{u(x)^K} + \frac{\beta}{u(x)^2} + \frac{\gamma}{u(x)} = 0, \quad 0 < x < 1, \quad K = 3, 4,$$
$$u(0) = 1, \quad u'(0) = 0, \quad u''(1) = 0, \quad u'''(1) = 0.$$

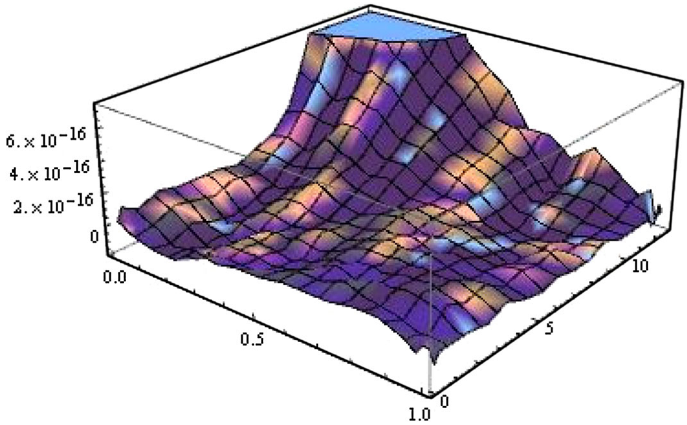

Fig. 6 Error remainder function $E(\theta_{3,3}(x,t))$ for $\varepsilon = 0$ in Example 4

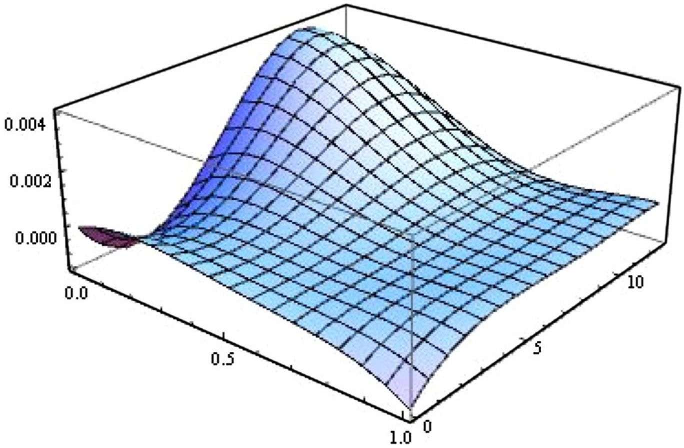

Fig. 7 Error remainder function $E(\theta_{3,3}(x,t))$ for $\varepsilon = -0.3$ in Example 4

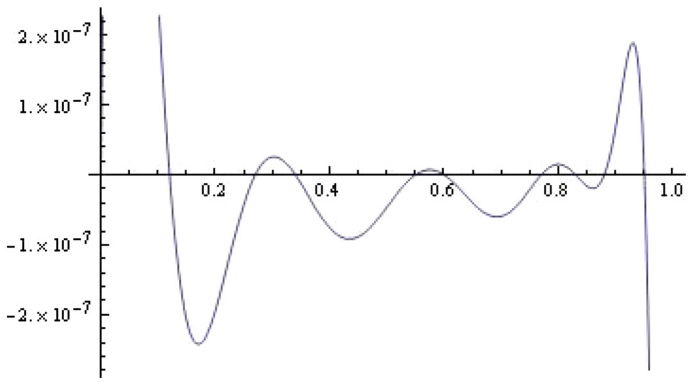

Fig. 8 Error remainder function $E_9(x)$ in Example 5

Table 4 Some values of error remainder function $E_m(x)$ in Example 5

x	m = 3	m = 6	m = 9
0	-3.33067×10^{-16}	4.44089×10^{-16}	-1.11022×10^{-16}
0.1	0.00804987	−0.0000442405	2.60384×10^{-7}
0.2	0.00900237	7.77156×10^{-16}	-1.94849×10^{-7}
0.3	0.0054599	-5.55112×10^{-16}	2.56746×10^{-8}
0.4	5.55112×10^{-16}	1.11022×10^{-16}	-7.52284×10^{-8}
0.5	−0.00461963	0.0000104312	-4.60867×10^{-8}
0.6	−0.00546145	−0.0000327157	1.66533×10^{-16}
0.7	2.22045×10^{-16}	−0.0000327157	-5.88821×10^{-8}
0.8	0.0117656	-2.22045×10^{-16}	1.49767×10^{-8}
0.9	0.0216599	0.000763789	6.87119×10^{-8}
1	4.44089×10^{-16}	-6.66134×10^{-16}	-7.27368×10^{-6}

Table 5 Some values of error remainder function $E_m(x)$ in Example 6

x	m = 3	m = 6	m = 9
0	-1.11022×10^{-16}	0	-1.11022×10^{-16}
0.1	0.00157288	3.50132×10^{-6}	-7.64006×10^{-8}
0.2	0.00159817	-8.35215×10^{-7}	3.81028×10^{-8}
0.3	0.000458466	1.96164×10^{-7}	-2.79425×10^{-9}
0.4	−0.00106802	-9.31966×10^{-7}	2.93434×10^{-9}
0.5	−0.00207336	-1.46916×10^{-6}	-5.20285×10^{-10}
0.6	0.00180238	0	2.22045×10^{-16}
0.7	0	-2.4264×10^{-7}	-3.49627×10^{-9}
0.8	0.0026551	3.09624×10^{-6}	9.70546×10^{-10}
0.9	0.00403533	0.0000161642	4.33997×10^{-9}
1	2.22045×10^{-16}	−0.0000917497	-4.18427×10^{-7}

We take $K = 3$, for the range of separation where the van der Waals force predominates, and $\alpha_K = 0.2,\ \beta = 0.5,\ \gamma = 0.25$ to compute the approximate solution. Applying the presented method with $m = 3,\ 6,\ 9$, we list some values of the error remainder function

$$E_m(x) = u_m^{(4)}(x) + \frac{\alpha_K}{u_m(x)^K} + \frac{\beta}{u_m(x)^2} + \frac{\gamma}{u_m(x)},$$

in Table 5. In contrast to our new approach, in Table 1 from [11] it is expressed that the maximal error remainder parameter for $m = 15$ is 0.0000188629, whereas the error remainder function $E_m(x)$ for $m = 9$ is a multiple of 10^{-7} from Table 5. New method does not depend on any parameter too.

Example 7 Consider the nonlinear differential equation for a double cantilever nanoelectro mechanical system (NEMS) [11,20]

Table 6 Some values of error remainder function $E_m(x)$ in Example 7

x	m = 3	m = 6	m = 9
0	-4.44089×10^{-16}	0	0
0.1	-0.000226047	-0.0000120871	3.50729×10^{-7}
0.2	-0.000192261	3.06549×10^{-6}	-2.4583×10^{-7}
0.3	-0.0000473781	-7.76787×10^{-7}	2.83649×10^{-8}
0.4	0.0000964918	4.14385×10^{-6}	-6.83721×10^{-8}
0.5	0.00016594	8.10947×10^{-6}	-3.26103×10^{-8}
0.6	0.000129179	8.88178×10^{-16}	4.44089×10^{-16}
0.7	0	-4.40265×10^{-6}	-2.23903×10^{-8}
0.8	-0.000157015	5.94871×10^{-6}	4.13628×10^{-9}
0.9	-0.000219751	-2.42261×10^{-6}	1.7564×10^{-8}
1	0	0	0

$$u^{(4)}(x) + \frac{\alpha_K}{u(x)^K} + \frac{\beta}{u(x)^2} + \frac{\gamma}{u(x)} = 0, \quad 0 < x < 1, \quad K = 3, 4,$$
$$u(0) = 1, \quad u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 0.$$

We take $K = 4$, for the range of separation where the Casimir force predominates, and $\alpha_K = 1$, $\beta = 1.5$, $\gamma = 0.5$ to compute the approximate solution. Applying the presented method with $m = 3, 6, 9$, we list in Table 6 some values of the error remainder function

$$E_m(x) = u_m^{(4)}(x) + \frac{\alpha_K}{u_m(x)^K} + \frac{\beta}{u_m(x)^2} + \frac{\gamma}{u_m(x)}.$$

In comparison with the new method, in Table 2 from [11] it is expressed that the maximal error remainder parameter for $m = 3$ which yields a polynomial of degree 12, is 0.0400777, whereas from Table 6 the error remainder function $E_m(x)$ for $m = 9$ (which yields a polynomial of degree 9) is a multiple of 10^{-7}. A clear advantage of this technique over the Adomian decomposition method is that no calculation of Adomians polynomials is needed.

Example 8 In [1,3], it is shown that the unsteady one-dimensional gas flow in a porous medium is modeled by a nonlinear ordinary differential equation as follows

$$y''(x) + \frac{2x}{\sqrt{1 - \alpha y(x)}} y'(x) = 0, \quad x > 0, \quad 0 < \alpha < 1,$$
$$y(0) = 1, \quad \lim_{x \to \infty} y(x) = 0.$$

A substantial amount of numerical and analytical work has been invested so far on this model [19,24]. This problem was also handled by Kidder [17]. Also, finite-difference Keller-box method and shooting method were employed to solve this problem [1]. In

Table 7 Residual values of Eq. 6.1 with $\alpha = 0.2$ for Example 8

t	m = 3	m = 6	m = 9	m = 12	m = 15
0.1	-0.0534597	0.0321189	-1.38778×10^{-17}	4.16334×10^{-17}	-5.20417×10^{-17}
0.2	-0.0767772	0.0139518	5.55112×10^{-17}	0.000122905	-2.77556×10^{-17}
0.3	1.66533×10^{-16}	5.55112×10^{-17}	1.11022×10^{-16}	1.11022×10^{-16}	-5.55112×10^{-17}
0.4	0.128366	-5.55112×10^{-17}	-0.00101015	-4.77933×10^{-6}	3.60791×10^{-6}
0.5	0.213035	0.000831948	0.000671637	6.68188×10^{-7}	5.55112×10^{-17}
0.6	0.174699	6.93889×10^{-18}	-0.00100417	-1.59595×10^{-16}	-6.245×10^{-17}
0.7	-2.77556×10^{-16}	-0.0000768896	0.00349653	-6.23937×10^{-7}	0
0.8	-0.227923	-0.0000624029	-3.33067×10^{-16}	0	1.11022×10^{-16}
0.9	-0.321388	0.000727156	-2.22045×10^{-16}	-3.33067×10^{-16}	0
1	0	0	4.44089×10^{-16}	-4.44089×10^{-16}	0

Table 8 Values of $u(0)$, $u(1)$ and $y'(0)$ with $\alpha = 0.2$ for Example 8

	m = 3	m = 6	m = 9	m = 12	m = 15
$u(0)$	0.0173401	0.00153705	-0.00029644	-0.0000112151	2.13671×10^{-7}
$u(1)$	1	1	1	1	1
$y'(0)$	-0.945112	-1.16978	-1.15863	-1.1501	-1.15024

order to approximate the solution of this problem, we first transfer main problem from interval $0 \le x < \infty$ to $0 < t \le 1$ by transformation function $t = e^{-x}$, thus our problem convert to

$$t^2u''(t) + tu'(t)\left(1 + \frac{2\ln t}{\sqrt{1-\alpha u(t)}}\right) = 0, \quad 0 < t < 1, \quad 0 < \alpha < 1,$$
$$u(1) = 1\,, \quad \lim_{t \to 0} u(t) = 0. \tag{6.1}$$

where $u(t) = y(-\ln t)$. We apply the presented method for problem 6.1 and propose the some residual values of Eq. 6.1 with $\alpha = 0.2,\ 0.5$ in Tables 7 and 9, respectively.

To compare our method with numerical findings presented in [1], values of $u(0)$, $u(1)$ and $y'(0)$ for $\alpha = 0.2,\ 0.5$ are exhibited in Tables 8 and 10, respectively, which $y'(x) = \frac{dy(x)}{dx} = -e^{-x}\frac{du(t)}{dt}|_{t=e^{-x}}$. Numerical results show that the high preciseness of our method and this method can be considered a good scale for comparison with other methods existing in [1].

Example 9 Consider the following high order nonlinear differential equation with multi-point boundary conditions on $[-1, 1]$

$$y^{(8)}(x) - y''(x)y^{(3)}(x) = -e^x(x+5)(11 + x + e^x(1 + 5x + 5x^2 + x^3)),$$
$$y(-1) + y(1) = 0,$$

Table 9 Residual values of Eq. 6.1 with $\alpha = 0.5$ for Example 8

t	m = 3	m = 6	m = 9	m = 12	m = 15
0.1	-0.0534597	0.0289134	1.73472×10^{-17}	-2.42861×10^{-17}	6.93889×10^{-18}
0.2	-0.0767772	0.0121143	1.11022×10^{-16}	0.000104543	0
0.3	1.66533×10^{-16}	5.55112×10^{-17}	5.55112×10^{-17}	0	-5.55112×10^{-17}
0.4	0.128366	-1.11022×10^{-16}	-0.000977605	-7.08453×10^{-6}	3.34621×10^{-6}
0.5	0.213035	0.000502075	0.000634453	7.82314×10^{-7}	5.55112×10^{-17}
0.6	0.174699	2.77556×10^{-17}	-0.000977605	-2.22045×10^{-16}	-9.71445×10^{-17}
0.7	-2.77556×10^{-16}	0.0000552033	0.0030881	-6.21959×10^{-7}	-2.77556×10^{-17}
0.8	-0.227923	-0.000149113	-3.33067×10^{-16}	0	-5.55112×10^{-17}
0.9	-0.321388	0.000903993	-2.22045×10^{-16}	0	-1.11022×10^{-16}
1	0	0	-4.44089×10^{-16}	-2.22045×10^{-16}	0

Table 10 Values of $u(0)$, $u(1)$ and $y'(0)$ with $\alpha = 0.5$ for Example 8

	m = 3	m = 6	m = 9	m = 12	m = 15
$u(0)$	0.0173401	0.00138089	-0.000277258	-0.0000111168	1.80426×10^{-7}
$u(1)$	1	1	1	1	1
$y'(0)$	-0.945112	-1.20923	-1.19967	-1.19143	-1.19157

$$
\begin{aligned}
y^{(1)}(-1) + y^{(1)}(1) &= 2e^{-1} - 2e,\\
y^{(2)}(-1) + y^{(2)}(1) &= 2e^{-1} - 6e,\\
y^{(3)}(-1) + y^{(3)}(1) &= -12e,\\
y^{(4)}(-1) + y^{(4)}(1) &= -4e^{-1} - 20e,\\
y^{(5)}(-1) + y^{(5)}(1) &= -10e^{-1} - 30e,\\
y^{(6)}(-1) + y^{(6)}(1) &= -18e^{-1} - 42e,\\
y^{(7)}(-1) + y^{(7)}(1) &= -28e^{-1} - 56e
\end{aligned}
$$

Analytic solution of the above differential system is $y^*(x) = (1 - x^2)e^x$. Using transformation function $x = 2t - 1$, the problem is altered from interval $x \in [-1, 1]$ to $t \in [0, 1]$ as follows

$$
\begin{aligned}
&\frac{u^{(8)}(t)}{2^8} - \frac{u^{(2)}(t)u^{(3)}(t)}{2^5}\\
&= -4e^{2t-2}(2+t)\left(e(5+t) + 2te^{2t}(-1+2t+2t^2)\right), \quad 0 \le t \le 1\\
&u(-1) + u(1) = 0,\\
&u^{(1)}(0) + u^{(1)}(1) = 4e^{-1} - 4e,\\
&u^{(2)}(0) + u^{(2)}(1) = 8e^{-1} - 24e,\\
&u^{(3)}(0) + u^{(3)}(1) = -96e,
\end{aligned}
$$

Table 11 Absolute error values of $u(t)$ for Example 9

t	m = 3	m = 6	m = 8
0	0.105451	0.000595088	4.78156×10^{-6}
0.1	0.0797359	0.000159328	2.29618×10^{-6}
0.2	0.0350764	0.000165537	1.06322×10^{-6}
0.3	0.0196851	0.0000449246	2.63573×10^{-7}
0.4	0.0754498	0.00026514	1.8448×10^{-7}
0.5	0.123493	0.000321529	1.34234×10^{-6}
0.6	0.156322	0.000307641	2.46215×10^{-6}
0.7	0.168943	0.000377997	1.76328×10^{-6}
0.8	0.160832	0.000438202	1.79086×10^{-6}
0.9	0.139058	0.000211407	1.8511×10^{-6}
1	0.123165	0.000347221	2.62644×10^{-6}
$\|\|u_m - u^*\|\|_\infty$	0.16902	0.000595088	4.78156×10^{-6}

$$
\begin{aligned}
u^{(4)}(0) + u^{(4)}(1) &= -64e^{-1} - 320e,\\
u^{(5)}(0) + u^{(5)}(1) &= -320e^{-1} - 960e,\\
u^{(6)}(0) + u^{(6)}(1) &= -1152e^{-1} - 2688e,\\
u^{(7)}(0) + u^{(7)}(1) &= -3584e^{-1} - 7168e,
\end{aligned}
$$

where $u(t) = y(2t - 1)$ and $\frac{d^k y(x)}{dx^k} = \frac{1}{2^k}\frac{d^k u(t)}{dt^k}$. In Table 11, some values of absolute error of $u(t)$ and $||u_m - u^*||_\infty$ of method are presented. Results show that the method is accurate and rapidly converges to the exact solution by increasing order of Bernstein polynomials.

Example 10 In this numerical test, the proposed method is applied for the following sixth order nonlinear equation [7]

$$
y^{(6)}(x) = e^{-x} y^2(x), \qquad x \in [0, 1],
$$

subject to the initial conditions $y^{(j)}(0) = 1$ for j = 0, 1, 2, 3, 4, 5, whose exact solution is $y^*(x) = e^x$. This example was numerically solved by Birkhoff-type interpolation method in [7]. In order to investigate the performance of our method and compare with [7], numerical results for $m = 3,\ 6,\ 8$ are presented in Table 12 which the results support the efficiency and accuracy of our method.

Example 11 Let us to consider the nonlinear stiff equation [7]

$$
\begin{cases}
y_1'(x) = -(2 + \varepsilon^{-1})y_1(x) + \varepsilon^{-1} y_2(x)^2, & y_1(0) = 1,\\
y_2'(x) = y_1(x) - y_2(x) - y_2(x)^2, & y_2(0) = 1,
\end{cases}
$$

which $\varepsilon = 10^3$. The exact solutions are

$$
y_1^*(x) = e^{-2x}, \qquad y_2^*(x) = e^{-x}.
$$

Table 12 Absolute error values of $u(t)$ for Example 10

t	m = 3	m = 6	m = 8
0	0.000106159	9.62369×10^{-8}	9.41371×10^{-11}
0.1	0.0000115318	2.42432×10^{-8}	1.68736×10^{-11}
0.2	0.0000442877	3.08218×10^{-8}	3.19678×10^{-12}
0.3	0.0000267931	4.02436×10^{-10}	1.86302×10^{-11}
0.4	8.61142×10^{-6}	2.82524×10^{-8}	2.26732×10^{-11}
0.5	0.0000347962	1.86932×10^{-9}	1.28475×10^{-12}
0.6	0.0000340629	2.79955×10^{-8}	2.35025×10^{-11}
0.7	3.86032×10^{-6}	3.18008×10^{-9}	1.68412×10^{-11}
0.8	0.0000358469	3.10703×10^{-8}	5.4694×10^{-12}
0.9	0.0000333831	2.18672×10^{-8}	1.84124×10^{-11}
1	0.000105853	9.61924×10^{-8}	9.40821×10^{-11}
$\|\|y_m - y^*\|\|_\infty$	0.000105853	9.62369×10^{-8}	9.41371×10^{-11}

In Tables 13 and 14, the numerical results with $m = 3,\ 6,\ 9,\ 12$ of using the present method are showed for y_1 and y_2, respectively. In comparison with Birkhoff-type interpolation method in [7], our method gives solutions with higher precision. This high accuracy of results guaranties the good performance of the method.

Example 12 Consider the nonlinear singular initial value problem (IVP) [15]

$$u''(x) + \frac{10}{x\sin(x)}u'(x) + \frac{6\,u(x)}{\cos(\sqrt{x})} + \sin(\sqrt{u(x)}) = f(x), \quad 0 < x \leq 1$$
$$u(0) = 2, \quad u'(0) = 0,$$

which

$$f(x) = x\Big[10(3-4x)x\sin x + \cos\sqrt{x}\,\Big(6(2+x^3-x^4) + \sin x\,(\,6x(1-2x) + \sin\sqrt{2+x^3-x^4}\,)\Big)\Big],$$

and the exact solution is $u^*(x) = x^3 - x^4 + 2$.

In [15], combination of homotopy perturbation method and reproducing kernel Hilbert space method was employed for solving this nonlinear singular initial value problem. Presented method is applied for $m = 3,\ 4$ and the error values in some points are proposed in Table 15 and the relative error function $\left(\left|\frac{u_m(x)-u^*(x)}{u^*(x)}\right|\right)$ for $m = 4$ is plotted in Fig. 9. Approximate solution using Bernstein polynomials of order $m = 4$ is

$$u_4(x) = x^3 - x^4 + 2 - 3.46945 \times 10^{-17}x + 9.36751 \times 10^{-17}x^2.$$

Table 13 Absolute error values of $y_1(t)$ for Example 11

t	m = 3	m = 6	m = 9	m = 12
0	0.00375937	2.82865×10^{-6}	5.76849×10^{-10}	4.72955×10^{-14}
0.1	0.00104388	6.05455×10^{-7}	1.7536×10^{-10}	1.11022×10^{-14}
0.2	0.00140951	9.09535×10^{-7}	1.4502×10^{-10}	1.92069×10^{-14}
0.3	0.000135927	1.43139×10^{-7}	6.71196×10^{-11}	1.66533×10^{-15}
0.4	0.00105819	8.08512×10^{-7}	6.23511×10^{-11}	1.249×10^{-14}
0.5	0.0013381	1.08361×10^{-7}	1.40687×10^{-10}	5.32907×10^{-15}
0.6	0.000592825	8.23244×10^{-7}	8.52606×10^{-11}	1.60982×10^{-15}
0.7	0.000696093	6.46967×10^{-8}	4.34841×10^{-11}	1.06859×10^{-14}
0.8	0.00156197	8.93715×10^{-7}	1.34435×10^{-10}	3.747×10^{-15}
0.9	0.000640929	7.4555×10^{-7}	1.69992×10^{-10}	2.63678×10^{-15}
1	0.0037561	2.82836×10^{-6}	5.76809×10^{-10}	4.50751×10^{-14}
$\|\|y_{1_m} - y_{1^*}\|\|_\infty$	0.00375937	2.82865×10^{-6}	5.76849×10^{-10}	4.73437×10^{-14}

Table 14 Absolute error values of $y_2(t)$ for Example 11

t	m = 3	m = 6	m = 9	m = 12
0	0.000380749	6.72605×10^{-8}	7.7236×10^{-12}	3.10862×10^{-15}
0.1	0.000114529	1.10369×10^{-7}	1.5547×10^{-10}	1.77525×10^{-13}
0.2	0.000130775	1.5991×10^{-7}	1.34249×10^{-10}	1.4766×10^{-13}
0.3	0.000219129	1.10979×10^{-7}	9.40644×10^{-11}	1.16351×10^{-13}
0.4	0.000251242	3.96129×10^{-8}	8.4298×10^{-11}	9.10383×10^{-14}
0.5	0.000172968	9.74315×10^{-9}	6.23439×10^{-11}	7.41629×10^{-14}
0.6	2.69114×10^{-6}	2.43575×10^{-8}	4.3533×10^{-11}	5.973×10^{-14}
0.7	0.000201982	5.01974×10^{-8}	4.6628×10^{-11}	4.90719×10^{-14}
0.8	0.000296177	5.20433×10^{-8}	2.97835×10^{-11}	4.10783×10^{-14}
0.9	0.000106808	2.12784×10^{-8}	6.3155×10^{-12}	3.45279×10^{-14}
1	0.0005896	3.51521×10^{-7}	5.43202×10^{-11}	3.17524×10^{-14}
$\|\|y_{2_m} - y_{2^*}\|\|_\infty$	0.0005896	3.51521×10^{-7}	1.6367×10^{-10}	1.8003×10^{-13}

In contrast to the new method, we note that in [15] the minimum approximation relative error is a multiple of 10^{-7}, while the maximum approximation relative error of the present method with $m = 4$ is a multiple of 10^{-16} which it shows the good performance of method in comparison with other methods.

Example 13 Consider the nonlinear stiff equation [13]

$$\begin{cases} y_1''(x) = y_1(x)y_2'(x) - x, & y_1(0) = 1, \quad y_2(0) = 0 \\ y_2''(x) = y_2^2(x) - y_1(x)y_2(x) + xe^x, & y_1'(0) = 2, \; y_2'(0) = 1 \end{cases}$$

whose exact solutions are $y_1^*(x) = x + e^x, \qquad y_2^*(x) = x.$

Table 15 Error values of $u(x)$ for $m = 3,\ 4$ in Example 12

x	m = 3	m = 4
0	0.0144406	0
0.1	−0.00311637	0
0.2	−0.00556828	0
0.3	−0.00131791	0
0.4	0.00363197	0
0.5	0.00567859	0
0.6	0.00361915	0
0.7	−0.00134911	0
0.8	−0.00562898	0
0.9	−0.00322324	0
1	0.0142653	0

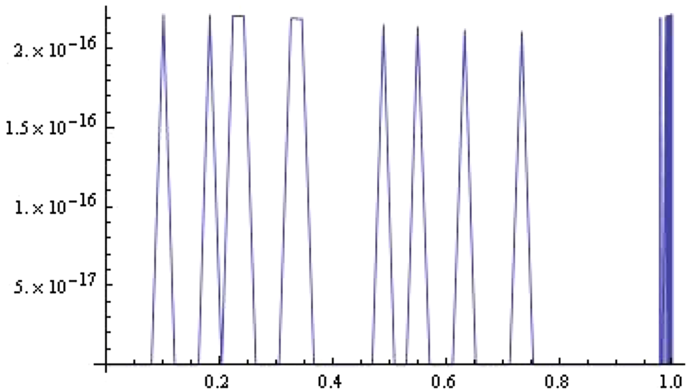

Fig. 9 Relative error function for $m = 4$ in Example 12

In [13], this problem was solved using a method which is based on the Banach fixed point theorem. Achieved minimum absolute error in [13] for $y_1,\ y_2$ are a multiple of 10^{-7} and 10^{-11}, respectively, with $n = 33, m = 5$ (which n, m indicate the numbers of nodes and of iterations). In Tables 16 and 17, the numerical results with $m = 3,\ 6,\ 9$ of using the present method are exhibited for y_1 and y_2, respectively. From numerical findings, maximum absolute error $y_1,\ y_2$ are a multiple of 10^{-12} and 10^{-14}, respectively, with $m = 9$ (which m indicates Bernstein polynomial order).

Example 14 Consider the following nonlinear fourth-order boundary value problem [14]

$$u^{(4)}(x) - e^x u''(x) + u(x) + \sin(u(x)) = 1 + \sin(1 + \sinh(x)) - (e^x - 2)\sinh(x)$$
$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1 + \sinh(1), \quad u'(1) = \cosh(1),$$

whose exact solution is $u^*(x) = 1 + \sinh(x)$. This BVP has been solved in [14] by the reproducing kernel Hilbert space method which its maximum absolute error is a mul-

Table 16 Absolute error values of $y_1(t)$ for Example 13

t	m = 3	m = 6	m = 9
0	0.000999768	9.62487×10^{-8}	2.47546×10^{-12}
0.1	0.000203671	2.42446×10^{-8}	7.37854×10^{-13}
0.2	0.000416156	3.08378×10^{-8}	5.90639×10^{-13}
0.3	0.000152583	4.28627×10^{-10}	2.12941×10^{-13}
0.4	0.000194499	2.82128×10^{-8}	3.40172×10^{-13}
0.5	0.000367751	1.79385×10^{-9}	6.15508×10^{-13}
0.6	0.000259265	2.8135×10^{-8}	2.80442×10^{-13}
0.7	0.0000737182	3.39824×10^{-9}	2.43805×10^{-13}
0.8	0.000391444	3.07798×10^{-8}	6.44373×10^{-13}
0.9	0.000252446	2.22242×10^{-8}	7.07878×10^{-13}
1	0.00100767	9.57076×10^{-8}	2.42295×10^{-12}

Table 17 Absolute error values of $y_2(t)$ for Example 13

t	m = 3	m = 6	m = 9
0	3.18746×10^{-6}	2.4232×10^{-11}	1.40799×10^{-15}
0.1	3.90853×10^{-7}	4.65937×10^{-11}	3.91354×10^{-15}
0.2	2.05134×10^{-6}	8.65468×10^{-11}	1.24623×10^{-14}
0.3	2.13112×10^{-6}	1.01278×10^{-10}	2.04836×10^{-14}
0.4	9.67303×10^{-7}	1.51306×10^{-10}	2.93099×10^{-14}
0.5	1.103×10^{-6}	2.59996×10^{-10}	3.71925×10^{-14}
0.6	3.74268×10^{-6}	3.82077×10^{-10}	4.36318×10^{-14}
0.7	6.61462×10^{-6}	4.32679×10^{-10}	5.05151×10^{-14}
0.8	9.3817×10^{-6}	3.76907×10^{-10}	5.68434×10^{-14}
0.9	0.0000117068	3.79939×10^{-10}	5.52891×10^{-14}
1	0.0000132529	1.01766×10^{-9}	6.26166×10^{-14}

Table 18 Absolute error values for Example 14

t	m = 3	m = 6	m = 9
0	0.00031637	6.58204×10^{-8}	7.82374×10^{-13}
0.1	0.0000577113	1.45506×10^{-8}	2.23419×10^{-13}
0.2	0.000147711	2.70465×10^{-8}	2.12469×10^{-13}
0.3	0.000089263	1.19681×10^{-8}	8.77076×10^{-15}
0.4	2.16739×10^{-6}	9.95623×10^{-10}	1.22125×10^{-15}
0.5	0.0000416166	2.84905×10^{-8}	3.78253×10^{-13}
0.6	0.0000147351	5.73902×10^{-8}	1.8574×10^{-13}
0.7	0.000158511	4.85464×10^{-8}	2.76335×10^{-13}
0.8	0.000317562	3.3788×10^{-8}	6.85341×10^{-13}
0.9	0.000343756	7.68594×10^{-8}	3.85025×10^{-13}
1	0	0	0

tiple of 10^{-7}. Numerical results in Table 18 show that approximate solution obtained is in good agreement with the exact solution and only a few order of Bernstein polynomial can be used to obtain a solution with a high degree of accuracy. Therefore, the present method can be applied as an accurate and reliable technique for the nonlinear BVPs.

7 Conclusion

In this paper the operational matrices of integration, differentiation, product and dual of Bernstein polynomials basis are utilized to reduce the nonlinear boundary value problems to the solution of algebraic equations. In Bernstein polynomials of mth degree, there are $m + 1$ mth degree polynomials which this causes to increase the efficiency and performance of the new method in comparison with methods which use of other polynomials. Comparisons between spectral method based on operational matrices of Bernstein polynomials using collocation method and modified Adomian decomposition method, Birkhoff-type interpolation method, reproducing kernel Hilbert space method, fixed point method, finite-difference Keller-box method, multilevel augmentation method and shooting method for solving the nonlinear ordinary and partial differential equations with multi-point boundary conditions are made. These comparisons enhance the use of the new method if we wish to obtain an accurate approximate solution that converges faster to the exact solution. Moreover, new presented method is also effective for solving appropriate higher order differential equations even for singular differential equations.

We have applied the proposed method to solve the exponential second-order nonlinear differential equation, fourth-order nonlinear differential equation with a product nonlinearity, fourth-order nonlinear differential equation with an exponential nonlinearity, second-order homogeneous partial differential equation, nonlinear third-order differential equation, differential equation related to a cantilever nano-electro mechanical system, differential equation related to a double cantilever nano-electro mechanical system, differential equation related to the unsteady one-dimensional gas flow in a porous medium, eighth order nonlinear differential equation, sixth order nonlinear differential equation, nonlinear stiff differential equation, singular initial value problem, nonlinear stiff differential equation, nonlinear fourth-order boundary value problem in a straightforward procedure, respectively. The obtained numerical tests demonstrate the practicality and efficiency of our new method and show that the method produces acceptable results. Independence of the new method on any parameter can be considered as a strength point of the this method. Our expository examples have demonstrated that only a low-order of Bernstein polynomial does provide an excellent approximation even for the cases of nonlinear and nonlinear singular BVPs. A clear advantage of this technique over the Adomian decomposition method is that no calculation of Adomians polynomials is needed. This method can be considered a good scale for comparison with other existing methods.

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