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Superconvergence and a posteriori error estimates for the Stokes eigenvalue problems

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Abstract In this paper we consider the finite element approximation of the Stokes eigenvalue problems based on projection method, and derive some superconvergence results and the related recovery type a posteriori error estimators. The projection method is a postprocessing procedure that constructs a new approximation by using the least squares strategy. The results are based on some regularity assumptions for the Stokes equations, and are applicable to the finite element approximations of the Stokes eigenvalue problems with general quasi-regular partitions. Numerical results are presented to verify the superconvergence results and the efficiency of the recovery type a posteriori error estimators.

Keywords Stokes eigenvalue problems · Superconvergence · A posteriori error estimates

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1 Introduction

In recent years the finite element approximations of eigenvalue problems have been extensively studied in the literature, especially the elliptic eigenvalue problems. The convergence and superconvergence for finite element approximations of elliptic eigenvalue problems have been provided in [2, 3, 5, 19, 23, 39]. A posteriori error estimates for the finite element approximations of elliptic eigenvalue problems have been extensively studied in [4, 11, 13, 14, 17, 21, 24, 27, 30, 36]. On the contrary, only few results are reported on the error analysis of finite element methods for the Stokes eigenvalue problems. Lovadina, Lyly and Stenberg [26] have proposed a suitable residual type a posteriori error analysis for the Stokes eigenvalue problems. Chen and Lin [7], Jia, Xie, Yin and Gao [20] have analyzed the Richardson extrapolation for the Stokes eigenvalue problems. In [9] Chen, Jia and Xie have proposed a postprocessing method that improved the convergence rate for the numerical approximations of the Stokes eigenvalue problems.

Recovery techniques such as the projection method have been widely used in the finite element superconvergence analysis and recovery type a posteriori error estimates for PDEs. The projection method is a postprocessing procedure that constructs a new approximation by using the method of least squares surface fitting, beginning with the pioneering work in [33]. The superconvergence based on least squares fitting is discussed for elliptic equation in [8, 16, 33] and for Stokes equation in [22, 34, 41]. The asymptotically exact a posteriori error estimators for the pointwise gradient error are provided in [8, 18, 32] using the similar technique. The estimators of this kind have been proved to be efficient and reliable both theoretically and numerically for solving above boundary value problems not only for structured meshes but also for irregular meshes. Superconvergence analysis and recovery type a posteriori error estimates for elliptic eigenvalue problem have been derived by Liu and Sun [24], Liu and Yan [25]. There has been a lack of superconvergence analysis and asymptotically exact a posteriori error estimates using projection method for finite element approximation of the Stokes eigenvalue problems, which is especially suitable for irregular meshes. To our best knowledge, the work here represents a first attempt to superconvergence analysis and recovery type a posteriori error estimates for the Stokes eigenvalue problems.

In this paper, we report superconvergence results and recovery type a posteriori error estimators for the finite element approximation of the Stokes eigenvalue problems by using the projection method. The results are based on some regularity assumptions for the Stokes problems and are applicable to the mixed finite element approximations of the Stokes eigenvalue problems with quasi-regular partitions. Therefore, the results of this paper can be employed to provide useful a posteriori error estimators in practical computing under unstructured meshes. Based on superconvergence results of the eigenfunctions, we derive a posteriori error estimators for the Stokes eigenvalues and design an adaptive algorithms for the finite element computation of Stokes eigenvalue problems.

This paper is organized as follows. In Sect. 2, some notations and the finite element methods for Stokes eigenvalue problems are introduced, some well known properties are also presented. In Sect. 3, we provide the theoretical analysis of the eigenfunctions for finite element approximation of the Stokes eigenvalue problems by projection methods, some superconvergence results on the eigenfunctions and eigenvalues are obtained. In Sect. 4, based on the superconvergence results provided in Sect. 3, the recovery type a posteriori error estimators of Stokes eigenvalue approximation are derived. In Sect. 5, some numerical examples are reported to support our theory. Finally, some conclusions are given at the end of the paper.

2 Mixed finite element discretization

In this paper, we shall use the standard notation for Sobolev spaces $W^{m,p}(\Omega)$ and their associated norms and seminorms in [1]. For $p=2$, we denote $H^m(\Omega)=W^{m,p}(\Omega)$ and $H_0^1(\Omega)=\{v\in H^1(\Omega):v|_{\partial\Omega}=0\}$, where $v|_{\partial\Omega}=0$ is understood in the sense of trace, $\|\cdot\|_m=\|\cdot\|_{m,2,\Omega}$, and $(\cdot,\cdot)$ is the standard L^2 inner product. We shall use the letter C to denote a positive constant which may stand for different value at its different occurrence and is independent of the mesh parameters.

We consider, as a model problem, the eigenvalue problem for the Stokes system with homogeneous Dirichlet boundary conditions, i.e.: find $(\mathbf{u},p;\lambda)$, with $\mathbf{u}\neq 0$ and $\lambda\in\mathbb{R}$, such that

$$\begin{cases} -\Delta\mathbf{u}+\nabla p=\lambda\mathbf{u} & \text{in } \Omega,\\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega,\\ \mathbf{u}=0 & \text{on } \Gamma,\\ \|\mathbf{u}\|_0=1, \end{cases} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ and, Δ, ∇, $\nabla\cdot$ denote the Laplacian, gradient and divergence operators, respectively.

The variational problem associated with (2.1) is given by: find $(u,p;\lambda)\in\mathbf{V}\times W\times\mathbb{R}$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v})-b(\mathbf{v},p)=\lambda(\mathbf{u},\mathbf{v}), & \forall\mathbf{v}\in\mathbf{V},\\ b(\mathbf{u},q)=0, & \forall q\in W,\\ \|\mathbf{u}\|_0=1, \end{cases} \tag{2.2}$$

where

$$\mathbf{V}=\left(H_0^1(\Omega)\right)^2,\qquad W=\left\{q\in L^2(\Omega):\int_\Omega q\,dx=0\right\},$$

$$a(\mathbf{u},\mathbf{v})=(\nabla\mathbf{u},\nabla\mathbf{v})=\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{v}\,dxdy,$$

$$b(\mathbf{v},p)=(\nabla\cdot\mathbf{v},p)=\int_\Omega\nabla\cdot\mathbf{v}\,p\,dxdy.$$

For the above problem we know that the following Babuška-Brezzi condition holds (see [15]):

$$\sup_{\mathbf{v}\in\mathbf{V}}\frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1}\geq C\|q\|_0,\quad \forall q\in W,$$

where C is a constant independent of $\mathbf{v}$ and q. It is known (see, e.g., [3]) that (2.2) has a countable sequence of real eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots$$

and corresponding eigenfunctions

$$(\mathbf{u}_1, p_1), (\mathbf{u}_2, p_2), \ldots,$$

which can be assumed to satisfy

$$(\mathbf{u}_i, \mathbf{u}_j) = \delta_{ij}, \quad i, j = 1, 2, 3, \ldots.$$

The superconvergence analysis to be presented in next section requires a certain regularity assumption for the Stokes problem. We consider a more general Stokes problem which seeks $(\mathbf{y}, s) \in \mathbf{V} \times W$ satisfying

$$\begin{cases} a(\mathbf{y}, \mathbf{v}) - b(\mathbf{v}, s) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ \qquad\qquad b(\mathbf{y}, q) = (g, q), & \forall q \in W, \end{cases} \tag{2.3}$$

where $g \in W$ and $\mathbf{f} \in (L^2(\Omega))^2$ are given functions. Assume that the domain Ω is regular enough to ensure a H^2-regularity for the solution of (2.3). In other words, for any $\mathbf{f} \in (L^2(\Omega))^2$ and $g \in H^1(\Omega) \cap W$ the problem (2.3) admits a unique solution $\mathbf{y} \in (H_0^1(\Omega) \cap H^2(\Omega))^2$ and $s \in H^1(\Omega) \cap W$ satisfying the following a priori estimate:

$$\|\mathbf{y}\|_2 + \|s\|_1 \le C\big(\|\mathbf{f}\|_0 + \|g\|_1\big), \tag{2.4}$$

where C is a constant independent of the data $\mathbf{f}$ and g.

Let T_h be a regular decomposition of the domain Ω into elements T which are allowed to be triangles or convex quadrilaterals. There exists a constant γ such that

$$\frac{h_T}{\rho_T} \le \gamma, \quad \forall T \in T_h,$$

where, for each $T \in T_h$, h_T is the diameter of T and ρ_T is the diameter of the biggest ball contained in T, $h = \max\{h_T : T \in T_h\}$.

Now, let's define the mixed finite element approximations of problem (2.2). Let $\mathbf{V}_h \subset \mathbf{V}$ and $W_h \subset W$ be two finite element spaces for velocity and pressure, respectively, associated with the partition T_h. Let P_r be the set of polynomials of degree no more than r with $r \ge 0$ and Q_r be the set of polynomials with the form $\sum_{i,j=0}^r a_{i,j} x^i y^j$. Assume that the polynomial space in the construction of $\mathbf{V}_h$ contains P_k, $k \ge 1$ for triangular element and Q_k for quadrilateral element, and that of W_h contain P_{k-1}. The two finite element spaces $\mathbf{V}_h$ and W_h are assumed to satisfy the following approximation properties:

Property P1 (Approximation property of $\mathbf{V}_h$) For any $\mathbf{u} \in (H^{m+1}(\Omega))^2$ there holds

$$\inf_{\mathbf{v}_h \in V_h} \big(\|\mathbf{u} - \mathbf{v}_h\|_0 + h\|\mathbf{u} - \mathbf{v}_h\|_1\big) \le Ch^{m+1}\|\mathbf{u}\|_{m+1}, \quad 0 \le m \le k. \tag{2.5}$$

Property P2 (Approximation property of W_h) For any $p \in H^m(\Omega)$ there holds

$$\inf_{q_h \in W_h} \|p - q_h\|_0 \le Ch^m\|p\|_m, \quad 0 \le m \le k. \tag{2.6}$$

Property P3 (Uniform Babuška-Brezzi condition)

$$\sup_{\mathbf{v_h}\in\mathbf{V}_h}\frac{b(\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_1}\geq C\|q_h\|_0,\quad \forall q_h\in W_h. \tag{2.7}$$

The above assumptions are satisfied by several mixed finite elements, see, e.g., [15] for more details.

The mixed finite element discretization for (2.2) reads: find $(\mathbf{u}_h,p_h;\lambda_h)\in\mathbf{V}_h\times W_h\times\mathbb{R}$ such that

$$\begin{cases} a(\mathbf{u}_h,\mathbf{v}_h)-b(\mathbf{v}_h,p_h)=\lambda_h(\mathbf{u}_h,\mathbf{v}_h), & \forall\mathbf{v}_h\in\mathbf{V}_h,\\ b(\mathbf{u}_h,q_h)=0, & \forall q_h\in W_h,\\ \|\mathbf{u}_h\|_0=1. \end{cases} \tag{2.8}$$

It is well known (see [3]) that (2.5) has a finite sequence of eigenvalues

$$0<\lambda_{1,h}\leq\lambda_{2,h}\leq\cdots\leq\lambda_{N,h}$$

and corresponding eigenfunctions

$$(\mathbf{u}_{1,h},p_{1,h}),(\mathbf{u}_{2,h},p_{2,h}),\ldots,(\mathbf{u}_{N,h},p_{N,h}),$$

where

$$(\mathbf{u}_{i,h},\mathbf{u}_{j,h})=\delta_{ij},\quad 1\leq i\leq j\leq N.$$

The eigenvalue approximation λ_h and the corresponding eigenfunction approximation $(\mathbf{u}_h,p_h)$ have the following bound (see, e.g., [2, 3, 6, 9, 28]):

$$|\lambda-\lambda_h|\leq C\Big(\inf_{\mathbf{v}\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}\|_1+\inf_{q\in W_h}\|p-q\|_0\Big)^2,$$

$$\|\mathbf{u}-\mathbf{u}_h\|_1\leq C\Big(\inf_{\mathbf{v}\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}\|_1+\inf_{q\in W_h}\|p-q\|_0\Big),$$

$$\|p-p_h\|_0\leq C\Big(\inf_{\mathbf{v}\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}\|_1+\inf_{q\in W_h}\|p-q\|_0\Big).$$

In particular, if $(\mathbf{u},p)\in(H^{k+1}(\Omega))^2\times H^k(\Omega)$, it follows from (2.5) and (2.6) that

$$|\lambda-\lambda_h|\leq Ch^{2k}\big(\|\mathbf{u}\|_{k+1}+\|p\|_k\big)^2, \tag{2.9}$$

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|p-p_h\|_0\leq Ch^k\big(\|\mathbf{u}\|_{k+1}+\|p\|_k\big). \tag{2.10}$$

Moreover, if the problem (2.3) has H^2-regularity (2.4), then

$$\|\mathbf{u}-\mathbf{u}_h\|_0\leq Ch^{k+1}\big(\|\mathbf{u}\|_{k+1}+\|p\|_k\big). \tag{2.11}$$

In the following we need to introduce the error expansions of the eigenvalues by the Rayleigh quotient formula. The identity of eigenvalue and eigenfunction approximation is crucial for our method.

Lemma 1 *Let $(\mathbf{u},p;\lambda)$ be the exact solution of the Stokes eigenvalue problem (2.2). Then for any $\mathbf{w}\in\mathbf{V}\setminus\{0\}$ and $\phi\in W$, there holds*

$$\frac{a(\mathbf{w},\mathbf{w})-2b(\mathbf{w},\phi)}{\|\mathbf{w}\|_0^2}-\lambda=\frac{\|\mathbf{w}-\mathbf{u}\|_a^2-\lambda\|\mathbf{w}-\mathbf{u}\|_0^2+2b(\mathbf{w}-\mathbf{u},\phi-p)}{\|\mathbf{w}\|_0^2}, \tag{2.12}$$

where $\|\mathbf{w}-\mathbf{u}\|_a=a(\mathbf{w}-\mathbf{u},\mathbf{w}-\mathbf{u})^{1/2}$.

Proof Recalling (2.2), we obtain

$$a(\mathbf{u}, \mathbf{w}) = b(\mathbf{w}, p) + \lambda(\mathbf{u}, \mathbf{w}), \tag{2.13}$$

$$a(\mathbf{u}, \mathbf{u}) = \lambda. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\begin{aligned}
&\|\mathbf{w}-\mathbf{u}\|_a^2 - \lambda\|\mathbf{w}-\mathbf{u}\|_0^2 \\
&\quad = a(\mathbf{w}, \mathbf{w}) - 2a(\mathbf{u}, \mathbf{w}) + a(\mathbf{u}, \mathbf{u}) - \lambda(\mathbf{w}, \mathbf{w}) + 2\lambda(\mathbf{u}, \mathbf{w}) - \lambda \\
&\quad = a(\mathbf{w}, \mathbf{w}) - \lambda(\mathbf{w}, \mathbf{w}) - 2b(\mathbf{w}, p) \\
&\quad = a(\mathbf{w}, \mathbf{w}) - \lambda(\mathbf{w}, \mathbf{w}) + 2b(\mathbf{w}-\mathbf{u}, \phi - p) - 2b(\mathbf{w}, \phi).
\end{aligned} \tag{2.15}$$

From (2.15), we have

$$\begin{aligned}
&a(\mathbf{w}, \mathbf{w}) - 2b(\mathbf{w}, \phi) - \lambda(\mathbf{w}, \mathbf{w}) \\
&\quad = \|\mathbf{w}-\mathbf{u}\|_a^2 - \lambda\|\mathbf{w}-\mathbf{u}\|_0^2 + 2b(\mathbf{w}-\mathbf{u}, \phi - p).
\end{aligned} \tag{2.16}$$

Dividing both sides of (2.16) by $\|w\|_0^2$ gives (2.12). □

Especially, when we set $\mathbf{w} = \mathbf{u}_h$ and $\phi = p_h$ in (2.12) we can conclude from the discrete Stokes eigenvalue problem (2.8) that

$$\lambda_h - \lambda = \|\mathbf{u}_h - \mathbf{u}\|_a^2 - \lambda\|\mathbf{u}_h - \mathbf{u}\|_0^2 + 2b(\mathbf{u}_h - \mathbf{u}, p_h - p). \tag{2.17}$$

3 Superconvergence analysis

In the following we will construct the recovery approximation on coarse meshes for the eigenfunction $(\mathbf{u}, p)$. To begin with, let T_τ be an another finite element partition of the domain Ω with mesh size $\tau > h$. It will be essential to our argument to allow τ to be sufficiently large compared to h. In this paper, we construct the partition T_τ such that they are quasi-uniform, i.e., they are regular and satisfy the inverse assumption (see [10]). Assume that τ is related to the original mesh size h by

$$\tau = h^\alpha, \tag{3.1}$$

where $\alpha \in (0, 1)$ is a parameter to be determined later. Let $\mathbf{V}_\tau \subset (H_0^1(\Omega))^2$ and $W_\tau \in L^2(\Omega)$ be two new finite element spaces consisting of piecewise polynomials of degree r and $r-1$ associated with the partition T_τ, respectively. Define Q_τ and R_τ to be the L^2-projection operators from $L^2(\Omega)$ onto the finite element spaces $\mathbf{V}_\tau$ and W_τ, respectively.

In the following, we will provide two important lemmas, which will be useful for establishing the superconvergence and the recovery type a posteriori error estimates in this and following sections. The following lemmas can be considered as a generalization of the results of Wang and Ye ([34]) from the Stokes problems to the Stokes eigenvalue problems.

Lemma 2 *Assume that* $\mathbf{V}_\tau \subset (H_0^1(\Omega))^2$ *be a finite element space of order* r, $r > k \geq 1$, $\tau = h^\alpha$. *Let* $(\mathbf{u}, p) \in (H^{r+1}(\Omega) \cap H_0^1(\Omega))^2 \times (H^k(\Omega) \cap W)$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ *be the solutions of* (2.2) *and* (2.8) *such that* (2.10) *and* (2.11) *hold true. Assume that the related problem* (2.3) *has* H^2*-regularity* (2.4). *Then, for* $\alpha = (k+1)/(r+1)$ *we have*

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \leq C h^{k+1}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big), \tag{3.2}$$

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a \leq C h^{k+\rho}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big), \tag{3.3}$$

where $\rho = (r-k)/(r+1)$.

Proof Using the triangle inequality we obtain

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \leq \|\mathbf{u} - Q_\tau \mathbf{u}\|_0 + \|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h\|_0. \tag{3.4}$$

From the approximation property of L^2-projection operator we get

$$\|\mathbf{u} - Q_\tau \mathbf{u}\|_0 \leq C\tau^{r+1}\|\mathbf{u}\|_{r+1} \leq C h^{\alpha(r+1)}\|\mathbf{u}\|_{r+1}. \tag{3.5}$$

Moreover, the stability property of L^2-projection yields

$$\|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \leq \|\mathbf{u} - \mathbf{u}_h\|_0. \tag{3.6}$$

Thus, we derive from (2.11) and (3.6) that

$$\|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \leq C h^{k+1}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big). \tag{3.7}$$

It follows from (3.4), (3.5) and (3.7) that

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_{0,\Omega} \leq C h^{\alpha(r+1)}\|\mathbf{u}\|_{r+1} + C h^{k+1}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big).$$

Then the error estimate (3.2) can be obtained by choosing α such that $\alpha(r+1) = k+1$, i.e.,

$$\alpha = \frac{k+1}{r+1}.$$

In the following we prove (3.3). It is easy to see that

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a \leq \|\mathbf{u} - Q_\tau \mathbf{u}\|_a + \|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h\|_a. \tag{3.8}$$

It is well known that

$$\|\mathbf{u} - Q_\tau \mathbf{u}\|_a \leq C\tau^r\|\mathbf{u}\|_{r+1,\Omega} = C h^{\alpha r}\|\mathbf{u}\|_{r+1}. \tag{3.9}$$

Using the inverse inequality (see, e.g., [10]) and (3.7) we arrive at

$$\begin{aligned}
&\|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h\|_a \\
&\quad = \big\|\nabla(Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h)\big\|_0 \\
&\quad \leq C\tau^{-1}\|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \leq C\tau^{-1}h^{k+1}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big) \\
&\quad = C h^{k+1-\alpha}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big).
\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10), we have

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a \leq C h^{\alpha r}\|\mathbf{u}\|_{r+1} + C h^{k+1-\alpha}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big). \tag{3.11}$$

Again, we choose

$$\alpha = \frac{k+1}{r+1},$$

which optimizes the two terms in (3.11) such that $\alpha r = k + 1 - \alpha$. Then the corresponding error estimate is given by

$$\begin{aligned}\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a &\le Ch^{\frac{(k+1)r}{r+1}}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big)\\ &= Ch^{k+\rho}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big),\end{aligned}$$

where $\rho = (r-k)/(r+1)$. This completes the proof of the lemma. □

Lemma 3 *Assume that $W_\tau \subset L^2(\Omega)$ be a finite element space of order $r-1$, $\tau = h^\alpha$. Let $(\mathbf{u}, p) \in (H^{k+1}(\Omega) \cap H_0^1(\Omega))^2 \times (H^r(\Omega) \cap W)$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ be the solutions of (2.2) and (2.8) such that (2.10) and (2.11) hold true. Assume that the related problem (2.3) has H^2-regularity (2.4). Then, for $\alpha = (k+1)/(r+1)$ we have*

$$\|p - R_\tau p_h\|_0 \le Ch^{k+\rho}\big(\|\mathbf{u}\|_{k+1} + \|p\|_r\big), \tag{3.12}$$

where $\rho = (r-k)/(r+1)$.

Proof Using the triangle inequality it is not difficult to get

$$\|p - R_\tau p_h\|_0 \le \|p - R_\tau p\|_0 + \|R_\tau p - R_\tau p_h\|_0. \tag{3.13}$$

Based on the property of L^2-projection operator, we have

$$\|p - R_\tau p\|_0 \le C\tau^r\|p\|_r \le Ch^{\alpha r}\|p\|_r. \tag{3.14}$$

In the following, we estimate the second term of the right side hand in (3.13). The definitions of $\|\cdot\|_0$ and R_τ give

$$\|R_\tau p - R_\tau p_h\|_0 = \sup_{\psi \in L^2(\Omega), \|\psi\|_0 = 1} \big|(R_\tau p - R_\tau p_h, \psi)\big|$$

and

$$(R_\tau p - R_\tau p_h, \psi) = (p - p_h, R_\tau \psi).$$

Then

$$\|R_\tau p - R_\tau p_h\|_0 = \sup_{\psi \in L^2(\Omega), \|\psi\|_0 = 1} \big|(p - p_h, R_\tau \psi)\big|.$$

Consider the following problem that seeks $(\mathbf{y}, s) \in \mathbf{V} \times W$ such that

$$a(\mathbf{y}, \mathbf{v}) - b(\mathbf{v}, s) = 0 \quad \forall \mathbf{v} \in \mathbf{V}, \tag{3.15}$$

$$b(\mathbf{y}, q) = (R_\tau \psi, q) \quad \forall q \in W. \tag{3.16}$$

Replacing q in (3.16) by $p - p_h$ and using (2.2), (2.8) and (3.15) we can obtain

$$
\begin{aligned}
&(p-p_h, R_\tau\psi) \\
&\quad = b(\mathbf{y}, p-p_h) = b(\mathbf{y}-\mathbf{y}_I, p-p_h) + b(\mathbf{y}_I, p-p_h) \\
&\quad = b(\mathbf{y}-\mathbf{y}_I, p-p_h) + a(\mathbf{u}-\mathbf{u}_h, \mathbf{y}_I) - (\lambda\mathbf{u}-\lambda_h\mathbf{u}_h, \mathbf{y}_I) \\
&\quad = b(\mathbf{y}-\mathbf{y}_I, p-p_h) + a(\mathbf{u}-\mathbf{u}_h, \mathbf{y}_I-\mathbf{y}) + a(\mathbf{u}-\mathbf{u}_h, \mathbf{y}) \\
&\qquad - \big(\lambda(\mathbf{u}-\mathbf{u}_h), \mathbf{y}_I\big) - \big((\lambda-\lambda_h)\mathbf{u}_h, \mathbf{y}_I\big) \\
&\quad = b(\mathbf{y}-\mathbf{y}_I, p-p_h) + a(\mathbf{u}-\mathbf{u}_h, \mathbf{y}_I-\mathbf{y}) + b(\mathbf{u}-\mathbf{u}_h, s-s_I) \\
&\qquad - \big(\lambda(\mathbf{u}-\mathbf{u}_h), \mathbf{y}_I\big) - \big((\lambda-\lambda_h)\mathbf{u}_h, \mathbf{y}_I\big),
\end{aligned}
$$

where $\mathbf{y}_I := Q_h\mathbf{y} \in \mathbf{V}_h$ is the L^2 projection of $\mathbf{y}$ onto $\mathbf{V}_h$ and $s_I := Q_h s \in W_h$ is the L^2 projection of s onto W_h. Using Schwarz inequality and the approximation properties (2.5), (2.6), (2.9), (2.10) and (2.11), we obtain

$$
\begin{aligned}
(p-p_h, R_\tau\psi) &\le \|\mathbf{y}-\mathbf{y}_I\|_1\|p-p_h\|_0 + \|\mathbf{u}-\mathbf{u}_h\|_1\|\mathbf{y}_I-\mathbf{y}\|_1 \\
&\quad + \|\mathbf{u}-\mathbf{u}_h\|_1\|s-s_I\|_0 + \lambda\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{y}_I\|_0 + |\lambda-\lambda_h|\|\mathbf{u}_h\|_0\|\mathbf{y}_I\|_0 \\
&\le Ch^{k+1}\big(\|\mathbf{y}\|_2 + \|s\|_1\big)\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big) \\
&\le Ch^{k+1}\|R_\tau\psi\|_1\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big) \\
&\le Ch^{k+1}\tau^{-1}\|\psi\|_{0,\Omega}.
\end{aligned}
$$

Therefore

$$
\|R_\tau p - R_\tau p_h\|_{0,\Omega} \le Ch^{k+1}\tau^{-1}. \tag{3.17}
$$

Combining (3.13), (3.14) and (3.17), we have

$$
\|p - R_\tau p_h\|_0 \le Ch^{\alpha r}\|p\|_r + Ch^{k+1-\alpha}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big). \tag{3.18}
$$

Again, we choose

$$
\alpha = \frac{k+1}{r+1},
$$

which optimizes the two terms in (3.18) such that $\alpha r = k+1-\alpha$. Then the corresponding error estimate is given by

$$
\begin{aligned}
\|p - R_\tau p_h\|_0 &\le Ch^{\frac{(k+1)r}{r+1}}\big(\|\mathbf{u}\|_{k+1} + \|p\|_r\big) \\
&= Ch^{k+\rho}\big(\|\mathbf{u}\|_{k+1} + \|p\|_r\big),
\end{aligned} \tag{3.19}
$$

where $\rho = (r-k)/(r+1)$, this proves (3.12). □

Lemmas 2 and 3 show that the convergence order of $\|\mathbf{u} - Q_\tau\mathbf{u}_h\|_a + \|p - R_\tau p_h\|_0$ is better than the optimal error $O(h^k)$. With $k=1$ and $r=2$, we have the following error estimate for the eigenfunction approximation:

$$
\|\mathbf{u} - Q_\tau\mathbf{u}_h\|_a + \|p - R_\tau p_h\|_0 \le Ch^{4/3}.
$$

Assume that the exact eigenfunction is sufficiently smooth, then it is not difficult to see that

$$
\|\mathbf{u} - Q_\tau\mathbf{u}_h\|_a + \|p - R_\tau p_h\|_0 \simeq O\big(h^2\big), \quad \text{as } r \to \infty.
$$

Moreover, comparing with the local recovery techniques such as SPR (see [31, 43]) or PPR (see [29, 36, 42]), the projection method can be viewed as a semi-local method. It requires a global projection of the finite element solution onto a much coarser mesh. Thus, its computational cost is higher than the local recovery techniques such as SPR or PPR. However, it should be pointed out that we can obtain superconvergence using projection method on arbitrary meshes instead of the almost uniform meshes as for general superconvergence analysis.

For problems with reentrant corners in the domain, the H^2 regularity for the Stokes problem (2.3) is not satisfied. Instead, the Stokes problem (2.3) has the $H^{1+\sigma}(\Omega)$ regularity for some $\sigma \in (0, 1)$. For sufficiently smooth eigenfunction $(\mathbf{u}, p)$ and for any $\epsilon > 0$, it can be proven that

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h^{\sigma-\epsilon}\|\mathbf{u} - \mathbf{u_h}\|_a \leq Ch^{k+\sigma-\epsilon}.$$

One is able to determine a value of α in $\tau = h^\alpha$ such that

$$\alpha = \frac{k + \sigma - \epsilon}{r + 1}.$$

Then similar to Lemmas 2 and 3, we can obtain

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \leq Ch^{k+\sigma-\epsilon},$$
$$\|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a \leq Ch^{k+\hat{\rho}},$$
$$\|p - R_\tau p_h\|_a \leq Ch^{k+\hat{\rho}},$$

where $\hat{\rho} = \frac{r(\sigma-\epsilon)-k}{r+1}$.

Using above results, we propose two enhanced eigenvalue approximations based on projection method. The first one is

$$\lambda_\bullet = \frac{\|Q_\tau \mathbf{u}_h\|_a^2 - 2b(Q_\tau \mathbf{u}_h, R_\tau p_h)}{\|Q_\tau \mathbf{u}_h\|_0^2},$$

which uses the standard Rayleigh quotient acceleration technique and the improved eigenfunction by means of the projection method based on Lemmas 2 and 3. The Rayleigh quotient acceleration technique has been widely used for the superconvergence of the eigenvalue problems based on the interpolation postprocessing approach or the two-grid discretization scheme in [9, 23, 37, 38, 40]. The another one is the new scheme:

$$\lambda_* = \|Q_\tau \mathbf{u}_h\|_a^2 - 2b(Q_\tau \mathbf{u}_h, R_\tau p_h),$$

where we again apply the improved eigenfunction $(Q_\tau \mathbf{u}_h, R_\tau p_h)$ based on the projection method, but the Rayleigh quotient technique is not used anymore. It is easy to see from the definitions of λ_* and $\lambda_\bullet$ that the cost of computation for λ_* is cheaper than the one for $\lambda_\bullet$.

Theorem 1 *Suppose that all conditions of Lemmas 2 and 3 are valid. Let $\lambda_\bullet$ and λ_* are defined as above. Then, we have*

$$|\lambda_\bullet - \lambda| \leq Ch^{2k+2\rho} \tag{3.20}$$

and

$$|\lambda_* - \lambda| \leq Ch^{2k+2\rho}. \tag{3.21}$$

Proof Recalling the identity (2.12), setting $\mathbf{w} = Q_\tau \mathbf{u}_h$ and $\phi = R_\tau p_h$ in (2.12) yields

$$\lambda_\bullet - \lambda = \frac{\|Q_\tau \mathbf{u}_h\|_a^2 - 2b(Q_\tau \mathbf{u}_h, R_\tau p_h)}{\|Q_\tau \mathbf{u}_h\|_0^2} - \lambda$$
$$= \frac{\|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a^2 - \lambda \|Q_\tau \mathbf{u}_h - \mathbf{u}\|_0^2 + 2b(Q_\tau \mathbf{u}_h - \mathbf{u}, R_\tau p_h - p)}{\|Q_\tau \mathbf{u}_h\|_0^2}. \quad (3.22)$$

Using the definition of L^2 projection we have

$$\|Q_\tau \mathbf{u}_h\|_0^2 \le \|\mathbf{u}_h\|_0^2 = 1.$$

Therefore, it follows from (3.22), Lemmas 2 and 3 that

$$\begin{aligned}|\lambda_\bullet - \lambda| &\le \left| \|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a^2 - \lambda \|Q_\tau \mathbf{u}_h - \mathbf{u}\|_0^2 + 2b(Q_\tau \mathbf{u}_h - \mathbf{u}, R_\tau p_h - p) \right| \\ &\le Ch^{2k+2\rho}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big)^2 + Ch^{2k+2}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big)^2 \\ &\quad + Ch^{2k+2\rho}\big(\|\mathbf{u}\|_{r+1} + \|p\|_k\big)\big(\|\mathbf{u}\|_{k+1} + \|p\|_r\big) \\ &\le Ch^{2k+2\rho}\big(\|\mathbf{u}\|_{r+1} + \|p\|_r\big)^2.\end{aligned}$$

This completes the proof of estimate (3.20). In the following we prove (3.21). It is easy to see that

$$\begin{aligned}|\lambda_* - \lambda| &= \big|\lambda_\bullet - \lambda - (\lambda_\bullet - \lambda_*)\big| \\ &\le |\lambda_\bullet - \lambda| + |\lambda_\bullet - \lambda_*| \\ &\le Ch^{2k+2\rho} + \left| \frac{\|Q_\tau \mathbf{u}_h\|_a^2 - 2b(Q_\tau \mathbf{u}_h, R_\tau p_h)}{\|Q_\tau \mathbf{u}_h\|_0^2}\big(1 - \|Q_\tau \mathbf{u}_h\|_0^2\big) \right|. \quad (3.23)\end{aligned}$$

We estimate the second term in the right side hand of (3.23). Using $\|u_h\|_0 = 1$ and the definition of L^2 projection, we have

$$\begin{aligned}1 - \|Q_\tau \mathbf{u}_h\|_0^2 &= \|\mathbf{u}_h\|_0^2 - \|Q_\tau \mathbf{u}_h\|_0^2 \\ &= (\mathbf{u}_h + Q_\tau \mathbf{u}_h, \mathbf{u}_h - Q_\tau \mathbf{u}_h) \\ &= (\mathbf{u}_h, \mathbf{u}_h - Q_\tau \mathbf{u}_h) \\ &= (\mathbf{u}_h - Q_\tau \mathbf{u}_h, \mathbf{u}_h - Q_\tau \mathbf{u}_h) \\ &= \|\mathbf{u}_h - Q_\tau \mathbf{u}_h\|_0^2. \quad (3.24)\end{aligned}$$

We can conclude from the error estimates (2.11) and (3.2) that

$$\|\mathbf{u}_h - Q_\tau \mathbf{u}_h\|_0 \le \|\mathbf{u}_h - \mathbf{u}\|_0 + \|\mathbf{u} - Q_\tau \mathbf{u}_h\|_0 \le Ch^{k+1}. \quad (3.25)$$

Then we can derive from (3.24) and (3.25) that

$$1 - \|Q_\tau \mathbf{u}_h\|_0^2 = \|\mathbf{u}_h - Q_\tau \mathbf{u}_h\|_0^2 \le Ch^{2k+2}. \quad (3.26)$$

It follows that

$$\left| \frac{\|Q_\tau \mathbf{u}_h\|_a^2 - 2b(Q_\tau \mathbf{u}_h, R_\tau p_h)}{\|Q_\tau \mathbf{u}_h\|_0^2}\big(1 - \|Q_\tau \mathbf{u}_h\|_0^2\big) \right| \le Ch^{2k+2}. \quad (3.27)$$

Summing up, (3.23) and (3.27) imply that

$$|\lambda_* - \lambda| \le Ch^{2k+2\rho} + Ch^{2k+2} \le Ch^{2k+2\rho}.$$

This completes the proof. □

4 Recovery type a posteriori error estimates

Using the superconvergence results obtained in the third section we are now able to derive the recovery type a posteriori error estimates.

In order to derive a posteriori error estimates for eigenfunctions we require that the exact solution $(\mathbf{u}, p)$ satisfies the following nondegeneracy property: There exists a constant $c_1 > 0$ independent of h such that

$$\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0 \ge c_1 h^k. \tag{4.1}$$

As argued by Dörfler and Nochetto [12], this is not a very restrictive condition in practice; it is guaranteed, for instance, if $|D^k \mathbf{u}(x)| + |D^{k-1} p(x)| \ge c > 0$ for all x in a fixed region of Ω. This is also a basic assumption in recovery type a posteriori error estimates method (see, e.g., [18, 27, 30]).

Based on the recovery operators Q_τ and R_τ defined above, we can define the recovery type a posteriori error estimator for eigenfunctions:

$$\eta_f := \|\mathbf{u}_h - Q_\tau \mathbf{u}_h\|_a + \|p_h - R_\tau p_h\|_0. \tag{4.2}$$

Then the following a posteriori error estimates for eigenfunctions can be proved.

Theorem 2 *Suppose that all conditions of Lemmas 2 and 3 are valid. Under the nondegeneracy condition* (4.1), *we have*

$$\left| \frac{\eta_f}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0} - 1 \right| \le Ch^\rho. \tag{4.3}$$

Proof We can conclude from Theorem 1 and the assumption in (4.1) that

$$\begin{aligned}
&\frac{\eta_f}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0} \\
&\quad = \frac{\|\mathbf{u}_h - \mathbf{u} + \mathbf{u} - Q_\tau \mathbf{u}_h\|_a + \|p_h - p + p - R_\tau p_h\|_0}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0} \\
&\quad \le \frac{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0 + \|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a + \|p - R_\tau p_h\|_0}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0} \\
&\quad \le 1 + \frac{Ch^{k+\rho}}{c_1 h^k} = 1 + Ch^\rho.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{\eta_f}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0} \\
&\quad = \frac{\|\mathbf{u}_h - \mathbf{u} + \mathbf{u} - Q_\tau \mathbf{u}_h\|_a + \|p_h - p + p - R_\tau p_h\|_0}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0}
\end{aligned}$$

$$
\geq \frac{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0 - \|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a - \|p - R_\tau p_h\|_0}{\|\mathbf{u}_h - \mathbf{u}\|_a + \|p_h - p\|_0}
$$

$$
\geq 1 - \frac{Ch^{k+\rho}}{c_1 h^k} = 1 - Ch^\rho.
$$

This completes the proof. □

Similarly, in order to derive a posteriori error estimates for eigenvalues we require that the exact solution λ satisfies the following nondegeneracy property: There exists a constant $c_2 > 0$ independent of h such that

$$
|\lambda_h - \lambda| \geq c_2 h^{2k}. \tag{4.4}
$$

Based on the recovery operators Q_τ and R_τ defined above, we can define the recovery type a posteriori error estimator for eigenvalues:

$$
\eta_e \equiv \|\mathbf{u}_h - Q_\tau \mathbf{u}_h\|_a^2 + 2b(\mathbf{u}_h - Q_\tau \mathbf{u}_h, p_h - R_\tau p_h). \tag{4.5}
$$

Then the following a posteriori error estimates can be proved.

Theorem 3 *Suppose that all conditions of Lemmas 2 and 3 are valid. Under the nondegeneracy condition* (4.4), *we have*

$$
\left| \frac{|\eta_e|}{|\lambda_h - \lambda|} - 1 \right| \leq Ch^\rho. \tag{4.6}
$$

Proof Recalling the identity (2.17), we have

$$
\begin{aligned}
\lambda_h - \lambda &= \|\mathbf{u}_h - \mathbf{u}\|_a^2 - \lambda \|\mathbf{u}_h - \mathbf{u}\|_0^2 + 2b(\mathbf{u}_h - \mathbf{u}, p_h - p) \\
&= \eta_e + \sum_{i=1}^{6} \xi_i,
\end{aligned} \tag{4.7}
$$

where

$$
\begin{aligned}
\xi_1 &= \|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a^2, \\
\xi_2 &= 2a(Q_\tau \mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - Q_\tau \mathbf{u}_h), \\
\xi_3 &= 2b(\mathbf{u}_h - Q_\tau \mathbf{u}_h, R_\tau p_h - p), \\
\xi_4 &= 2b(Q_\tau \mathbf{u}_h - \mathbf{u}, p_h - R_\tau p_h), \\
\xi_5 &= 2b(Q_\tau \mathbf{u}_h - \mathbf{u}, R_\tau p_h - p), \\
\xi_6 &= -\lambda \|\mathbf{u}_h - \mathbf{u}\|_0^2.
\end{aligned}
$$

Now, let us estimate the term ξ_i $(i = 2, \ldots, 6)$ one by one. We can deduce from (2.10), (3.3) and (3.12) that

$$
\begin{aligned}
|\xi_2| &\leq C\|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a \|Q_\tau \mathbf{u}_h - \mathbf{u}_h\|_a \\
&\leq C\|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a \big(\|\mathbf{u}_h - \mathbf{u}\|_a + \|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a\big) \\
&\leq Ch^{k+\rho}\big(Ch^k + Ch^{k+\rho}\big) \leq Ch^{2k+\rho},
\end{aligned} \tag{4.8}
$$

$$\begin{aligned}|\xi_3| &\leq C\|\mathbf{u}_h - Q_\tau \mathbf{u}_h\|_a \|R_\tau p_h - p\|_0 \\ &\leq C\big(\|\mathbf{u}_h - \mathbf{u}\|_a + \|\mathbf{u} - Q_\tau \mathbf{u}_h\|_a\big)\|R_\tau p_h - p\|_0 \\ &\leq \big(Ch^k + Ch^{k+\rho}\big)Ch^{k+\rho} \leq Ch^{2k+\rho}\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}|\xi_4| &\leq C\|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a \|p_h - R_\tau p_h\|_0 \\ &\leq C\|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a\big(\|p_h - p\|_0 + \|p - R_\tau p_h\|_0\big) \\ &\leq Ch^{k+\rho}\big(Ch^k + Ch^{k+\rho}\big) \leq Ch^{2k+\rho}.\end{aligned} \tag{4.10}$$

Similarly, applying (2.11), (3.3) and (3.12), we have

$$\begin{aligned}|\xi_5| &\leq C\|Q_\tau \mathbf{u}_h - \mathbf{u}\|_a \|R_\tau p_h - p\|_0 \\ &\leq Ch^{k+\rho} \cdot Ch^{k+\rho} \leq Ch^{2k+2\rho}\end{aligned} \tag{4.11}$$

and

$$|\xi_6| = \lambda\|\mathbf{u}_h - \mathbf{u}\|_0^2 \leq Ch^{2k+2}. \tag{4.12}$$

It follows from (3.3) and (4.8)–(4.12) that

$$\left|\sum_{i=1}^{6} \xi_i\right| \leq \sum_{i=1}^{6} |\xi_i| \leq Ch^{2k+\rho}. \tag{4.13}$$

By the nondegeneracy property (4.4) and (4.7), (4.13) we obtain

$$\left|\frac{|\eta_e|}{|\lambda_h - \lambda|} - 1\right| \leq \frac{|\eta_e - (\lambda_h - \lambda)|}{|\lambda_h - \lambda|} = \frac{|\sum_{i=1}^6 \xi_i|}{|\lambda_h - \lambda|} \leq \frac{Ch^{2k+\rho}}{c_2 h^{2k}} \leq Ch^{\rho},$$

this completes the proof. □

Based on the superconvergence results obtained in Sect. 3, Theorems 2 and 3 show that the a posteriori error estimators η_f and η_e are asymptotically exact to the eigenfunction and eigenvalue, respectively. Noting that Theorems 2 and 3 are valid under general regular meshes and the a posteriori error estimators η_f and η_e are computable, they can be viewed as indicators for mesh refinement in adaptive finite element procedure.

Although the above recovery type a posteriori error estimates were established for the smooth eigenfunction, they can be extended to non-smooth eigenfunction on non-convex domains by using similar techniques. The global regularity assumption used in the proof doesn't hold for non-smooth eigenfunction on the non-convex domains, however, it is possible to make only local regularity assumptions so that some theoretical analysis of the results apply. For the singular solution of elliptic problems, numerical results show that recovery type a posteriori error estimates still provide useful information and form a reliable basis for adaptive refinement (see, e.g., [24, 25, 35, 36]). The theoretical analysis of recovery type a posteriori error estimators of non-smooth solution on non-convex domains has been well established, see, e.g., [8, 33, 35, 36]. The theoretical analysis for the singular eigenvector problem will be postponed to our future work.

5 Numerical examples

In this section we give three numerical examples to illustrate the theoretical results obtained in above sections. Our numerical examples will be given for the two-dimensional problem with the linear triangular MINI element for which the velocity and pressure spaces are defined as

$$V_h = \left\{ \mathbf{v} \in \mathbf{V} : \mathbf{v}|_T \in \left(P_1 + \mathrm{span}\{\lambda_1\lambda_2\lambda_3\}\right)^2, \forall T \in T_h \right\}$$

and

$$W_h = \left\{ q \in W \cap H^1(\Omega) : q|_T \in P_1, \forall T \in T_h \right\},$$

where λ_i $(i = 1, 2, 3)$ denotes the barycentric coordinate and P_1 consists of first order polynomials defined on the element T.

In the following, we shall prove that the upper bounds of eigenvalues of the Stokes problem are obtained by using MINI element. Denote p_I the standard piecewise linear Lagrange interpolation of p in the finite element space W_h, we have the following well known interpolation error estimate:

$$\|p_I - p\|_0 \leq Ch^2\|p\|_2. \tag{5.1}$$

Using (2.8) and (5.1), we have

$$\begin{aligned} \left|b(\mathbf{u}_h - \mathbf{u}, p_h - p)\right| &= \left|b(\mathbf{u}_h - \mathbf{u}, p_h - p_I + p_I - p)\right| \\ &= \left|b(\mathbf{u}_h - \mathbf{u}, p_I - p)\right| \\ &\leq \|\mathbf{u}_h - \mathbf{u}\|_1 \|p_I - p\|_0 \\ &\leq Ch^3\|\mathbf{u}\|_2\|p\|_2. \end{aligned} \tag{5.2}$$

It follows from (2.17), (5.2) and the nondegeneracy property (4.1) that

$$\lambda_h - \lambda \geq c_1 h^2 - Ch^4 - Ch^3. \tag{5.3}$$

The inequality (5.3) implies that the approximate eigenvalues are greater than the exact ones when h is small enough.

Let $\mathbf{V}_\tau \subset (H_0^1(\Omega))^2$ and $W_\tau \in L^2(\Omega)$ be two new finite element spaces consisting of continuous, piecewise quadratic polynomials and piecewise linear polynomials associated with the partition T_τ, respectively. It is shown from Lemmas 2 and 3 that theoretically, we should choose $\tau = h^{2/3}$ to guarantee the optimal superconvergence with ρ equal to $1/3$. But in our numerical examples, we only choose $\tau = 2h$ or $\tau = 4h$ for the recovery operators Q_τ, R_τ and obtain satisfied superconvergence results. The fine mesh T_h is always produced from T_τ, specifically, each coarse mesh element in T_τ is refined into 4 elements by connecting the middle points of the edge or refined uniformly twice to produce 16 elements for two-dimensional problems.

Suppose a sequence of meshes T_h, given by either a uniform or an adaptive refinement, we use a function $F(N) = C(V_N)^{-p/2}$ to estimate the order of convergence p. The convergence rate is measured by the total number of vertices V_N:

$$p := \frac{2\log(e_N) - 2\log(e_{N-1})}{\log(V_{N-1}) - \log(V_N)},$$

Table 1 Errors and orders of convergence for Example 5.1 on uniform meshes

NOV	81	289	1089	4225	16641
$\lambda_h - \lambda$	1.290e0	3.142e-1	7.760e-2	1.940e-2	5.000e-3
order		2.22	2.11	2.05	1.98
η_e	1.158e0	2.764e-1	6.898e-2	1.720e-2	4.827e-3
order		2.25	2.09	2.05	1.85
$\lambda_* - \lambda$	3.605e-1	3.755e-2	3.887e-3	6.521e-4	1.604e-4
order		3.56	3.42	2.63	2.05

Table 2 Errors and orders of convergence for Example 5.1 on adaptive meshes

NOV	65	265	1089	3617	13137
$\lambda_h - \lambda$	1.290e0	3.027e-1	6.450e-2	1.850e-2	4.900e-3
order		2.064	2.19	2.08	2.06
η_e	1.158e0	2.510e-1	5.316e-2	1.541e-2	4.042e-3
order		2.18	2.20	2.06	2.07

where e_N is the error on the mesh of level N with number of vertices V_N. In the following tables, "order" represents the convergence order of the error, while "NOV" represents the number of vertices. In the following, the reference solution has been extrapolated from the numerical results by assuming that the error $\lambda_h - \lambda$ behaves as Ch^r for some constants C and r independent of h (see [5, 26]).

Example 5.1 In our first example we will consider problems defined on $\Omega = (-1, 1) \times (-1, 1)$ with homogenous Dirichlet boundary conditions imposed on the velocity. We take $\lambda = 13.0861$ as the first reference eigenvalue.

In this example, the solution is very smooth, which satisfies all the assumptions of our theory. In Table 1 we present the convergence results on the uniformly refined meshes. It is shown quite clearly that order two is obtained for the errors $\lambda_h - \lambda$. It shows also the superconvergence results for $\lambda_* - \lambda$. Moreover, we can observe that the recovery type a posteriori error estimate is asymptotically exact. In Table 2 we record the convergence results on adaptively refined meshes. This adaptive refinement procedure is based on longest-edge bisection, which results to a sequence of unstructured, nonuniform and shape regular meshes. One can also observe that the adaptively refined meshes are almost uniformly distributed because the exact eigenfunction is smooth, but the adaptive method is still more efficient than the uniform refinement strategy. For example, for the uniform refinement method, we have that $\lambda_h - \lambda = 5.000\text{e-}3$ on the mesh with 16641 nodes, while for the adaptive method, the values of this error is 4.900e-3 on the mesh with only 13137 nodes. In Fig. 1 we present the log-log plots of the rate of convergence on the uniformly refined and adaptively refined meshes. In Fig. 2 the adaptive mesh is plotted.

Example 5.2 In our previous example, the domain was a convex polygon. Let us change the domain to a L-shaped domain $\Omega = [-1, 1] \times [-1, 1] \setminus (0, 1] \times (0, 1]$ with

Fig. 1 Log-log plot of convergence in Example 5.1

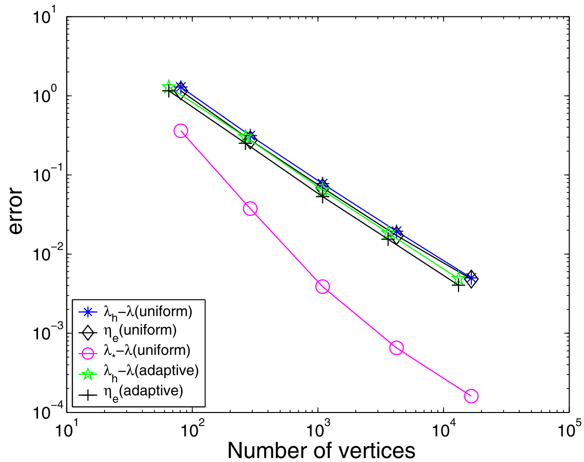
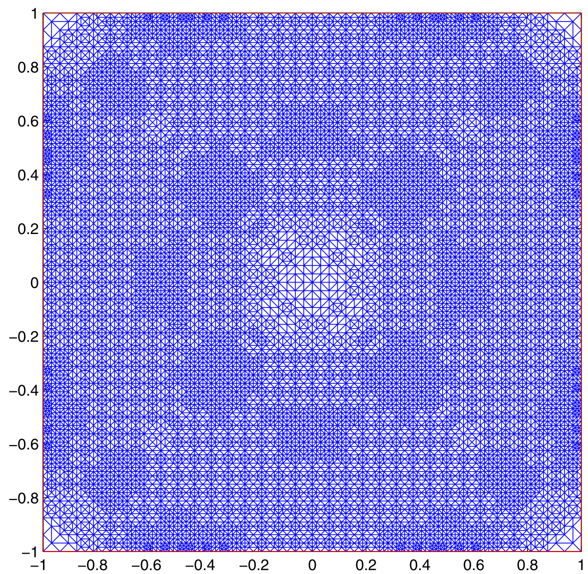


Fig. 2 Adaptively refined mesh with 13137 vertices in Example 5.1



reentrant corners. We take $\lambda = 48.9844$ as the fourth reference eigenvalue. The dual problem of this example has $H^{1+\sigma}$ regularity for $\sigma \in (0, 1)$.

In Table 3 we report the convergence results on the uniformly refined meshes. We can also observe two order convergence for the errors $\lambda_h - \lambda$ and superconvergence result for $\lambda_* - \lambda$. Moreover, we find that the recovery type a posteriori error estimate is asymptotically exact. Form the convergence and superconvergence results we conclude that the exact eigenfunction of the fourth eigenvalue is very smooth. In Table 4 we record the convergence results on adaptively refined meshes. The adaptive method is again more efficient than the uniform refinement strategy. For example, for the uniform refinement method we have that $\lambda_h - \lambda = 8.930$e-2 on the mesh with

Table 3 Errors and orders of convergence for Example 5.2 on uniform meshes

NOV	45	153	561	2145	8385
$\lambda_h - \lambda$	2.191e1	5.737e0	1.439e0	3.587e-1	8.930e-2
order		2.19	2.13	2.07	2.04
η_e	3.048e1	5.629e0	1.266e0	3.188e-1	8.038e-2
order		2.76	2.30	2.06	2.02
$\lambda_* - \lambda$	1.492e1	1.642e0	2.160e-1	2.178e-2	2.516e-3
order		3.61	3.12	3.42	3.17

Table 4 Errors and orders of convergence for Example 5.2 on adaptive meshes

NOV	45	181	577	2095	8107
$\lambda_h - \lambda$	2.191e1	5.449e0	1.204e0	3.138e-1	7.950e-2
order		2.00	2.60	2.09	2.03
η_e	3.048e1	5.315e0	1.050e0	2.669e-1	6.794e-2
order		2.51	2.79	2.12	2.02

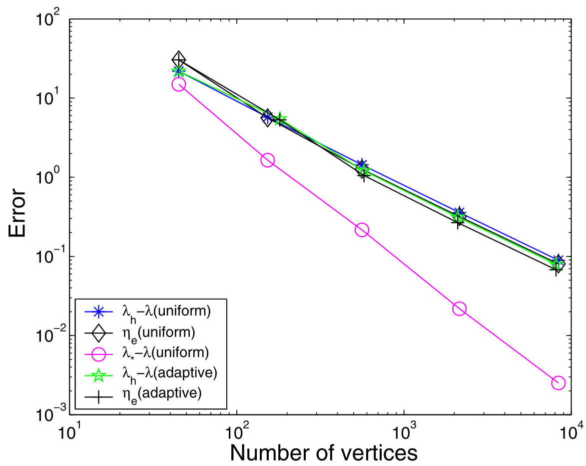


Fig. 3 Log-log plot of convergence in Example 5.2

8385 nodes. However, for the adaptive method, the values of this error is 7.950e-2 on the mesh with only 8107 nodes. In Fig. 3 we present the log-log plot of the rate of convergence on the uniformly refined and adaptively refined meshes. In Fig. 4 the adaptive mesh is plotted.

Example 5.3 In the above examples the eigenfunction **u** was all analytic. We consider a unit square with a 45°-crack. We take $\lambda = 31.2444$ as the first reference eigenvalue.

Convergence results on uniformly and adaptively refined meshes are reported in Tables 5 and 6. The exact eigenfunction **u** is not smooth in this case, and this can be confirmed from the numerical results. For the case of uniform refinement with 8385 nodes, we see quite clearly the order of convergence for $\lambda_h - \lambda$ is 1.16. In

Fig. 4 Adaptively refined mesh with 8107 vertices in Example 5.2

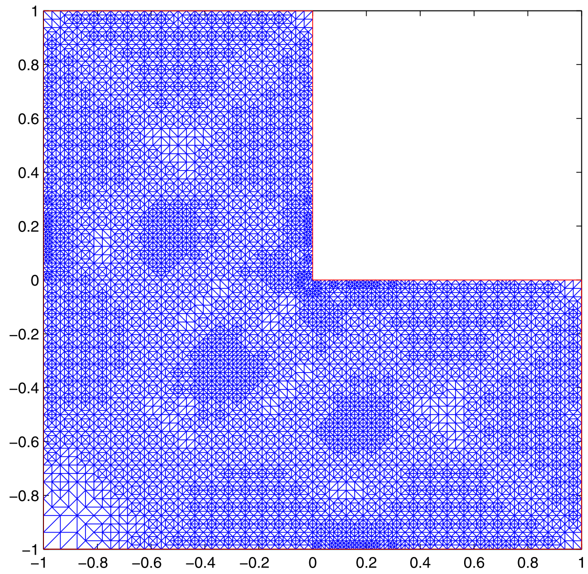


Table 5 Errors and orders of convergence for Example 5.3 on uniform meshes

NOV	45	153	561	2145	8385
$\lambda_h - \lambda$	8.036e0	2.655e0	8.828e-1	3.292e-1	1.491e-1
order		1.81	1.70	1.47	1.16
η_e	1.263e1	2.543e0	8.111e-1	2.933e-1	1.139e-1
order		2.62	1.76	1.52	1.39
$\lambda_* - \lambda$	3.478e0	1.039e0	2.691e-1	1.174e-1	7.382e-2
order		1.97	2.08	1.24	0.68

case of uniform meshes we can not expect superconvergence for the approximation λ_*. For the adaptive meshes we used η_e as a posteriori error estimator, the order of convergence is improved and approaches order two for $\lambda_h - \lambda$. This numerical behavior is typical for a reasonable adaptive refinement procedure. Nevertheless, η_e is still close to $\lambda_h - \lambda$. For the uniform refinement method, we have that $\lambda_h - \lambda =$ 1.491e-1 on the mesh with 8385 nodes. However, for the adaptive method, this error is 1.090e-1 on the mesh with only 3351 nodes. Since the exact eigenfunction is not so smooth as the ones in previous examples, it is more efficient to use the adaptive meshes than the uniform meshes. In Fig. 5 we present the log-log plots of the rate of convergence. In Fig. 6 the adaptive mesh is plotted. The distributions of nodes in adaptive mesh clearly show the accumulation of nodes in the vicinity of the singular point $(0, 0)$.

Table 6 Errors and orders of convergence for Example 5.3 on adaptive meshes

NOV	45	164	557	1300	3351
$\lambda_h - \lambda$	8.036e0	2.499e0	6.866e-1	3.191e-1	1.090e-1
order		1.81	2.11	1.81	2.27
η_e	1.263e1	2.134e0	5.470e-1	2.318e-1	8.698e-2
order		2.75	2.23	2.03	2.07

Fig. 5 Log-log plot of convergence in Example 5.3

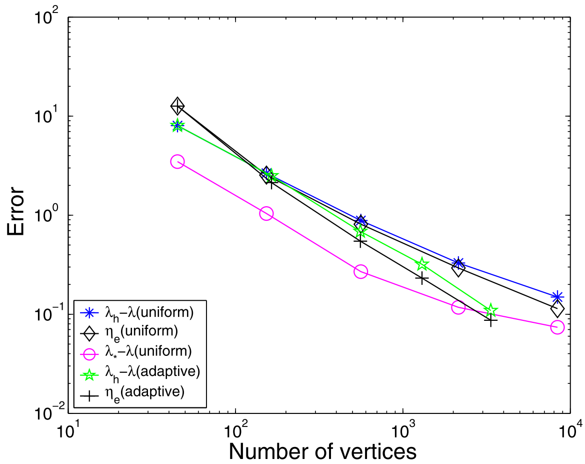
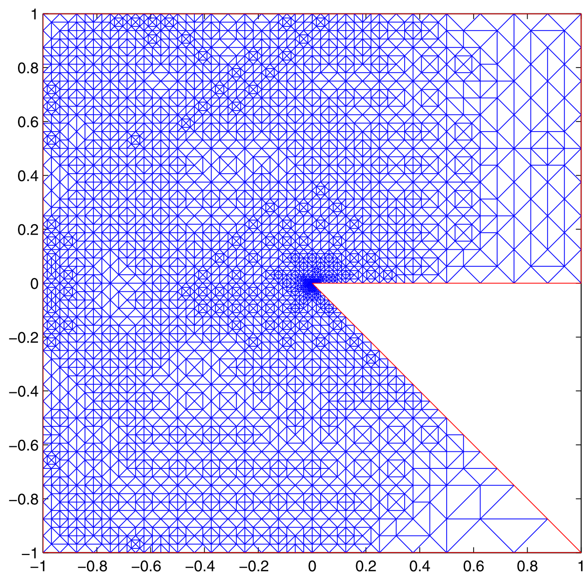


Fig. 6 Adaptively refined mesh with 3351 vertices in Example 5.3



6 Conclusions

In this paper, we presented the superconvergence result and the related recovery type a posteriori error estimators based on projection method for finite element approximation of the Stokes eigenvalue problems. The projection method is a postprocessing procedure that constructs a new approximation by using the least squares method. The results are based on some regularity assumptions for the Stokes equations, and are applicable to the finite element approximations of Stokes eigenvalue problems with general quasi-regular partitions.

The numerical examples show that the recovered eigenvalue λ_* superconverges to λ if the exact eigenfunction u is smooth enough (as shown in Examples 5.1 and 5.2). It is shown in Example 5.2 that λ_* enhances the accuracy of λ_h although the related Stokes problem does not satisfy H^2-regularity assumption. Furthermore, Example 5.3 shows that λ_* is still more accurate than λ_h even though the solutions are not smooth and superconvergence can not be obtained. Moreover, one can observe that the adaptive method using our a posteriori error estimators is more efficient relative to the uniform refinement strategy, especially for singular problems. The effectiveness of our adaptive scheme does not necessarily depend on either the uniformity of the mesh or the global regularity of the solution. However, theoretically there is still much work to do to fill in the gaps. Although the numerical results in this work are solely for the two-dimensional problems and the conforming finite element methods, the idea is nevertheless applicable to more general eigenvalue problems.

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