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Explicit description of 2D parametric solution sets

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Abstract Consider a linear system $A(p) \cdot x = b(p)$, where the elements of the matrix and the right-hand side vector depend linearly on a m-tuple of parameters $p = (p_1, \dots, p_m)$, the exact values of which are unknown but bounded within given intervals. Apart from quantifier elimination, the only known general way of describing the solution set $\{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\}$ is a lengthy and non-unique Fourier-Motzkin-type parameter elimination process that leads to a description of the solution set by exponentially many inequalities. In this work we modify the parameter elimination process in a way that has a significant impact on the representation of the inequalities describing the solution set and their number. An explicit minimal description of the solution set to 2D parametric linear systems is derived. It generalizes the Oettli-Prager theorem for non-parametric linear systems. The number of the inequalities describing the solution set grows linearly with the number of the parameters involved simultaneously in both equations of the system. The boundary of any 2D parametric solution set is described by polynomial equations of at most second degree. It is proven that when the general parameter elimination process is applied to two equations of a system in higher dimension, some inequalities become redundant.

Keywords Linear systems · Solution set · Interval parameters

Mathematics Subject Classification (2000) 65F05 · 65G99

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1 Introduction

Consider the linear algebraic system

$$A(p) \cdot x = b(p), \quad p = (p_1, \ldots, p_m)^\top, \tag{1.1}$$

$$\begin{aligned} &a_{ij}(p) := a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu} p_\mu, \qquad b_i(p) := b_{i,0} + \sum_{\mu=1}^{m} b_{i,\mu} p_\mu, \\ &a_{ij,\mu}, b_{i,\mu} \in \mathbb{R}, \quad \mu = 0, \ldots, m, \ i, j = 1, \ldots, n. \end{aligned} \tag{1.2}$$

The only information that we have about the values of the parameters p_μ, $\mu = 1, \ldots, m$ is that they lie within given intervals

$$p \in [p] = \big([p]_1, \ldots, [p]_m\big)^\top. \tag{1.3}$$

Such systems are common in many engineering analysis or design problems [5, 6], control engineering [1], robust Monte Carlo simulations [7], etc., where there are complicated dependencies between the model parameters which are uncertain. The uncertainty in the model parameters could originate from measurement imprecision, round-off errors, and various other kinds of inexact knowledge. A set of solutions to (1.1)–(1.3), called *united parametric solution set*, is

$$\Sigma^p = \Sigma\big(A(p), b(p), [p]\big) := \big\{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\big\}. \tag{1.4}$$

Denote by $\mathbb{R}^n, \mathbb{R}^{n\times m}$ the set of real vectors with n components and the set of real $n \times m$ matrices, respectively. A real compact interval is $[a] = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote the sets of interval n-vectors and interval $n \times m$ matrices, respectively. Interval vectors $[b] = ([b]_i) = ([\underline{b}_i, \overline{b}_i]) \in \mathbb{IR}^n$ and interval matrices $[A] = ([a]_{ij}) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in \mathbb{IR}^{n\times m}$ are vectors and matrices, respectively, with interval entries. For $[a] = [\underline{a}, \overline{a}]$, define *mid-point* $\dot{a} := (\underline{a} + \overline{a})/2$ and *radius* $\hat{a} := (\overline{a} - \underline{a})/2$. These functionals are applied to interval vectors and matrices componentwise.

The well-known non-parametric interval linear system $[A]x = [b]$, which is the most studied in the interval literature (cf. [8]), can be considered as a special case of the parametric linear system with $n^2 + n$ independent parameters $a_{ij} \in [a]_{ij}$, $b_i \in [b]_i$, $i, j = 1, \ldots, n$. For a parametric system (1.1)–(1.3), the *corresponding* non-parametric one with $[A] = ([a]_{ij}) \in \mathbb{IR}^{n\times n}$, $[b] = ([b]_i) \in \mathbb{IR}^n$ can be obtained as

$$[a]_{ij} = a_{ij}\big([p]\big) = a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu}[p]_\mu, \qquad [b]_i = b_i\big([p]\big) = b_{i,0} + \sum_{\mu=1}^{m} b_{i,\mu}[p]_\mu.$$

The non-parametric solution set, called united solution set, is defined as

$$\Sigma\big([A], [b]\big) := \big\{x \in \mathbb{R}^n \mid \exists A \in [A], \exists b \in [b], A \cdot x = b\big\}.$$

This set is well studied with many results concerning its characterization and properties, see, e.g., [2, 8]. In particular, the Oettli-Prager theorem [9] describes this solution set by the inequalities

$$|A(\dot{p})x - b(\dot{p})| \le \hat{A}\big([p]\big)|x| + \hat{b}\big([p]\big). \tag{1.5}$$

Characterizing the solution set (1.4) by inequalities not involving the interval parameters is a fundamental problem of considerable practical importance. It is useful for visualizing the solution set, exploring its properties and for computing componentwise boundaries. The description of the parametric solution set is related to quantifier elimination which stimulated a tremendous amount of research. Since Tarski's general theory [13] is EXPSPACE-hard [4], a lot of research is devoted to special cases with polynomial-time decidability. So far, the only known general way of describing the parametric solution set is a lengthy Fourier-Motzkin-type parameter elimination process presented in [3]. Several open questions concerning this process and the maximal degree of the polynomials in the solution set describing inequalities are also formulated therein.

In this work we introduce a modification of the parameter elimination that has a significant impact on the representation of the inequalities describing the solution set and their number. Based on the modified parameter elimination process, we study the parameter elimination in 2-dimensional linear systems involving an arbitrary number of parameters. Because the general parameter elimination is a long process leading to an exponential number of characterizing inequalities, minimal explicit descriptions of the parametric solution sets are of particular interest. By proving superfluous and redundant character of some inequalities we derive a minimal explicit description of 2D parametric solution sets. This also allows us to determine the shape of these solution sets, i.e., the maximal degree of the polynomial equations describing the solution set boundary. The explicit solution set characterization is illustrated on some numerical examples and compared to descriptions obtained by other approaches.

2 Fourier-Motzkin-type elimination of parameters

The solution set (1.4) is characterized as follows, by a trivial set of inequalities

$$\Sigma^p = \{x \in \mathbb{R}^n \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \ldots, m : \text{(2.1)–(2.2) hold}\},$$

where for $i = 1, \ldots, n$

$$\sum_{j=1}^{n}\left(a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu} p_\mu\right) x_j \le b_{i,0} + \sum_{\mu=1}^{m} b_{i,\mu} p_\mu \le \sum_{j=1}^{n}\left(a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu} p_\mu\right) x_j, \tag{2.1}$$

$$p_\mu^- \le p_\mu \le p_\mu^+, \quad \mu = 1, \ldots, m. \tag{2.2}$$

Starting from such a description, Theorem 2.1 below shows how the parameters in this set can be eliminated successively in order to obtain a new description not involving p_μ, $\mu = 1, \ldots, m$.

Theorem 2.1 (Alefeld et al. [3]) *Let* $f_{\lambda\mu}$, g_λ, $\lambda = 1, \dots, k$ (≥ 2), $\mu = 1, \dots, m$, *be real-valued functions of* $x = (x_1, \dots, x_n)^\top$ *on some subset* $D \subseteq \mathbb{R}^n$. *Assume that there is a positive integer* $k_1 < k$ *such that*: $f_{\lambda 1}(x) \not\equiv 0$ *for all* $\lambda \in \{1, \dots, k\}$; $f_{\lambda 1}(x) \geq 0$ *for all* $x \in D$ *and all* $\lambda \in \{1, \dots, k\}$; *for each* $x \in D$ *there is an index* $\beta^* = \beta^*(x) \in \{1, \dots, k_1\}$ *with* $f_{\beta^* 1}(x) > 0$ *and an index* $\gamma^* = \gamma^*(x) \in \{k_1 + 1, \dots, k\}$ *with* $f_{\gamma^* 1}(x) > 0$. *For m parameters* $p_1, \dots, p_m$ *varying in* $\mathbb{R}$ *and for x varying in D define the sets* S_1, S_2 *by*

$$S_1 := \left\{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \dots, m : (2.3), (2.4) \textit{ hold}\right\},$$
$$S_2 := \left\{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = 2, \dots, m : (2.5) \textit{ holds}\right\},$$

where inequalities (2.3), (2.4) *and* (2.5), *respectively, are given by*

$$g_\beta(x) + \sum_{\mu=2}^{m} f_{\beta\mu}(x) p_\mu \leq f_{\beta 1}(x) p_1, \quad \beta = 1, \dots, k_1, \tag{2.3}$$

$$f_{\gamma 1}(x) p_1 \leq g_\gamma(x) + \sum_{\mu=2}^{m} f_{\gamma\mu}(x) p_\mu, \quad \gamma = k_1 + 1, \dots, k \tag{2.4}$$

and for $\beta = 1, \dots, k_1$, $\gamma = k_1 + 1, \dots, k$

$$g_\beta(x) f_{\gamma 1}(x) + \sum_{\mu=2}^{m} f_{\beta\mu}(x) f_{\gamma 1}(x) p_\mu \leq g_\gamma(x) f_{\beta 1}(x) + \sum_{\mu=2}^{m} f_{\gamma\mu}(x) f_{\beta 1}(x) p_\mu. \tag{2.5}$$

(*Trivial inequalities such as* $0 \leq 0$ *can be omitted.*) *Then* $S_1 = S_2$.

Theorem 2.1 defines the transition from inequalities (2.3), (2.4) to inequalities (2.5), so that the parameter p_1 does not occur in the set S_2. This parameter elimination process resembles the so-called Fourier-Motzkin elimination of variables, see, e.g., [12]. Its application based on the end-point inequalities (2.2) leads to a very large number of solution set characterizing inequalities. Some illustrative examples can be found in [2, 10].

A new classification of the parameters involved in a parametric linear system was introduced in [10]. Therein the parameters were classified into three classes (zeroth, first and second class) with respect to the way they participate in the equations of the system. In this paper we consider only two classes of parameters by joining the parameters of the zeroth and first class into the first class.

With the notations $A_{\bullet\bullet\mu} := (a_{ij,\mu}) \in \mathbb{R}^{n\times n}$, $b_{\bullet\mu} := (b_{i,\mu}) \in \mathbb{R}^n$, $\mu = 0, \dots, m$ the system (1.1) can be rewritten equivalently as

$$\left(A_{\bullet\bullet 0} + \sum_{\mu=1}^{m} p_\mu A_{\bullet\bullet\mu}\right) x = b_{\bullet 0} + \sum_{\mu=1}^{m} p_\mu b_{\bullet\mu}.$$

For a matrix $A \in \mathbb{R}^{n\times n}$, $A_{m\bullet}$ denotes the m-th row of A.

Definition 2.1 *A parameter p_μ, $1 \le \mu \le m$, is of* 1*st class* if it occurs in only one equation of the system (1.1).

A parameter p_μ is of 1st class iff $b_{\bullet\mu} - A_{\bullet\bullet\mu}x$ has only one nonzero component (that is $b_{i\mu} - A_{i\bullet\mu}x \neq 0$ for exactly one i, $1 \le i \le n$). It does not matter how many times a 1st class parameter occurs within an equation. For example, the parameters b_1 and b_2 involved in the system from Example 5.1 are parameters of 1st class. An efficient elimination procedure for parameters of 1st class was developed in [10]. An explicit characterization of the solution set to parametric linear systems involving only 1st class parameters is also given therein.

Definition 2.2 *A parameter p_μ, $1 \le \mu \le m$, is of* 2*nd class* if it is involved in more than one equation of the system (1.1).

A parameter p_μ is of 2nd class iff the vector $b_{\bullet\mu} - A_{\bullet\bullet\mu}x$ has more than one nonzero components. In this work, we study the elimination process involving 2nd class parameters.

3 Modified elimination of parameters

The trivial set of characterizing inequalities (2.1), (2.2) can be rewritten equivalently as

$$A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu=1}^{m}(A_{\bullet\bullet\mu}x - b_{\bullet\mu})p_\mu \le 0 \le A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu=1}^{m}(A_{\bullet\bullet\mu}x - b_{\bullet\mu})p_\mu, \tag{3.1}$$

$$\dot{p}_\mu - \hat{p}_\mu \le p_\mu \le \dot{p}_\mu + \hat{p}_\mu, \quad \mu = 1, \dots, m, \tag{3.2}$$

where the parameter characterizing inequalities (2.2) are replaced by the equivalent inequalities (3.2). In the parameter elimination process we shall apply the following relation

$$t\dot{p}_\mu - |t|\hat{p}_\mu \le tp_\mu \le t\dot{p}_\mu + |t|\hat{p}_\mu, \quad \text{for } t \in \mathbb{R}, \tag{3.3}$$

without the necessity to consider the particular sign of t.

Denote $f_{\lambda 0}(x) := A_{\lambda\bullet 0}x - b_{\lambda 0}$, $f_{\lambda\mu}(x) := A_{\lambda\bullet\mu}x - b_{\lambda\mu}$, $\lambda = 1, \dots, n$, which are real-valued functions of $x = (x_1, \dots, x_n)^\top$. Denote by $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ three index sets such that $\mathcal{M} := \{1, \dots, m\} = \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$, p_μ is a parameter of 1st class if $p_\mu \in \mathcal{M}_1$ and p_μ is a parameter of 2nd class if $p_\mu \in \mathcal{M}_2$. For the elimination of each parameter of 1st class p_k, $k \in \mathcal{M}_1$ the inequalities (3.1) are rewritten equivalently as

$$A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu\in\mathcal{M}\setminus\{k\}}(A_{\bullet\bullet\mu}x - b_{\bullet\mu})p_\mu \le -(A_{\bullet\bullet k}x - b_{\bullet k})p_k, \tag{3.4}$$

$$-(A_{\bullet\bullet k}x - b_{\bullet k})p_k \le A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu\in\mathcal{M}\setminus\{k\}}(A_{\bullet\bullet\mu}x - b_{\bullet\mu})p_\mu. \tag{3.5}$$

Then we apply Theorem 2.1 by combining the inequality (3.4) with the right side inequality of (3.2) and by combining the inequality (3.5) with the left side inequality of (3.2). After multiplying the inequalities (3.2) by the corresponding coefficient function $f_{\lambda k}(x)$, the relation (3.3) is applied. In this way, we obtain the following equivalent set of characterizing inequalities, where all 1st class parameters are eliminated.

$$f_{i0}(x) + \sum_{\mu\in\mathcal{M}_1} (f_{i\mu}(x)\dot{p}_\mu - |f_{i\mu}(x)|\hat{p}_\mu) + \sum_{\mu\in\mathcal{M}_2} f_{i\mu}(x)p_\mu \leq 0, \quad i = 1,\dots,n, \tag{3.6}$$

$$0 \leq f_{i0}(x) + \sum_{\mu\in\mathcal{M}_1} (f_{i\mu}(x)\dot{p}_\mu + |f_{i\mu}(x)|\hat{p}_\mu) + \sum_{\mu\in\mathcal{M}_2} f_{i\mu}(x)p_\mu, \quad i = 1,\dots,n, \tag{3.7}$$

$$\dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu \in \mathcal{M}_2. \tag{3.8}$$

We notice that the expression in the left side of inequality (3.6) and the expression in the right side of inequality (3.7) differ only in the sign of the terms involving $\hat{p}_\mu$. Therefore we use a $\mp$ notation and rewrite equivalently the corresponding inequalities (3.6), (3.7) as *one inequality pair* in a more compact form representing the two expressions only on the left side of the inequality pair.

$$f_{i0}(x) + \sum_{\mu\in\mathcal{M}_1} f_{i\mu}(x)\dot{p}_\mu \mp \sum_{\mu\in\mathcal{M}_1} |f_{i\mu}(x)|\hat{p}_\mu + \sum_{\mu\in\mathcal{M}_2} f_{i\mu}(x)p_\mu \leq 0 \leq \cdots,$$
$$i = 1,\dots,n, \tag{3.9}$$

wherein "$\cdots$" denote the whole expression in the left side inequality with the bottom sign of $\mp$. If $\mathcal{M}_2 = \emptyset$, (3.9) is equivalent to

$$|A(\dot{p})x - b(\dot{p})| \leq \sum_{\mu=1}^{m} |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\hat{p}_\mu, \tag{3.10}$$

which generalizes the Oettli-Prager characterization (1.5) to parametric systems involving only 1st class parameters. This explicit description was derived in [10] and compared to the characterization obtained by the original parameter elimination process. This result shows that the elimination of 1st class parameters does not expand the number of characterizing inequalities. The shape of the solution sets to systems involving only 1st class parameters remain linear.

Let $1 \leq \mathrm{Card}(\mathcal{M}_1) < m$. Starting from the inequalities (3.6)–(3.8), we eliminate a 2nd class parameter p_k, $k \in \mathcal{M}_2$. Let $\mathcal{T}$ be the index set of the inequality pairs involving this parameter. By combining each inequality pair $\alpha \in \mathcal{T}$ with the corresponding parameter characterizing inequalities $\dot{p}_k - \hat{p}_k \leq p_k \leq \dot{p}_k + \hat{p}_k$, we obtain the so-called end-point characterizing inequalities. For $\mathcal{T} = \{1,\dots,n\}$, these are

$$f_{i0}(x) + \sum_{\mu\in\mathcal{M}_1\cup\{k\}} (f_{i\mu}(x)\dot{p}_\mu \mp |f_{i\mu}(x)|\hat{p}_\mu) + \sum_{\mu\in\mathcal{M}_2\setminus\{k\}} f_{i\mu}(x)p_\mu \leq 0 \leq \cdots,$$
$$i = 1,\dots,n.$$

For each $\alpha, \beta \in \mathcal{T}$, $\alpha < \beta$, we have to eliminate p_k from the following inequality pairs

$$f_{\alpha 0}(x) + \sum_{\mu \in \mathcal{M}_1} (f_{\alpha\mu}(x)\dot{p}_\mu \mp |f_{\alpha\mu}(x)|\hat{p}_\mu) + \sum_{\mu \in \mathcal{M}_2\setminus\{k\}} f_{\alpha\mu}(x)p_\mu \le -f_{\alpha k}(x)p_k \le \cdots, \tag{3.11}$$

$$f_{\beta 0}(x) + \sum_{\mu \in \mathcal{M}_1} (f_{\beta\mu}(x)\dot{p}_\mu \mp |f_{\beta\mu}(x)|\hat{p}_\mu) + \sum_{\mu \in \mathcal{M}_2\setminus\{k\}} f_{\beta\mu}(x)p_\mu \le -f_{\beta k}(x)p_k \le \cdots. \tag{3.12}$$

In general $f_{\alpha k}(x)$ and $f_{\beta k}(x)$ do not have a common factor.[1] Thus, (3.11) multiplied by $f_{\beta k}(x)$ gives

$$f_{\alpha 0}(x)f_{\beta k}(x) + \sum_{\mu \in \mathcal{M}_1} f_{\alpha\mu}(x)f_{\beta k}(x)\dot{p}_\mu \mp \delta \sum_{\mu \in \mathcal{M}_1} |f_{\alpha\mu}(x)|f_{\beta k}(x)\hat{p}_\mu + \sum_{\mu \in \mathcal{M}_2\setminus\{k\}} f_{\alpha\mu}(x)f_{\beta k}(x)p_\mu \le -f_{\alpha k}(x)f_{\beta k}(x)p_k \le \cdots,$$

wherein $\delta = \{1 \text{ if } f_{\beta k}(x) \ge 0, -1 \text{ otherwise}\}$. The last inequality pair is equivalent to

$$f_{\alpha 0}(x)f_{\beta k}(x) + \sum_{\mu \in \mathcal{M}_1} (f_{\alpha\mu}(x)f_{\beta k}(x)\dot{p}_\mu \mp |f_{\alpha\mu}(x)||f_{\beta k}(x)|\hat{p}_\mu) + \sum_{\mu \in \mathcal{M}_2\setminus\{k\}} f_{\alpha\mu}(x)f_{\beta k}(x)p_\mu \le -f_{\alpha k}(x)f_{\beta k}(x)p_k \le \cdots. \tag{3.13}$$

Similarly, the inequality pair (3.12) is multiplied by $f_{\alpha k}(x)$. Combining the left side of (3.13) with the right side of (3.12)$*f_{\alpha k}(x)$, and respectively the opposite inequality sides, we get the following cross inequality pair

$$f_{\alpha 0}(x)f_{\beta k}(x) - f_{\beta 0}(x)f_{\alpha k}(x) + \sum_{\mu \in \mathcal{M}_1} (f_{\alpha\mu}(x)f_{\beta k}(x) - f_{\beta\mu}(x)f_{\alpha k}(x))\dot{p}_\mu \mp \sum_{\mu \in \mathcal{M}_1} (|f_{\alpha\mu}(x)||f_{\beta k}(x)| + |f_{\beta\mu}(x)||f_{\alpha k}(x)|)\hat{p}_\mu + \sum_{\mu \in \mathcal{M}_2\setminus\{k\}} (f_{\alpha\mu}(x)f_{\beta k}(x) - f_{\beta\mu}(x)f_{\alpha k}(x))p_\mu \le 0 \le \cdots \tag{3.14}$$

for $\alpha, \beta \in \mathcal{T}$, $\alpha < \beta$. We call the inequality pairs (3.14) "cross inequality pairs" to distinguish them from the end-point characterizing inequalities.

[1] If the coefficient functions of a parameter in two inequalities have a common factor, each inequality is multiplied by the quotient of the corresponding coefficient function and the greater common factor of both functions, see [2, Corollary 1].

Thus, by using the parameter inequalities (3.2), instead of the equivalent inequalities (2.2), and the relations (3.3) we can give a new modified formulation of the parameter elimination Theorem 2.1.

Theorem 3.1 *Let $g_\lambda(x)$, $f_{\lambda\nu,1}(x)$, $f_{\lambda\nu,2}(x)$, $f_{\lambda\mu}(x)$, $\lambda = 1,\dots,k$ $(\geq n)$ be real-valued functions of $x = (x_1,\dots,x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that $m_1 - 1$ parameters are eliminated, where $m_1 \geq 1$, and there exists a non-empty set $\mathcal{T} \subseteq \{1,\dots,k\}$ such that $f_{\lambda m_1}(x) \not\equiv 0$ for all $\lambda \in \mathcal{T}$. For the parameters p_μ, $\mu = m_1,\dots,m$ varying in $\mathbb{R}$ and for x varying in D define the sets S_1, S_2 by*

$$S_1 := \big\{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = m_1,\dots,m : (3.15), (3.16)\ \mathit{hold}\big\},$$
$$S_2 := \big\{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = m_1+1,\dots,m : (3.17), (3.18), (3.19)\ \mathit{hold}\big\},$$

where inequalities (3.15), (3.16) and (3.17), (3.18), (3.19), respectively, are given by

$$g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + \sum_{\mu=m_1+1}^{m} f_{\lambda\mu}(x)p_\mu \leq -f_{\lambda m_1}(x)p_{m_1} \leq \cdots, \quad \lambda = 1,\dots,k, \tag{3.15}$$

$$\dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu = m_1,\dots,m, \tag{3.16}$$

$$g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + f_{\lambda m_1}(x)\dot{p}_{m_1} \mp |f_{\lambda m_1}(x)|\hat{p}_{m_1} + \sum_{\mu=m_1+1}^{m} f_{\lambda\mu}(x)p_\mu \leq 0 \leq \cdots, \quad \lambda = 1,\dots,k \tag{3.17}$$

and for $\alpha, \beta \in \mathcal{T}$, $\alpha < \beta$

$$g_\alpha(x)f_{\beta m_1}(x) - g_\beta(x)f_{\alpha m_1}(x) + \sum_{\nu=1}^{m_1-1}(f_{\beta m_1}(x)f_{\alpha\nu,1}(x) - f_{\alpha m_1}(x)f_{\beta\nu,1}(x))\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1}(|f_{\beta m_1}(x)|f_{\alpha\nu,2}(x) + |f_{\alpha m_1}(x)|f_{\beta\nu,2}(x))\hat{p}_\nu + \sum_{\mu=m_1+1}^{m}(f_{\alpha\mu}(x)f_{\beta m_1}(x) - f_{\beta\mu}(x)f_{\alpha m_1}(x))p_\mu \leq 0 \leq \cdots, \tag{3.18}$$

$$\dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu = m_1+1,\dots,m. \tag{3.19}$$

Then $S_1 = S_2$.

The proof of Theorem 3.1 is similar to that of Theorem 2.1 given in [3].

Theorem 3.1 is more detailed in the characterizing inequalities than the original Theorem 2.1 and involves explicitly the parameter inequalities (3.16), (3.19). This allows weakening the requirements of Theorem 2.1, in particular not requiring positiveness of the parameter coefficient functions. Therefore, the modified parameter elimination does not depend on a particular orthant. Besides, Theorem 3.1 gives a compact representation of the characterizing inequalities. Due to the absolute values of the expressions involved in these inequalities, the parametric solution set is described by fewer inequalities than when applying Theorem 2.1. Note that the resulting inequalities (3.17) and (3.18) have the form (3.15) which allows the elimination process to continue with the next parameters.

Although the number of solution set describing inequalities generated by Theorem 3.1 is smaller than the number of inequalities generated by the application of Theorem 2.1, this number grows exponentially with the number of 2nd class parameters involved simultaneously in the equations of a system. Neither Theorem 2.1 nor Theorem 3.1 prescribe the order of parameter elimination. If all 1st class parameters are eliminated before the elimination of the 2nd class parameters, the elimination of 1st class parameters does not generate cross inequalities. However, if a 2nd class parameter, say p_2, is eliminated before the 1st class parameters p_μ, $\mu \in \mathcal{M}_1$ involved in the equations containing p_2, then p_μ will be involved also in the cross inequality pairs for p_2. Thus, the elimination of these 1st class parameters p_μ will be done as the elimination of 2nd class parameters. This implies that two sets of characterizing inequalities, generated by two different orders of parameter elimination, may contain a different number of inequalities. Therefore, in the next section we consider in more details the parameter elimination for 2D parametric linear systems.

4 Characterization of 2D solution sets

Consider a linear system (1.1)–(1.3), where $A(p) \in \mathbb{R}^{2\times 2}$ and $b(p) \in \mathbb{R}^2$ involve $m+s$ parameters. Without loss of generality we can re-number the parameters and assume that the first $m \geq 1$ parameters are of 2nd class while the next $s \geq 0$ parameters are of 1st class. The trivial inequalities explicitly characterizing the solution set are

$$f_0(x) + \sum_{\mu=1}^{m+s} f_\mu(x) p_\mu \leq 0 \leq f_0(x) + \sum_{\mu=1}^{m+s} f_\mu(x) p_\mu, \tag{4.1}$$

$$\text{and for } \mu = 1, \ldots, m+s, \quad \dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \tag{4.2}$$

where $f_\lambda(x) := (f_{\lambda 1}(x), f_{\lambda 2}(x))^\top$, $\lambda = 0, \ldots, m+s$ and $f_{\lambda i}(x) := A_{i\bullet\lambda} x - b_{i\lambda}$, $i = 1, 2$. For simplicity of the exposition we first eliminate all 2nd class parameters.

Let $m \geq 2$. After eliminating the first 2nd class parameter p_1, we have the following cross inequality pair

$$\Delta_{0,1}(x) + \sum_{\mu=2}^{m+s} \Delta_{\mu,1}(x) p_\mu \leq 0 \leq \Delta_{0,1}(x) + \sum_{\mu=2}^{m+s} \Delta_{\mu,1}(x) p_\mu, \tag{$e_{1,2}(1)$}$$

where $\Delta_{\alpha,\beta}(x) := f_{\alpha,1}(x) f_{\beta,2}(x) - f_{\alpha,2}(x) f_{\beta,1}(x)$, $\Delta_{\alpha,\beta}(x) = -\Delta_{\beta,\alpha}(x)$. From now on we label the inequalities by $(e_{i,j}(u, \dots, v))$, where the subscripts i, j denote which inequalities are combined, while $u, \dots, v$ denote the indices of the parameters that are eliminated starting from the rightmost one. In what follows we shall omit the argument of the functions Δ and f. Combining (4.1) with the parameter inequalities (4.2) for p_1 we get the following endpoint inequality pairs

$$f_{0,i} + f_{1,i}\dot{p}_1 \mp |f_{1,i}|\hat{p}_1 + \sum_{\mu=2}^{m} f_{\mu,i} p_\mu + \sum_{\nu \in N_1} f_{\nu,1} p_\nu + \sum_{\nu \in N_2} f_{\nu,2} p_\nu \le 0 \le \cdots, \qquad (e_i(1))$$

where $i = 1, 2$, N_1, N_2 are the index sets of 1st class parameters involved in the corresponding equations of the system, $N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 = \{m+1, \dots, m+s\}$. The elimination of each successive parameter "*updates*" the latter endpoint inequality pairs, and the hitherto existing cross inequality pairs, correspondingly. When all the parameters are eliminated, the endpoint inequality pairs can equivalently be written as a single inequality (3.10). Due to their simplicity, in what follows we shall not list the updated endpoint inequalities.

The elimination of p_2 updates $(e_{1,2}(1))$ by combining it with the parameter inequalities for p_2, thus yielding

$$\Delta_{0,1} + \Delta_{\mu,1}\dot{p}_2 \mp |\Delta_{\mu,1}|\hat{p}_2 + \sum_{\mu=3}^{m+s} \Delta_{\mu,1} p_\mu \le 0 \le \cdots, \qquad (e_{1,2}(2, \tilde{1}))$$

and introduces three more cross inequality pairs.

$$\Delta_{0,2} + \Delta_{1,2}\dot{p}_1 \mp (|f_{1,1} f_{2,2}| + |f_{1,2} f_{2,1}|)\hat{p}_1 + \sum_{\mu=3}^{m} \Delta_{\mu,2} p_\mu + \sum_{\nu \in N_1} f_{\nu,1} f_{2,2} p_\nu - \sum_{\nu \in N_2} f_{\nu,2} f_{2,1} p_\nu \le 0 \le \cdots, \qquad (e_{1,2}(2, 1))$$

which can be written as[2]

$$\left| \Delta_{0,2} + \Delta_{1,2}\dot{p}_1 + \sum_{\mu=3}^{m+s} \Delta_{\mu,2} p_\mu \right| \le (|f_{1,1} f_{2,2}| + |f_{1,2} f_{2,1}|)\hat{p}_1 \qquad (e_{1,2}(2, 1))$$

and for $i = 1, 2$

$$f_{0,i}\Delta_{2,1} - f_{2,i}\Delta_{0,1} + f_{1,i}\Delta_{2,1}\dot{p}_1 \mp |f_{1,i}||\Delta_{2,1}|\hat{p}_1$$

[2]The advantage of the modified parameter elimination is that cross inequality pairs can equivalently be rewritten as single absolute value inequalities.

$$+\sum_{\mu=3}^{m+s}(f_{\mu,i}\Delta_{2,1}-f_{2,i}\Delta_{\mu,1})p_\mu \ \leq 0 \leq \cdots . \qquad (e_{i,(1,2)}(2,1))$$

After some simplification, the latter cross inequality pairs can be written as

$$|f_{1,i}|\left|\Delta_{0,2}+\Delta_{1,2}\dot{p}_1+\sum_{\mu=3}^{m+s}\Delta_{\mu,2}p_\mu\right| \leq |f_{1,i}||\Delta_{1,2}|\hat{p}_1. \qquad (e_{i,(1,2)}(2,1))$$

The inequalities $(e_{1,(1,2)}(2,1))/|f_{1,1}|$ and $(e_{2,(1,2)}(2,1))/|f_{1,2}|$ are equivalent. Therefore one of them is superfluous. Since $|\Delta_{1,2}| = |f_{1,1}f_{2,2} - f_{1,2}f_{2,1}| \leq |f_{1,1}f_{2,2}| + |f_{1,2}f_{2,1}|$, the right side of inequality $e_{1,2}(2,1)$ is greater than or equal to the right sides of the inequalities $(e_{i,(1,2)}(2,1))/|f_{1,i}|$, $i = 1,2$. The relations between these cross inequalities remain valid for all $p_\mu \in [p_\mu]$, $\mu = 3,\ldots,m+s$, after the elimination of these parameters by combining $(e_{1,2}(2,1))$, $(e_{i,(1,2)}(2,1))/|f_{1,i}|$, $i = 1,2$ with the corresponding endpoint parameter inequalities (4.2). Therefore the inequality $(e_{1,2}(2,1))$ will not contribute to the boundary of the parametric solution set and $(e_{1,2}(2,1))$ is redundant.[3] Thus, instead of considering three additional cross inequalities $(e_{1,2}(2,1))$, $(e_{i,(1,2)}(2,1))$, $i = 1,2$, we continue the parameter elimination process considering only one additional cross inequality pair $e_{1,2}(\tilde{2},1) := (e_{i,(1,2)}(2,1))/|f_{1,i}|$. After eliminating all the remaining parameters, the two cross inequalities $e_{1,2}(2,\tilde{1})$ and $e_{1,2}(\tilde{2},1)$ become

$$\left|\Delta_{0,k}+\sum_{\mu\in M_k}\Delta_{\mu,k}\dot{p}_\mu\right| \leq \sum_{\mu\in M_k}|\Delta_{\mu,k}|\hat{p}_\mu, \quad k=1,2, \qquad (e^{(k)}_{1,2}(\tilde{2},1))$$

where $M_\mu := \{1,\ldots,m+s\}\setminus\{\mu\}$ for $1 \leq \mu \leq m$.

By induction on the number of 2nd class parameters we shall prove that the elimination of each such parameter expands the number of characterizing inequalities by only one cross inequality. Let $m-1$ parameters of 2nd class be eliminated and the set of characterizing inequalities be

$$f_{0,i}+\sum_{\mu=1}^{m-1}(f_{\mu,i}\dot{p}_\mu \mp |f_{\mu,i}|\hat{p}_\mu)+\sum_{\nu\in N_1}f_{\nu,1}p_\nu$$
$$+\sum_{\nu\in N_2}f_{\nu,2}p_\nu \ \leq -f_{m,i}p_m \leq \cdots, \quad i=1,2, \qquad (e_i(m-1))$$

$$\Delta_{0,k}+\sum_{\mu\in M_m,\mu\neq k}(\Delta_{\mu,k}\dot{p}_\mu \mp |\Delta_{\mu,k}|\hat{p}_\mu)+\sum_{\nu\in N_1}f_{\nu,1}f_{k,2}p_\nu$$
$$-\sum_{\nu\in N_2}f_{\nu,2}f_{k,1}p_\nu \leq -\Delta_{m,k}p_m \leq \cdots, \qquad k\in M_m, \qquad (e^{(k)}_{1,2}(m-1))$$

[3] Inequalities which are equivalent are called superfluous, while the inequalities which do not contribute to the boundary of the solution set are called redundant.

where $M_m = \{1, \ldots, m-1\}$ and, for simplicity, the inequalities are labeled $e_i(m-1)$, $e_{1,2}^{(k)}(m-1)$ instead of $e_i(m-1,\ldots,1)$, $e_{1,2}^{(k)}(m-1,\ldots,1)$ respectively.

Beside updating the above inequalities by combining them with the parameter endpoint inequalities $\dot{p}_m - \hat{p}_m \le p_m \le \dot{p}_m + \hat{p}_m$, the elimination of p_m introduces the following additional cross inequalities. One cross inequality is obtained by combining $e_1(m-1)$ and $e_2(m-1)$. After some simplification, it can be written as

$$\left|\Delta_{0,m} + \sum_{\mu=1}^{m-1} \Delta_{\mu,m}\dot{p}_\mu + \sum_{\nu\in N_1\cup N_2} \Delta_{\nu,m}p_\nu\right| \le \sum_{\mu=1}^{m-1}(|f_{\mu,1}f_{m,2}| + |f_{\mu,2}f_{m,1}|)\hat{p}_\mu. \qquad (e_{1,2}^{(m)}(m))$$

By combining $e_1(m-1)$ with every one of the inequalities $e_{1,2}^{(k)}(m-1)$, $k \in M_m$, we obtain $m-1$ additional cross inequalities. After some simplification and dividing by $(-f_{k,1})$, these inequalities become

$$\left|\Delta_{0,m} + \sum_{\mu=1}^{m-1} \Delta_{\mu,m}\dot{p}_\mu + \sum_{\nu\in N_1\cup N_2} \Delta_{\nu,m}p_\nu\right| \le \sum_{\mu=1,\mu\neq k}^{m-1} \hat{p}_\mu(|f_{\mu,1}\Delta_{m,k}| + |f_{m,1}\Delta_{\mu,k}|)/|f_{k,1}| + |\Delta_{k,m}|\hat{p}_k, \quad k \in M_m. \qquad (e_{1,(1,2)}^{(k)}(m))$$

Similarly, by combining $e_2(m-1)$ with every one of the inequalities $e_{1,2}^{(k)}(m-1)$, $k \in M_m$, we obtain $m-1$ additional cross inequalities. After some simplification and dividing by $(-f_{k,2})$, these inequalities become

$$\left|\Delta_{0,m} + \sum_{\mu=1}^{m-1} \Delta_{\mu,m}\dot{p}_\mu + \sum_{\nu\in N_1\cup N_2} \Delta_{\nu,m}p_\nu\right| \le \sum_{\mu=1,\mu\neq k}^{m-1} \hat{p}_\mu(|f_{\mu,2}\Delta_{m,k}| + |f_{m,2}\Delta_{\mu,k}|)/|f_{k,2}| + |\Delta_{k,m}|\hat{p}_k, \quad k \in M_m. \qquad (e_{2,(1,2)}^{(k)}(m))$$

Finally, by combining every two of the inequalities $e_{1,2}^{(k)}(m-1)$, $k \in M_m$, we obtain $(m-1)(m-2)/2$ additional cross inequalities. After some simplification and dividing by $\Delta_{\alpha,\beta}$, the latter become

$$\left|\Delta_{0,m} + \sum_{\mu\in M_m} \Delta_{\mu,m}\dot{p}_\mu + \sum_{\nu\in N_1\cup N_2} \Delta_{\nu,m}p_\nu\right| \le |\Delta_{\alpha,m}|\hat{p}_\alpha + |\Delta_{\beta,m}|\hat{p}_\beta + \sum_{\mu\in M_m,\mu\neq\alpha,\beta} \hat{p}_\mu(|\Delta_{\mu,\alpha}\Delta_{m,\beta}| + |\Delta_{\mu,\beta}\Delta_{m,\alpha}|)/|\Delta_{\alpha,\beta}|, \qquad (e_{(1,2),(1,2)}^{(\alpha,\beta)}(m))$$

where $\alpha, \beta \in M_m$, $\alpha < \beta$.

Consider the last set of cross inequalities $(e^{(\alpha,\beta)}_{(1,2),(1,2)}(m))$, $\alpha, \beta \in M_m$, $\alpha < \beta$. The smallest right side of these inequalities is the right side of the following inequality

$$\left|\Delta_{0,m} + \sum_{\mu\in M_m} \Delta_{\mu,m}\dot{p}_\mu + \sum_{\nu\in N_1\cup N_2} \Delta_{\nu,m}p_\nu\right| \le \sum_{\mu\in M_m} |\Delta_{\mu,m}|\hat{p}_\mu \qquad (e_{(1,2)}(\hat{m}))$$

obtained when

$$|\Delta_{\mu,\alpha}\Delta_{m,\beta}| + |\Delta_{\mu,\beta}\Delta_{m,\alpha}| = |\Delta_{\mu,\alpha}\Delta_{m,\beta} - \Delta_{\mu,\beta}\Delta_{m,\alpha}| = |\Delta_{\alpha,\beta}||\Delta_{\mu,m}|$$
$$\text{for } \operatorname{sign}\Delta_{\mu,\alpha}\Delta_{m,\beta} \ne \operatorname{sign}\Delta_{\mu,\beta}\Delta_{m,\alpha}. \qquad (*)$$

We will prove that in the set of cross inequalities $(e^{(\alpha,\beta)}_{(1,2),(1,2)}(m))$, $\alpha, \beta \in M_m$, $\alpha < \beta$, there is at least one inequality $(e_{(1,2)}(\hat{m}))$.

Each triple of indices μ, α, β can be ordered in three different ways: $\mu < \alpha < \beta$, $\alpha < \mu < \beta$, $\alpha < \beta < \mu$. The three different orderings correspond to different cross inequalities involving the expressions $|\Delta_{\mu,\alpha}\Delta_{m,\beta}| + |\Delta_{\mu,\beta}\Delta_{m,\alpha}|$. For each fixed μ, two of the μ, α, β orderings imply (*) for the third μ, α, β ordering, since $\Delta_{u,v} = -\Delta_{v,u}$. We illustrate the last implication by the example $m - 1 = 3$, where the set of cross inequalities $(e^{(\alpha,\beta)}_{(1,2),(1,2)}(m))$ consists of three inequalities having the following right sides

$$|\Delta_{1,m}|\hat{p}_1 + |\Delta_{2,m}|\hat{p}_2 + (|\Delta_{3,1}\Delta_{m,2}| + |\Delta_{3,2}\Delta_{m,1}|)/|\Delta_{1,2}|, \qquad (e^{(1,2)}_{(1,2),(1,2)}(m))$$

$$|\Delta_{1,m}|\hat{p}_1 + |\Delta_{3,m}|\hat{p}_3 + (|\Delta_{2,1}\Delta_{m,3}| + |\Delta_{2,3}\Delta_{m,1}|)/|\Delta_{1,3}|, \qquad (e^{(1,3)}_{(1,2),(1,2)}(m))$$

$$|\Delta_{2,m}|\hat{p}_2 + |\Delta_{3,m}|\hat{p}_3 + (|\Delta_{1,2}\Delta_{m,3}| + |\Delta_{1,3}\Delta_{m,2}|)/|\Delta_{2,3}|. \qquad (e^{(2,3)}_{(1,2),(1,2)}(m))$$

For $\mu = 1$ that is $(e^{(2,3)}_{(1,2),(1,2)}(m))$, the relation (*) follows from $(e^{(1,2)}_{(1,2),(1,2)}(m))$ and $(e^{(1,3)}_{(1,2),(1,2)}(m))$, where

$$\operatorname{sign}(\Delta_{3,1}\Delta_{m,2}) = \operatorname{sign}(\Delta_{3,2}\Delta_{m,1}) = -\operatorname{sign}(\Delta_{2,3}\Delta_{m,1}),$$
$$\operatorname{sign}(\Delta_{2,1}\Delta_{m,3}) = \operatorname{sign}(\Delta_{2,3}\Delta_{m,1})$$

imply $\operatorname{sign}(\Delta_{3,1}\Delta_{m,2}) \ne \operatorname{sign}(\Delta_{2,1}\Delta_{m,3})$.

Since μ, α, β trace all combinations of indices in M_m, we have $(e_{(1,2)}(\hat{m}))$ for an arbitrary large M_m. It is not difficult to see that the right sides of the inequalities $e^{(m)}_{1,2}(m)$, $e^{(k)}_{1,(1,2)}(m)$ and $e^{(k)}_{2,(1,2)}(m)$ are greater than (or equal to) the right side of $e_{(1,2)}(\hat{m})$. Thus, after the elimination of all p_ν, $\nu \in N_1 \cup N_2$, $e_{(1,2)}(\hat{m})$ becomes

$$\left|\Delta_{0,m} + \sum_{\mu=1,\mu\ne m}^{m+s} \Delta_{\mu,m}\dot{p}_\mu\right| \le \sum_{\mu=1,\mu\ne m}^{m+s} |\Delta_{\mu,m}|\hat{p}_\mu$$

and could contribute to the boundary of the parametric solution set, while all other generated cross inequalities will be redundant or superfluous.

Now, we prove that the elimination of 1st class parameters does not introduce any cross inequalities. Without loss of generality, we assume that the system involves $m = 2$ parameters of 2nd class p_1, p_2 which are eliminated. We eliminate two 1st class parameters q_1, q_2 from the inequalities

$$f_{0,i} + \sum_{\mu=1}^{2} f_{\mu,i}\dot{p}_\mu \mp \sum_{\mu=1}^{2} |f_{\mu,i}|\hat{p}_\mu + \begin{cases} g_{1,1}q_1, & \text{for } i = 1 \\ g_{2,2}q_2, & \text{for } i = 2 \end{cases} \le 0 \le \cdots, \qquad (e_i(2,1))$$

$$\Delta_{0,1} + \Delta_{2,1}\dot{p}_2 \mp |\Delta_{2,1}|\hat{p}_2 + g_{1,1}f_{1,2}q_1 - g_{2,2}f_{1,1}q_2 \le 0 \le \cdots, \qquad (e_{1,2}(1))$$

$$\Delta_{0,2} + \Delta_{1,2}\dot{p}_1 \mp |\Delta_{1,2}|\hat{p}_1 + g_{1,1}f_{2,2}q_1 - g_{2,2}f_{2,1}q_2 \le 0 \le \cdots. \qquad (e_{1,2}(\tilde{2},1))$$

The elimination of q_1 generates the following additional cross inequalities.

$$\begin{aligned} &f_{0,1}f_{1,2} - \Delta_{0,1} + f_{1,1}f_{1,2}\dot{p}_1 \mp |f_{1,1}f_{1,2}|\hat{p}_1 \\ &\quad + (f_{2,1}f_{1,2} - \Delta_{2,1})\dot{p}_2 \mp (|f_{2,1}f_{1,2}| + |\Delta_{2,1}|)\hat{p}_2 \\ &\quad + g_{2,2}f_{1,1}q_2 \le 0 \le \cdots. \end{aligned} \qquad (e_{1,(1,2)}(1))$$

Since $|f_{2,1}f_{1,2}| + |\Delta_{2,1}| = |f_{2,1}f_{1,2}| + |-\Delta_{2,1}| \ge |f_{2,1}f_{1,2} - \Delta_{2,1}| = |f_{1,1}f_{2,2}|$, and $f_{0,1}f_{1,2} - \Delta_{0,1} = f_{0,2}f_{1,1}$ the polynomials defined in $(e_{1,(1,2)}(1))/f_{1,1}$ are less than, respectively greater than, or equal to those defined in $e_2(2,1)$. Therefore $(e_{1,(1,2)}(1))$ is redundant.

Similarly, the polynomials defined in the following additional cross inequality pair

$$\begin{aligned} &f_{0,1}f_{2,2} - \Delta_{0,2} + (f_{1,1}f_{2,2} - \Delta_{1,2})\dot{p}_1 \mp (|f_{1,1}f_{2,2}| + |\Delta_{1,2}|)\hat{p}_1 \\ &\quad + f_{2,2}f_{2,1}\dot{p}_2 \mp |f_{2,2}f_{2,1}|\hat{p}_2 + g_{2,2}f_{2,1}q_2 \le 0 \le \cdots, \end{aligned} \qquad (e_{1,(1,2)}(\tilde{2},1))$$

divided by $f_{2,1}$, are outside those defined in $e_2(2,1)$ and therefore $(e_{1,(1,2)}(\tilde{2},1))$ is also redundant.

The additional cross inequality

$$\begin{aligned} &\Delta_{0,1}f_{2,2} - \Delta_{0,2}f_{1,2} - \Delta_{1,2}f_{1,2}\dot{p}_1 \mp |\Delta_{1,2}f_{1,2}|\hat{p}_1 + \Delta_{2,1}f_{2,2}\dot{p}_2 \mp |\Delta_{2,1}f_{2,2}|\hat{p}_2 \\ &\quad - (g_{2,2}f_{1,1}f_{2,2} - g_{2,2}f_{2,1}f_{1,2})q_2 \le 0 \le \cdots \end{aligned} \qquad (e_{(1,2),(1,2)}(\tilde{2},1))$$

after simplifying the expressions becomes

$$\begin{aligned} &-\Delta_{1,2}f_{0,2} - \Delta_{1,2}f_{1,2}\dot{p}_1 \mp |\Delta_{1,2}f_{1,2}|\hat{p}_1 \\ &\quad + \Delta_{2,1}f_{2,2}\dot{p}_2 \mp |\Delta_{2,1}f_{2,2}|\hat{p}_2 - g_{2,2}\Delta_{1,2}q_2 \le 0 \le \cdots. \end{aligned} \qquad (e_{(1,2),(1,2)}(\tilde{2},1))$$

Since the latter divided by $-\Delta_{1,2}$ is equivalent to $e_2(2,1)$, then $(e_{(1,2),(1,2)}(\tilde{2},1))$ is superfluous.

Similarly, the elimination of q_2 does not introduce other characterizing inequalities.

Thus, we proved the following theorem.

Theorem 4.1 *For a 2-dimensional parametric linear system involving $m+s$ parameters, such that p_μ, $\mu \in M$, $Card(M) = m$, are 2nd class parameters, $x \in \Sigma(A(p), b(p), [p])$ iff*

$$|A(\dot{p})x - b(\dot{p})| \le \sum_{\mu=1}^{m+s} \left| f_\mu(x) \right| \hat{p}_\mu = \sum_{\mu=1}^{m+s} |A_{\bullet\bullet\mu} x - b_{\bullet\mu}| \hat{p}_\mu, \qquad (4.3)$$

$$\left| \Delta_{0,i} + \sum_{\mu=1,\mu\ne i}^{m+s} \Delta_{\mu,i} \dot{p}_\mu \right| \le \sum_{\mu=1,\mu\ne i}^{m+s} |\Delta_{\mu,i}| \hat{p}_\mu, \quad i \in M, \qquad (4.4)$$

where $\Delta_{\alpha,\beta}(x) := f_{\alpha,1}(x) f_{\beta,2}(x) - f_{\alpha,2}(x) f_{\beta,1}(x)$, $f_\alpha(x) = (f_{\alpha,1}(x), f_{\alpha,2}(x))^\top$ is the coefficient vector of the parameter p_α, $\dot{p}_\alpha = (\underline{p}_\alpha + \overline{p}_\alpha)/2$, $\hat{p}_\alpha = (\overline{p}_\alpha - \underline{p}_\alpha)/2$.

Theorem 4.1 gives an explicit unique representation of any 2D parametric solution set and generalizes the Oettli-Prager criterion (1.5) to arbitrary 2D parametric systems.

The order of eliminating the parameters is not important. The theorem can be proven if we first eliminate all 1st class parameters and then eliminate the 2nd class parameters, the proof is given in the Appendix. Thus, we have the following corollary about the order of eliminating 1st and 2nd class parameters.

Corollary 4.1 *The order of eliminating 1st and 2nd class parameters can be exchanged. Whatever the order of parameter elimination, the elimination of 1st class parameters does not generate characterizing cross inequalities.*

Theorem 4.1 clarifies also how complicated a 2D parametric solution set involving an arbitrary number of parameters can be. The shape of a 2D parametric solution set depends on the cross-inequalities (4.4), in particular, on the maximal degree of $\Delta_{\mu,i}(x)$, for $\mu = 0, \ldots, m+s$, $i \in M$, $\mu \ne i$.

Corollary 4.2 *The maximal degree of the polynomials in the inequalities characterizing any 2D parametric solution set is two.*

In some special cases the boundary of the parametric solution set may become linear. Without proving the superfluous and redundant inequalities by Theorem 4.1, the application of Theorem 3.1 to a 2D parametric system could possibly lead to increasing the degree of the polynomials in the solution set characterizing inequalities with the elimination of every 2nd class parameter.

The number of the inequalities describing a 2D parametric solution set grows linearly with the number of the involved 2nd class parameters. Without Theorem 4.1 proving the redundancy of inequalities, the application of Theorem 3.1 to a 2D parametric linear system involving m parameters of 2nd class would generate $O(2^m)$ cross inequalities.

5 Numerical examples

Example 5.1 Consider the following parametric system involving three 2nd class parameters and two 1st class parameters

$$\begin{pmatrix} 2p_1 - 1 & -p_2 \\ p_2 + p_3 & 2p_1 \end{pmatrix} x = \begin{pmatrix} p_3 + b_1 \\ p_3 + b_2 \end{pmatrix},$$

where $p_1 \in [1, 2]$, $p_2 \in [-1, 2]$, $p_3 \in [-2, 2]$, $b_1 \in [0.9, 1.1]$, $b_2 \in [0.9, 1.1]$. By Theorem 4.1 the description of the parametric solution set is

$$\begin{aligned}
&|-1 + 2x_1 - 0.5x_2| \le 2.1 + |x_1| + 1.5|x_2|,\\
&|-1 + 0.5x_1 + 3x_2| \le 0.1 + 2|x_1 - 1| + 1.5|x_1| + |x_2|,\\
&|2x_1 - 2x_2 - 2x_1x_2 + 0.5(-2x_1^2 - 2x_2^2)|\\
&\quad \le 0.2|x_1| + 2|-2(x_1 - 1)x_1 - 2x_2| + 0.2|x_2| + 3|-x_1^2 - x_2^2|,\\
&|-x_1 - x_2 + 3(x_1^2 + x_2^2)| \le 0.1|x_1| + 0.1|x_2| + 2|-x_1 + (x_1 - 1)x_2| + |x_1^2 + x_2^2|,\\
&|-x_1 - (x_1 - 1)x_1 + 3((x_1 - 1)x_1 + x_2) + 0.5(x_1 - (x_1 - 1)x_2)|\\
&\quad \le 0.1 + 0.1|1 - x_1| + 3|(x_1 - 1)x_1 + x_2| + 1.5|x_1 - (x_1 - 1)x_2|.
\end{aligned}$$

Although the system contains three 2nd class parameters, the maximal degree of the polynomials in the inequalities is two. We use the *Mathematica*® (v.6 and above) function `RegionPlot` to visualize the solution set described by the above inequalities. The filled region on Fig. 1 represents the parametric solution set. In order to confirm the obtained results we draw the boundary of the parametric solution set by

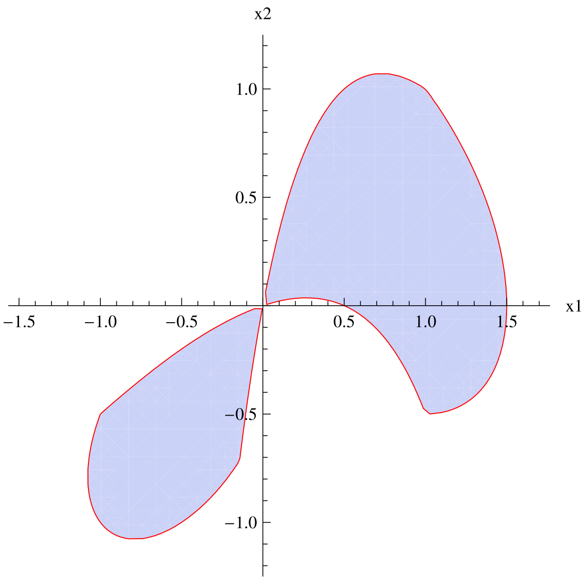

Fig. 1 Solution set of the linear system from Example 5.1

the method of parametric plots developed in [11]. Both the filled region plot and the boundary plot are presented on the same figure. We see that the two methods completely agree.

In the next examples we demonstrate the application of Theorem 4.1 to parametric systems in higher dimensions.

Example 5.2 Let $A \in \mathbb{R}^{3\times 3}$ be a Hankel matrix which is triangular with respect to the counterdiagonal

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad a_i \in [a_i],\ b_i \in [b_i],\ i = 1, 2, 3.$$

Since a_1, b_i, $i = 1, 2, 3$ are 1st class parameters, their elimination will not generate any cross inequalities. The elimination of a_2 generates $e_4 = e_{1,2}(2)$ given below. In the elimination of a_3 by Theorem 2.1 we have to consider the cross inequalities between each two inequality pairs involving a_3. That is $e_{\alpha,\beta}$ for every $\alpha, \beta \in \mathcal{T}$, $\alpha < \beta$, where $\mathcal{T} = \{e_1, \dots, e_4\}$. However, by Theorem 4.1, $e_{1,2}(3, 2)$ is redundant to $e_{1,4}(3, 2)/f_{3,1} \equiv e_{2,4}(3, 2)/f_{3,2} =: e_5$. Thus, the above 3D system with Hankel triangular matrix is described by the set of endpoint inequalities

$$\left|\sum_{i=1}^{3} x_i \dot{a}_i - \dot{b}_1\right| \le \sum_{i=1}^{3} |x_i|\hat{a}_i + \hat{b}_1,$$

$$\left|\sum_{i=2}^{3} x_{i-1} \dot{a}_i - \dot{b}_2\right| \le \sum_{i=2}^{3} |x_{i-1}|\hat{a}_i + \hat{b}_2,$$

$$|x_1\dot{a}_3 - \dot{b}_3| \le |x_1|\hat{a}_3 + \hat{b}_3$$

and the following cross inequalities

$$|x_1^2\dot{a}_1 - x_1\dot{b}_1 + x_2\dot{b}_2 + (x_1x_3 - x_2^2)\dot{a}_3| \le x_1^2\hat{a}_1 + |x_1|\hat{b}_1 + |x_2|\hat{b}_2 + |x_1x_3 - x_2^2|\hat{a}_3, \qquad (e_4)$$

$$|x_1x_2\dot{a}_1 - x_2\dot{b}_1 + x_3\dot{b}_2 + (x_2^2 - x_1x_3)\dot{a}_2| \le |x_1x_2|\hat{a}_1 + |x_2|\hat{b}_1 + |x_3|\hat{b}_2 + |x_1x_3 - x_2^2|\hat{a}_2, \qquad (e_5)$$

$$|x_1^2\dot{a}_1 - x_1\dot{b}_1 + x_3\dot{b}_3 + x_1x_2\dot{a}_2| \le x_1^2\hat{a}_1 + |x_1|\hat{b}_1 + |x_3|\hat{b}_3 + |x_1x_2|\hat{a}_2, \qquad (e_{1,3})$$

$$|x_1^2\dot{a}_2 - x_1\dot{b}_2 + x_2\dot{b}_3| \le x_1^2\hat{a}_2 + |x_1|\hat{b}_2 + |x_2|\hat{b}_3, \qquad (e_{2,3})$$

$$|x_1^3\dot{a}_1 - x_1^2\dot{b}_1 + x_1x_2\dot{b}_2 + (x_1x_3 - x_2^2)b_3| \le |x_1|^3\hat{a}_1 + x_1^2\hat{b}_1 + |x_1x_2|\hat{b}_2 + |x_1x_3 - x_2^2|\hat{b}_3. \qquad (e_{3,4})$$

The application of Theorem 4.1 to systems in higher dimensions will identify more superfluous and redundant inequalities.

Example 5.3 The system

$$\begin{pmatrix} d & 0 & 0 \\ s & d & 0 \\ l & s & d \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad d \in [1, 2],\ s \in [-4, -2],\ l \in [-8, -4],$$

with a Toeplitz matrix is considered in [3], where the solution set

$$1/2 \le x_1 \le 1, \qquad 2x_1^2 \le x_2 \le 4x_1^2, \qquad 4x_1^3 \le x_1x_3 - x_2^2 \le 8x_1^3 \tag{5.1}$$

is found by solving the parametric system. By applying elimination of the parameters and Theorem 4.1, we obtain the following description of the parametric solution set

$$|3x_1/2 - 1| \le |x_1|/2, \tag{e_1}$$

$$|-3x_1 + 3x_2/2| \le |x_1| + |x_2|/2, \tag{e_2}$$

$$|-6x_1 - 3x_2 + 3x_3/2| \le 2|x_1| + |x_2| + |x_3|/2, \tag{e_3}$$

$$|3(x_2^2 - x_1x_3)/2 + 6x_1^2| \le |x_2^2 - x_1x_3|/2 + 2x_1^2, \tag{$e_{2,3}(s)$}$$

$$|-x_2 + 3x_1^2| \le x_1^2, \tag{$e_{1,2}(d)$}$$

$$|x_3 - 6x_1^2 - 3x_1x_2| \le 2x_1^2 + |x_1x_2|, \tag{$e_{1,3}(d)$}$$

$$|x_2^2 - x_1x_3 + 6x_1^3| \le 2x_1^3, \tag{$e_{1,(2,3)}(d,s)$}$$

$$|3(x_2^2 - x_1x_3) + 6x_1x_2| \le |x_2^2 - x_1x_3| + 2|x_1x_2|, \tag{$e_{2,3}(d)$}$$

which is equivalent to (5.1). It is not difficult to see that the boundary of the parametric solution set is formed by e_1, $e_{1,2}(d)$ and $e_{1,(2,3)}(d, s)$.

6 Conclusion

The proposed modified parameter elimination process simplifies the representation of the inequalities describing the solution set. This facilitates the analytic derivations and also the software implementation of the elimination process. The solution set characterization does not depend on the particular orthants. It is presented by fewer inequalities compared to the elimination initially proposed in [3].

The derived explicit description of 2D parametric solution sets involves a minimal number of characterizing inequalities. The number of these inequalities grows linearly with the number of 2nd class parameters involved in the system and does not depend on the number of the involved 1st class parameters. The boundary of any 2D parametric solution set is described by polynomial equations of at most second degree.

Theorem 4.1 implicitly specifies which are the superfluous or redundant inequalities when the general elimination Theorem 3.1 is applied to two parametric equations. Since the elimination of 2nd class parameters from linear systems involving more equations consists of parameter elimination applied to each pair of inequalities containing the parameter, Theorem 4.1 is important for and applicable to arbitrary parametric systems. The above results serve as a background for further investigations on the description of particular or more general parametric solution sets.

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Appendix: Modifying the order of parameter elimination

Consider a 2D parametric linear system. Assume that the first m_1 parameters, $0 \le m_1 < m$, are 1st class parameters which are eliminated from the (trivial) inequalities (3.1) characterizing the parametric solution set. Then these inequalities have the form

$$f_{0,1} + \sum_{\nu\in N_1} f_{\nu,1}\dot{p}_\nu \mp \sum_{\nu\in N_1} |f_{\nu,1}|\dot{p}_\nu + \sum_{\mu=m_1+1}^{m} f_{\mu,1}p_\mu \le 0 \le \cdots,$$

$$f_{0,2} + \sum_{\nu\in N_2} f_{\nu,2}\dot{p}_\nu \mp \sum_{\nu\in N_2} |f_{\nu,2}|\dot{p}_\nu + \sum_{\mu=m_1+1}^{m} f_{\mu,2}p_\mu \le 0 \le \cdots,$$

where $f_\lambda(x) := (f_{\lambda 1}(x), f_{\lambda 2}(x))^\top$, $\lambda = 0,\dots,m$ and $f_{\lambda i}(x) := A_{i\bullet\lambda}x - b_{i\lambda}$, $i = 1,2$.

Eliminating the first 2nd class parameter p_{m_1+1} we get the following cross inequality pair.

$$\begin{aligned}
&f_{0,1}f_{m_1+1,2} - f_{0,2}f_{m_1+1,1} + \sum_{\nu=1}^{m_1}(f_{\nu,1}f_{m_1+1,2} - f_{\nu,2}f_{m_1+1,1})\dot{p}_\nu \\
&\quad \mp \sum_{\nu=1}^{m_1}(|f_{\nu,1}f_{m_1+1,2}| + |f_{\nu,2}f_{m_1+1,1}|)\hat{p}_\nu \\
&\quad + \sum_{\mu=m_1+2}^{m} (f_{\mu,1}f_{m_1+1,2} - f_{\mu,2}f_{m_1+1,1})p_\mu \le 0 \le \cdots,
\end{aligned}$$

that is

$$\Delta_{0,m_1+1} + \sum_{\nu=1}^{m_1} \Delta_{\nu,m_1+1}\dot{p}_\nu \mp \sum_{\nu=1}^{m_1} |\delta\Delta_{\nu,m_1+1}|\hat{p}_\nu + \sum_{\mu=m_1+2}^{m} \Delta_{\mu,m_1+1}p_\mu \le 0 \le \cdots, \qquad (e_{1,2}(1))$$

where $\delta = \{1 \text{ if } \nu \in N_1,\ -1 \text{ if } \nu \in N_2\}$.

Eliminating p_{m_1+2} we get three more cross inequality pairs. The first one is

$$\Delta_{0,m_1+2} + \sum_{\nu=1}^{m_1+1} \Delta_{\nu,m_1+2}\dot{p}_\nu \mp \sum_{\nu=1}^{m_1} |\delta\Delta_{\nu,m_1+2}|\hat{p}_\nu$$

$$\mp (|f_{m_1+1,1} f_{m_1+2,2}| + |f_{m_1+1,2} f_{m_1+2,1}|)\hat{p}_{m_1+1} + \sum_{\mu=m_1+3}^{m} \Delta_{\mu,m_1+2} p_\mu \le 0 \le \cdots, \qquad (e_{1,2}(2))$$

which can be written as a single absolute value inequality

$$\left|\Delta_{0,m_1+2} + \sum_{\mu=1,\mu\neq m_1+2}^{m} \Delta_{\mu,m_1+2}\dot{p}_\mu\right| \le \sum_{\nu\in N_1\cup N_2} |\delta\Delta_{\nu,m_1+2}|\hat{p}_\nu + (|f_{m_1+1,1} f_{m_1+2,2}| + |f_{m_1+1,2} f_{m_1+2,1}|)\hat{p}_{m_1+1} + \sum_{\mu=m_1+3}^{m} |\Delta_{\mu,m_1+2}|\hat{p}_\mu. \qquad (e_{1,2}(2))$$

And from the inequalities

$$f_{0,2} + \sum_{\nu\in N_2} f_{\nu,2}\dot{p}_\nu \mp \sum_{\nu\in N_2} |f_{\nu,2}|\hat{p}_\nu + f_{m_1+1,2}\dot{p}_{m_1+1} \mp |f_{m_1+1,2}|\hat{p}_{m_1+1} + \sum_{\mu=m_1+2}^{m} f_{\mu,2} p_\mu \le 0 \le \cdots,$$

$$\Delta_{0,m_1+1} + \sum_{\nu=1}^{m_1} \Delta_{\nu,m_1+1}\dot{p}_\nu \mp \sum_{\nu=1}^{m_1} |\delta\Delta_{\nu,m_1+1}|\hat{p}_\nu + \sum_{\mu=m_1+2}^{m} \Delta_{\mu,m_1+1} p_\mu \le 0 \le \cdots$$

we obtain the following cross inequality pair

$$f_{0,2}\Delta_{m_1+2,m_1+1} - \Delta_{0,m_1+1} f_{m_1+2,2} - \sum_{\nu\in N_1} \Delta_{\nu,m_1+1} f_{m_1+2,2}\dot{p}_\nu \mp \sum_{\nu\in N_1} |\Delta_{\nu,m_1+1} f_{m_1+2,2}|\hat{p}_\nu + \sum_{\nu\in N_2} (f_{\nu,2}\Delta_{m_1+2,m_1+1} - \Delta_{\nu,m_1+1} f_{m_1+2,2})\dot{p}_\nu \mp \sum_{\nu\in N_2} (|f_{\nu,2}\Delta_{m_1+2,m_1+1}| + |-\Delta_{\nu,m_1+1} f_{m_1+2,2}|)\hat{p}_\nu + f_{m_1+1,2}\Delta_{m_1+2,m_1+1}\dot{p}_{m_1+1} \mp |f_{m_1+1,2}\Delta_{m_1+2,m_1+1}|\hat{p}_{m_1+1} + \sum_{\mu=m_1+3}^{m} (f_{\mu,2}\Delta_{m_1+2,m_1+1} - \Delta_{\mu,m_1+1} f_{m_1+2,2}) p_\mu \le 0 \le \cdots. \qquad (e_{2,(1,2)}(2,1))$$

After simplifying the expressions and dividing by $f_{m_1+1,2}$ the last cross inequality pair, written as one absolute value inequality, is

$$\left|-\Delta_{0,m_1+2} - \sum_{\mu=1,\mu\neq m_1+2}^{m} \Delta_{\mu,m_1+2}\dot{p}_\mu\right|$$
$$\leq \sum_{\nu\in N_1} |\Delta_{\nu,m_1+2}|\hat{p}_\nu + \sum_{\mu=m_1+1,\mu\neq m_1+2}^{m} |\Delta_{\mu,m_1+2}|\hat{p}_\mu$$
$$+ \sum_{\nu\in N_2} |f_{\nu,2}|(|\Delta_{m_1+2,m_1+1}| + |f_{m_1+1,1}f_{m_1+2,2}|)/|f_{m_1+1,2}|\hat{p}_\nu. \qquad (e_{2,(1,2)}(2,1))$$

Now, we compare $(e_{2,(1,2)}(2,1))$ and $(e_{1,2}(2))$. The left sides of these inequalities are the same since $|a| = |-a|$ for any a. So, compare the right sides of these inequalities. We shall prove that instead of $(e_{2,(1,2)}(2,1))$ and $(e_{1,2}(2))$ we can consider

$$\left|\Delta_{0,m_1+2} + \sum_{\mu=1,\mu\neq m_1+2}^{m} \Delta_{\mu,m_1+2}\dot{p}_\mu\right| \leq \sum_{\mu=1,\mu\neq m_1+2}^{m} |\Delta_{\mu,m_1+2}|\hat{p}_\mu. \qquad (e_{1,2}(\tilde{2}))$$

Denote $s_{12} := \operatorname{sign}(f_{m_1+1,2}f_{m_1+2,1})$, $s_{11} := \operatorname{sign}(f_{m_1+1,1}f_{m_1+2,2})$, $s_{\Delta_{2,1}} := \operatorname{sign}(\Delta_{m_1+2,m_1+1})$.

– If $s_{12} \neq s_{11}$, then $|\Delta_{m_1+1,m_1+2}| = |f_{m_1+1,1}f_{m_1+2,2}| + |f_{m_1+1,2}f_{m_1+2,1}|$ and since

$$|\Delta_{m_1+2,m_1+1}| + |f_{m_1+1,1}f_{m_1+2,2}| \geq |\Delta_{m_1+2,m_1+1} + f_{m_1+1,1}f_{m_1+2,2}|$$
$$= |f_{m_1+1,2}f_{m_1+2,1}|,$$
$$|f_{\nu,2}|(|\Delta_{m_1+2,m_1+1}| + |f_{m_1+1,1}f_{m_1+2,2}|)/|f_{m_1+1,2}|$$
$$\geq |f_{\nu,2}||f_{m_1+1,2}f_{m_1+2,1}|/|f_{m_1+1,2}| = |-\Delta_{\nu,m_1+2}|$$

  for every $\nu \in N_2$. Thus, we consider $(e_{1,2}(2))$ which is equivalent to $(e_{1,2}(\tilde{2}))$.
– If $s_{12} = s_{11}$ and $s_{\Delta_{2,1}} = s_{11}$, then we consider $(e_{2,(1,2)}(2,1))$, where

$$|\Delta_{m_1+2,m_1+1}| + |f_{m_1+1,1}f_{m_1+2,2}| = |\Delta_{m_1+2,m_1+1} + f_{m_1+1,1}f_{m_1+2,2}|$$

  and $|\Delta_{m_1+1,m_1+2}| \leq |f_{m_1+1,1}f_{m_1+2,2}| + |f_{m_1+1,2}f_{m_1+2,1}|$.
– The case $s_{12} = s_{11}$ and $s_{\Delta_{2,1}} = s_{11}$ follows from the first case above since $s_{\Delta_{2,1}} = -s_{\Delta_{1,2}}$.

Similarly we prove that instead of $(e_{1,(1,2)}(2,1))$ and $(e_{1,2}(2))$ we can consider $(e_{1,2}(\tilde{2}))$.

By induction on the number of 2nd class parameters we prove that the order of eliminating 1st and 2nd class parameters can be exchanged. Moreover, any order of eliminating the parameters will generate the same system of characterizing inequalities.

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