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Convergence and stability of a stabilized finite volume method for the stationary Navier-Stokes equations

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Abstract In this paper, a new stabilized finite volume method is studied and developed for the stationary Navier-Stokes equations. This method is based on a local Gauss integration technique and uses the lowest equal order finite element pair P_1–P_1 (linear functions). Stability and convergence of the optimal order in the H^1-norm for velocity and the L^2-norm for pressure are obtained. A new duality for the Navier-Stokes equations is introduced to establish the convergence of the optimal order in the L^2-norm for velocity. Moreover, superconvergence between the conforming mixed finite element solution and the finite volume solution using the same finite element pair is derived. Numerical results are shown to support the developed convergence theory.

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1 Introduction

Stabilized mixed finite element methods using the equal order finite element pairs for the Stokes equations have recently received considerable attention. While they do not satisfy the inf-sup (LBB) stability conditions, these elements offer simple and practical uniform data structure and adequate accuracy. This stabilization technique of these methods use a local pressure projection [5, 13, 22]. Stabilizations of this type for the Stokes equations have been extensively studied in the past decades [4, 6, 26]. Recent studies have focused on stabilization of the lowest equal-order finite element pair P_1–P_1 (linear functions) or Q_1–Q_1 (bilinear functions) using the projection of the pressure onto the piecewise constant space [5, 22]. This stabilization technique is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and therefore can be implemented at the element level [5, 22, 24].

In this paper we consider an extension of this stabilized mixed finite element method to a finite volume method for the stationary Navier-Stokes equations. The finite volume method is intuitive in that it is directly based on local conservation of mass, momentum, or energy over volumes (control volumes or co-volumes). It lies somewhere between the finite element and the finite difference methods and has the flexibility similar to that of the finite element method for handling complicated geometries. Implementation is comparable to that of the finite difference method. The finite volume method is also referred to as the control volume method, the co-volume method, or the first order generalized difference method [7, 9–11, 14, 29]. Its theoretical analysis is much more complex than that of the finite element method.

Here we develop and study a finite volume method for the Navier-Stokes equations. This method is developed through its relationship with the conforming finite element method using the lowest equal-order element pair P_1–P_1. This pair of conforming elements does not satisfy the discrete inf-sup stability condition for these equations so the local pressure projection technique proposed in [5, 22] is therefore employed. Although the analysis of a finite volume method for the Stokes equations is established [21, 25], the analysis for the Navier-Stokes equations must take special care of the nonlinear discrete terms arising from the finite volume discretization; these nonlinear trilinear terms no longer satisfy the anti-symmetry properties. Therefore, compared to the finite element analysis, the most challenging aspect rests in the treatment of the nonlinear convection terms, which has a significant impact on the analysis, especially on the L^2-norm estimate for velocity. Up to this point, there is no result available for the optimal conversion rate for velocity in the L^2-norm for the stationary Navier-Stokes equations. Here, a new duality argument is introduced to establish the convergence of optimal order in this norm for the velocity of the nonlinear Navier-Stokes flow.

Existence and uniqueness of a solution to the finite volume approximation of the stationary Navier-Stokes equations is hereby presented in this paper. Moreover, stability and convergence of the optimal order in the H^1- and L^2-norm for velocity and the L^2-norm for pressure are obtained. For the first time, the convergence analysis further demonstrates an important superconvergence resulting from the conforming mixed finite element solution and the finite volume solution using the same finite element pair for the incompressible flow. Numerical calculations presented here support these convergence and superconvergence results.

This paper is organized as follows: In the next section, we introduce notation and the stationary Navier-Stokes equations. Then, in the third section, a stabilized finite element method for the Navier-Stokes equations is recalled. In the fourth section, a new stabilized finite volume method for these equations is developed. Stability and optimal order error estimates for this finite volume method are obtained in the fifth and sixth sections, respectively. Numerical calculations are presented in the seventh section. Finally, some conclusions are drawn in the last section.

2 Preliminaries

In this paper we focus on two dimensions. Let Ω be a bounded and convex domain in $\Re^2$, assumed to have a Lipschitz-continuous boundary Γ. The stationary Navier-Stokes equations are

$$-\nu\Delta u+\nabla p+(u\cdot\nabla)u+\frac{1}{2}(\operatorname{div} u)u=f \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} u=0 \quad \text{in } \Omega, \tag{2.2}$$

$$u=0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where $u=(u_1(x_1,x_2),u_2(x_1,x_2))$ represents the velocity vector, $p=p(x_1,x_2)$ the pressure, $f=f(x_1,x_2)$ the prescribed body force, and $\nu>0$ the viscosity. Note that the term $(\operatorname{div} u)u/2$ is added to ensure the dissipativity of the Navier-Stokes equations [27].

Set

$$X=\left(H_0^1(\Omega)\right)^2, \qquad Y=\left(L^2(\Omega)\right)^2, \qquad M=\left\{q\in L^2(\Omega):\int_\Omega q\,dx=0\right\}.$$

The spaces $(L^2(\Omega))^m$, $m=1,2$, or 4, are endowed with the L^2-scalar product $(\cdot,\cdot)$ and the L^2-norm $\|\cdot\|_0$, as appropriate. The space X is equipped with the usual scalar product $(\nabla u,\nabla v)$ and the norm $|\cdot|_1$. Standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$, with the norm $\|\cdot\|_{m,r}$ and the seminorm $|\cdot|_{m,r}$, $m,r\ge 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_m$ for $\|\cdot\|_{m,2}$.

The following inequalities will be used [1]:

$$\|v\|_{0,4}\le C_0\|v\|_0^{1/2}\|v\|_1^{1/2}, \qquad \|v\|_0\le C_1|v|_1 \quad \forall v\in X, \tag{2.4}$$

$$\|\nabla v\|_{0,4} \le C_0 \|v\|_1^{1/2} \|v\|_2^{1/2}, \qquad \|v\|_\infty \le C_2 \|v\|_0^{1/2} \|v\|_2^{1/2} \quad \forall v \in X \cap \left(H^2(\Omega)\right)^2. \tag{2.5}$$

The generic positive constant C (with or without a subscript) depends only on Ω. Subsequently, C will denote a generic positive constant depending at most on the data Ω, ν, and f.

We introduce the bilinear forms

$$a(u,v) = \nu(\nabla u, \nabla v), \quad u, v \in X,$$
$$d(v,p) = (\operatorname{div} v, p), \quad v \in X, \ p \in M,$$

and the trilinear form

$$b(u; v, w) = ((u \cdot \nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w)$$
$$= \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v), \quad u, v, w \in X. \tag{2.6}$$

Then the weak formulation of (2.1)–(2.3) is to find $(u, p) \in (X, M)$ such that

$$a(u,v) - d(v,p) + d(u,q) + b(u; u, v) = (f, v) \quad \forall (v,q) \in (X, M). \tag{2.7}$$

The trilinear form $b(\cdot; \cdot, \cdot)$ satisfies [17, 24, 27]:

$$b(u; v, w) = -b(u; w, v) \quad \forall u, v, w \in X, \tag{2.8}$$
$$|b(u; v, w)| \le C_3 \|u\|_1 \|v\|_1 \|w\|_1 \quad \forall u, v, w \in X, \tag{2.9}$$

and

$$|b(u; v, w)| + |b(v; u, w)| + |b(w; u, v)| \le C_3 \|u\|_1 \|v\|_2 \|w\|_0$$
$$\forall u \in X, v \in X \cap \left(H^2(\Omega)\right)^2, w \in Y. \tag{2.10}$$

Moreover, the bilinear form $d(\cdot, \cdot)$ satisfies the *inf-sup* condition [8, 15] for all $q \in M$

$$\sup_{v \in X} \frac{|d(v,q)|}{\|v\|_1} \ge \beta_1 \|q\|_0, \tag{2.11}$$

where β_1 is a positive constant depending only on Ω.

The bilinear form $\mathcal{B}$ on $(X, M) \times (X, M)$ associated with the Stokes equations will be used:

$$\mathcal{B}((u,p),(v,q)) = a(u,v) - d(v,p) + d(u,q), \quad (u,p),(v,q) \in (X, M).$$

It satisfies the continuity property and *inf-sup* condition [27]

$$|\mathcal{B}((u,p),(v,q))| \le C(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0), \quad (u,p),(v,q) \in (X, M), \tag{2.12}$$

$$\sup_{(v,q)\in(X,M)} \frac{|\mathcal{B}((u,p),(v,q))|}{\|v\|_1+\|q\|_0} \ge \beta_2(\|u\|_1+\|p\|_0), \quad (u,p)\in(X,M), \tag{2.13}$$

where $\beta_2 > 0$ depends only on Ω and ν.

Using this bilinear form, the variational formulation of the stationary Navier-Stokes equations can be rewritten: Find $(u,p)\in(X,M)$ such that

$$\mathcal{B}((u,p),(v,q)) + b(u;u,v) = (f,v) \quad \forall(v,q)\in(X,M). \tag{2.14}$$

Detailed results on existence and uniqueness of a solution to (2.1)–(2.3) are known [15, 27]. In particular, we state the next theorem.

Theorem 2.1 *If $\nu > 0$ and $f \in Y$ satisfy*

$$1 - \frac{C_1C_3}{\nu^2}\|f\|_0 > 0, \tag{2.15}$$

then the variational problem (2.14) admits a unique solution $(u,p)\in(D(A), H^1(\Omega)\cap M)$ satisfying

$$\|u\|_1 \le \frac{C_1}{\nu}\|f\|_0, \qquad \|u\|_2+\|p\|_1 \le C\|f\|_0, \tag{2.16}$$

where the positive constants C_1 and C_3 are given by (2.4) and (2.9), and

$$D(A) = (H^2(\Omega))^2 \cap V, \quad V = \{v\in X : \operatorname{div} v = 0\}.$$

In general, (2.14) has more than one solution. Uniqueness is guaranteed if the viscosity and body force satisfy (2.15).

3 Stabilized finite element method

Let K_h be a regular, quasi-uniform triangulation of the polygonal domain Ω into a union of triangles [8, 12]. We consider the finite element spaces

$$X_h = \left\{v=(v_1,v_2)\in X\cap C^0(\Omega)^2 : v_i|_K \in P_1(K), \quad i=1,2,\ K\in K_h\right\},$$

$$M_h = \left\{q\in M\cap C^0(\Omega) : q|_K\in P_1(K),\ K\in K_h\right\},$$

where $P_1(K)$ represents the space of linear functions on the set K.

Let I_h and J_h be two interpolation operators from $X\cap(C^0(\bar{\Omega}))^2$ and M into X_h and M_h, respectively, such that for $v\in X\cap(H^2(\Omega))^2$ and $q\in H^1(\Omega)\cap M$,

$$\|v-I_hv\|_i \le Ch^{2-i}|v|_2, \qquad \|q-J_hq\|_0 \le Ch|q|_1, \quad i=0,1. \tag{3.1}$$

In particular, the interpolation operator I_h satisfies

$$\|I_hv\|_1 \le C\|v\|_1. \tag{3.2}$$

Due to the quasi-uniformness of the triangulation K_h, the following properties hold [12, 27]:

$$\|v_h\|_1 \le C_4 h^{-1}\|v_h\|_0, \qquad \|v_h\|_\infty \le C_5|\log h|^{1/2}\|v_h\|_1 \quad \forall v_h \in X_h. \tag{3.3}$$

We now introduce a discrete analogue A_h of the Laplace operator A through the condition [19]

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \quad u_h, v_h \in X_h.$$

Define

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0 \ \forall q_h \in M_h\}.$$

The restriction of A_h to V_h is invertible, with the inverse A_h^{-1}. In addition, A_h is self-adjoint and positive definite. Moreover, we define the discrete Sobolev norm of $A_h^{1/2}$ on V_h by

$$\|v_h\|_{r,h} = \|A_h^{r/2} v_h\|_0, \quad v_h \in V_h.$$

The basic idea of a discrete analogue A_h is derived from [19]. Furthermore, (2.5) and (3.3) still hold.

The lowest equal-order pair $X_h \times M_h$ of the conforming finite elements does not satisfy the discrete *inf-sup* condition

$$\sup_{0\ne v_h\in X_h} \frac{d(v_h, q_h)}{\|v_h\|_1} \ge \beta_2\|q_h\|_0, \quad q_h \in M_h, \tag{3.4}$$

where the constant $\beta_2 > 0$ is independent of h. In order to fulfill this condition, the local Gauss integration term is used [21, 22, 24]:

$$G_h(p_h, q_h) = \sum_{K\in K_h} \left\{ \int_{K,2} p_h q_h \, dx - \int_{K,1} p_h q_h \, dx \right\}, \quad p_h, q_h \in M_h, \tag{3.5}$$

where $\int_{K,i} g(x)\, dx$ indicates an appropriate Gauss integral over K that is exact for polynomials of degree i, $i = 1, 2$, and $g(x) = p_h q_h$ is a polynomial of degree not greater than two. In particular, the trial function $p_h \in M_h$ must be projected to piecewise constant space W_h defined below when $i = 1$ for any $q_h \in M_h$. Consequently, we define the L^2-projection operator $\Pi_h : L^2(\Omega) \to W_h$:

$$(p, q_h) = (\Pi_h p, q_h) \quad \forall p \in L^2(\Omega), \ q_h \in W_h, \tag{3.6}$$

where $W_h \subset L^2(\Omega)$ denotes the piecewise constant space associated with the triangulation K_h. The following properties of the projection operator Π_h can be proved [12]:

$$\|\Pi_h p\|_0 \le C\|p\|_0 \quad \forall p \in L^2(\Omega), \tag{3.7}$$

$$\|p - \Pi_h p\|_0 \le Ch\|p\|_1 \quad \forall p \in H^1(\Omega). \tag{3.8}$$

The L^2-projection operator Π_h can be extended to the vector case.

As a result of (3.6), the bilinear form $G_h(\cdot,\cdot)$ can be expressed as

$$G_h(p_h, q_h) = (p_h - \Pi_h p_h, q_h) = (p_h - \Pi_h p_h, q_h - \Pi_h q_h), \quad p_h, q_h \in M_h. \tag{3.9}$$

Now, the bilinear form of the finite element method on $(X_h, M_h) \times (X_h, M_h)$ is given by

$$\mathcal{B}_h((u_h, p_h), (v_h, q_h)) = a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + G_h(p_h, q_h),$$
$$(u_h, p_h), (v_h, q_h) \in (X_h, M_h). \tag{3.10}$$

This bilinear form satisfies the continuity and weak coercivity properties [22]:

$$|\mathcal{B}_h((u_h, p_h), (v_h, q_h))| \le C\left(\|u_h\|_1 + \|p_h\|_0\right)\left(\|v_h\|_1 + \|q_h\|_0\right), \tag{3.11}$$

$$\sup_{(v_h, q_h) \in (X_h, M_h)} \frac{|\mathcal{B}_h((u_h, p_h), (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0} \ge \beta_3\left(\|u_h\|_1 + \|p_h\|_0\right), \tag{3.12}$$

where $\beta_3 > 0$ is independent of h.

The corresponding discrete variational formulation of (2.14) for the Navier–Stokes equations is recast: Find $(u_h, p_h) \in (X_h, M_h)$ such that

$$\mathcal{B}_h((u_h, p_h), (v_h, q_h)) + b(u_h; u_h, v_h) = (f, v_h) \quad \forall (v_h, q_h) \in (X_h, M_h). \tag{3.13}$$

Because of the properties (3.11) and (3.12), system (3.13) can be shown to have a unique solution [18]. Moreover, the optimal error estimate for the finite element solution (u_h, p_h) holds for sufficiently small $h > 0$ [17]:

$$\|u - u_h\|_0 + h\left(\|u - u_h\|_1 + \|p - p_h\|_0\right) \le Ch^2\left(\|u\|_2 + \|p\|_1 + \|f\|_0\right). \tag{3.14}$$

4 Stabilized finite volume method

The main purpose in this paper is to develop a stabilized finite volume method for the Navier-Stokes equations (2.1)–(2.3) and establish its stability and convergence results.

Let $\mathcal{P}$ be the set containing all the interior nodes associated with the triangulation K_h, and N be the total number of the nodes. To define the finite volume method, based on K_h, a dual mesh $\tilde{K}_h$ is introduced; the elements in $\tilde{K}_h$ are called control volumes. The dual mesh can be constructed by the following rule: For each element $K \in K_h$ with vertices P_j, $j = 1, 2, 3$, select its barycenter O and the midpoint M_j on each of the edges of K, and then construct the control volumes in $\tilde{K}_h$ by connecting O to M_j as shown in Fig. 1.

The dual finite element space is defined as

$$\tilde{X}_h = \left\{\tilde{v} \in \left(L^2(\Omega)\right)^2 : \tilde{v}|_{\tilde{K}} \in P_0(\tilde{K})\ \forall \tilde{K} \in \tilde{K}_h;\ \tilde{v}|_{\partial\tilde{K}} = 0\right\}.$$

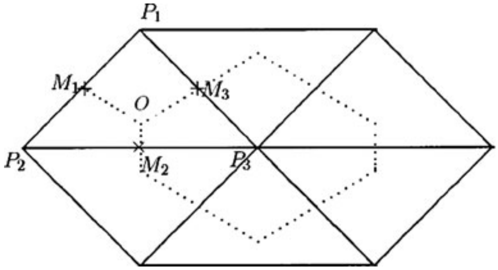

Fig. 1 Control volumes associated with triangles

Clearly, the dimensions of X_h and $\tilde{X}_h$ are the same. Furthermore, there exists an invertible linear mapping $\Gamma_h : X_h \to \tilde{X}_h$ such that for

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\phi_j(x), \quad x \in \Omega,\ v_h \in X_h, \tag{4.1}$$

we have

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \quad x \in \Omega,\ v_h \in X_h, \tag{4.2}$$

where $\{\phi_j\}$ indicates the basis for the finite element space X_h and $\{\chi_j\}$ denotes the basis for the finite volume space $\tilde{X}_h$ that are the characteristic functions associated with the dual partition $\tilde{K}_h$:

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in \tilde{K}_j \in \tilde{K}_h, \\ 0 & \text{otherwise.} \end{cases}$$

The above idea of connecting the trial and test spaces in different finite dimensional spaces through the mapping Γ_h was first introduced in [3, 20] in the context of elliptic problems. Furthermore, the mapping Γ_h satisfies the following properties [28]:

Lemma 4.1 *Let $K \in K_h$. If $v_h \in X_h$ and $1 \le r \le \infty$, then*

$$\int_K (v_h - \Gamma_h v_h)dx = 0, \tag{4.3}$$

$$\|v_h - \Gamma_h v_h\|_{0,r,K} \le C_6 h_K \|v_h\|_{1,r,K}, \tag{4.4}$$

$$\|\Gamma_h v_h\|_0 \le C_7 \|v_h\|_0, \tag{4.5}$$

where h_K is the diameter of the element K.

Multiplying (2.1) by $\Gamma_h v_h \in \tilde{X}_h$ and integrating over the dual elements $\tilde{K} \in \tilde{K}_h$, multiplying (2.2) by $q_h \in M_h$ and integrating over the primal elements $K \in K_h$, and applying Green's formula for both equations, we obtain the following bilinear forms for the finite volume method:

$$A(u_h, \Gamma_h v_h) = -\nu \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} \frac{\partial u_h}{\partial n} ds, \quad u_h,\ v_h \in X_h,$$

$$D(\Gamma_h v_h, p_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial \tilde{K}_j} p_h n\, ds, \quad v_h \in X_h,\ p_h \in M_h,$$

$$(f, \Gamma_h v_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f\, dx, \quad v_h \in X_h,$$

where n is the unit normal outward to $\partial \tilde{K}_j$. Using a technique similar to the trilinear form of the finite element method in the previous section, we define the trilinear form $b(\cdot;\cdot,\cdot) : X_h \times X_h \times \tilde{X}_h \to \Re$ of the finite volume method [16, 27]:

$$b(u_h; v_h, \Gamma_h w_h) = \left((u_h \cdot \nabla) v_h + \frac{1}{2} (\operatorname{div} u_h) v_h, \Gamma_h w_h \right) \quad \forall u_h, v_h, w_h \in X_h.$$

Now, the new stabilized finite volume method for the Navier-Stokes equations (2.1)–(2.3) is: Find $(\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h)$ such that

$$\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (v_h, q_h)) + b(\tilde{u_h}; \tilde{u_h}, \Gamma_h v_h) = (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h), \tag{4.6}$$

where we defined the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ on $(X_h, M_h) \times (X_h, M_h)$:

$$\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (v_h, q_h)) = A(\tilde{u_h}, \Gamma_h v_h) + D(\Gamma_h v_h, \tilde{p}_h) + d(\tilde{u}_h, q_h) + G_h(\tilde{p}_h, q_h). \tag{4.7}$$

The next lemma establishes a relationship between the finite element and finite volume methods [21, 29].

Lemma 4.2 *It holds that*

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h) \quad \forall u_h, v_h \in X_h, \tag{4.8}$$

with the following properties:

$$A(u_h, \Gamma_h v_h) = A(v_h, \Gamma_h u_h), \tag{4.9}$$

$$|A(u_h, \Gamma_h v_h)| \le C \|u_h\|_1 \|v_h\|_1, \tag{4.10}$$

$$|A(v_h, \Gamma_h v_h)| \ge C \|v_h\|_1^2. \tag{4.11}$$

Moreover, the bilinear form $D(\cdot,\cdot)$ satisfies

$$D(\Gamma_h v_h, q_h) = -d(v_h, q_h) \quad \forall (v_h, q_h) \in (X_h, M_h). \tag{4.12}$$

5 Stability

To establish existence and uniqueness of a solution to the finite volume method (4.6), the following result on the continuity and weak coercivity of the bilinear form $\mathcal{C}_h(\cdot,\cdot)$ will be used [21]:

Lemma 5.1 *It holds that*

$$|\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (v_h, q_h))| \le C\,(\|\tilde{u}_h\|_1 + \|\tilde{p}_h\|_0)\,(\|v_h\|_1 + \|q_h\|_0)$$
$$\forall (\tilde{u}_h, \tilde{p}_h),\ (v_h, q_h) \in (X_h, M_h). \tag{5.1}$$

Moreover,

$$\sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0} \ge \beta_4\,(\|\tilde{u}_h\|_1 + \|\tilde{p}_h\|_0)$$
$$\forall (\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h), \tag{5.2}$$

where $\beta_4 > 0$ is independent of h.

We are now in a position to show the well posedness of system (4.6). For this we define the mesh parameter

$$h_0(h) = \frac{4C_1C_5C_6C_7}{\nu^2}|\log h|^{1/2} h\|f\|_0.$$

In the following theorem, to prove existence of a solution to (4.6), the main idea is to linearize the convective term and apply the Brouwer fixed point theorem.

Theorem 5.2 *For each $h > 0$ such that*

$$0 < h_0 \le 1/2, \tag{5.3}$$

system (4.6) admits a solution $(\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h)$. Moreover, if the viscosity $\nu > 0$, the body force $f \in Y$, and the mesh size $h > 0$ satisfy

$$0 < h_0 \le \frac{1}{4}, \qquad 1 - \frac{4C_1C_3C_7}{\nu^2}\|f\|_0 > 0, \tag{5.4}$$

then the solution $(\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h)$ is unique. Furthermore, it satisfies

$$\|\tilde{u}_h\|_1 \le \frac{2C_1C_7}{\nu}\|f\|_0,$$
$$\|\tilde{p}_h\|_0 \le 2\beta_4^{-1}C_1C_7\|f\|_0\left(1 + \frac{2C_1C_3C_7}{\nu^2}\|f\|_0\right), \tag{5.5}$$
$$\|A_h\tilde{u}_h\|_0 \le \frac{2C_7}{\nu}\|f\|_0\left(1 + \frac{2^5C_1^4C_2^2C_7^4}{\nu^4}\|f\|_0^2\right).$$

Proof For fixed $f \in Y$, we introduce the set

$$B_M = \left\{ (\tilde{v}_h, \tilde{q}_h) \in (X_h, M_h) : \|\tilde{v}_h\|_1 \le \frac{2C_1C_7}{\nu}\|f\|_0, \right.$$
$$\left. \|\tilde{p}_h\|_0 \le 2\beta_4^{-1} C_1C_7\|f\|_0 \left(1 + \frac{2C_1C_3C_7}{\nu^2}\|f\|_0\right) \right\}.$$

Then we define the mapping $T_h : (X_h, M_h) \to (X_h, M_h)$ by

$$\mathcal{C}_h((T_h\tilde{v}_h, \tilde{p}_h), (v_h, q_h)) + b(\tilde{v}_h; T_h\tilde{v}_h, \Gamma_h v_h) = (f, \Gamma_h v_h), \quad (v_h, q_h) \in (X_h, M_h), \tag{5.6}$$

where $T_h(\tilde{v}_h, \tilde{p}_h) \to (T_h\tilde{v}_h, \tilde{p}_h) = (\tilde{u}_h, \tilde{p}_h)$. We will prove that T_h maps B_M into B_M.

First, taking $(v_h, q_h) = (\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h)$ in (5.6), we see that

$$\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (\tilde{u}_h, \tilde{p}_h)) + b(\tilde{v}_h; \tilde{u}_h, \Gamma_h\tilde{u}_h - \tilde{u}_h) + b(\tilde{v}_h; \tilde{u}_h, \tilde{u}_h) = (f, \Gamma_h\tilde{u}_h). \tag{5.7}$$

By using the definition of $b(\cdot;\cdot,\cdot)$ and $\mathcal{C}_h(\cdot,\cdot)$, (2.4), (3.3), Theorem 2.1, Lemmas 4.1–4.2, and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&|\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (\tilde{u}_h, \tilde{p}_h))| \ge \nu\|\tilde{u}_h\|_1^2, \\
&|b(\tilde{v}_h; \tilde{u}_h, \tilde{u}_h)| = 0, \\
&|b(\tilde{v}_h; \tilde{u}_h, \Gamma_h\tilde{u}_h - \tilde{u}_h)| \le \left(\|\tilde{v}_h\|_{L^\infty}\|\tilde{u}_h\|_1 + \frac{\sqrt{2}}{2}\|\tilde{v}_h\|_1\|\tilde{u}_h\|_{L^\infty}\right)\|\Gamma_h\tilde{u}_h - \tilde{u}_h\|_0 \\
&\qquad \le 2C_5C_6|\log h|^{1/2}h\|\tilde{v}_h\|_1\|\tilde{u}_h\|_1^2 \\
&\qquad \le \frac{4C_1C_5C_6C_7}{\nu}|\log h|^{1/2}h\|f\|_0\|\tilde{u}_h\|_1^2 \\
&\qquad \le \nu h_0\|\tilde{u}_h\|_1^2, \\
&|(f, \Gamma_h\tilde{u}_h)| \le \|f\|_0\|\Gamma_h\tilde{u}_h\|_0 \le C_1C_7\|f\|_0\|\tilde{u}_h\|_1,
\end{aligned} \tag{5.8}$$

which, together with (5.7), gives

$$\nu(1 - h_0)\|\tilde{u}_h\|_1 \le C_1C_7\|f\|_0.$$

Using (5.3), it holds

$$\|\tilde{u}_h\|_1 \le \frac{2C_1C_7}{\nu}\|f\|_0. \tag{5.9}$$

In view of (5.2), (5.3), (5.6), and (5.9), we see that

$$\begin{aligned}
\|\tilde{u}_h\|_1 + \|\tilde{p}_h\|_0 &\le \beta_4^{-1} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{C}_h((\tilde{u}_h, \tilde{p}_h), (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0} \\
&\le \beta_4^{-1}(\nu h_0\|\tilde{u}_h\|_1 + C_3\|\tilde{v}_h\|_1\|\tilde{u}_h\|_1 + C_1C_7\|f\|_0) \\
&\le 2\beta_4^{-1}C_1C_7\|f\|_0\left(1 + \frac{2C_1C_3C_7\|f\|_0}{\nu^2}\right).
\end{aligned} \tag{5.10}$$

Since the mapping T_h is well defined, it follows from Brouwer's fixed point theorem that there exists a solution to system (4.6).

To prove uniqueness, assume that $(\tilde{u}_1, \tilde{p}_1)$ and $(\tilde{u}_2, \tilde{p}_2)$ are two solutions to (4.6). Then we see that

$$\begin{aligned}&\mathcal{C}_h((\tilde{u}_1-\tilde{u}_2, \tilde{p}_1-\tilde{p}_2), (v_h, q_h)) + b(\tilde{u}_1-\tilde{u}_2; \tilde{u}_1, \Gamma_h v_h)\\&\quad + b(\tilde{u}_2; \tilde{u}_1-\tilde{u}_2, \Gamma_h v_h) = 0.\end{aligned} \tag{5.11}$$

Letting $(v_h, q_h) = (\tilde{u}_1-\tilde{u}_2, \tilde{p}_1-\tilde{p}_2) = (e, \eta)$, we obtain

$$\mathcal{C}_h((e,\eta),(e,\eta)) \geq \nu\|e\|_1^2. \tag{5.12}$$

By (2.8), (2.9), (3.3), (4.4), (5.5), and the same approach as (5.8), it follows that

$$\begin{aligned}&|b(e;\tilde{u}_1,\Gamma_h e) + b(\tilde{u}_2; e, \Gamma_h e)|\\&\quad = |b(e;\tilde{u}_1,\Gamma_h e - e) + b(e;\tilde{u}_1, e) + b(\tilde{u}_2; e, \Gamma_h e - e)|\\&\quad \leq C_3\|\tilde{u}_1\|_1\|e\|_1^2 + 2\nu h_0\|e\|_1^2\\&\quad \leq \left(\frac{2C_1C_3C_7}{\nu}\|f\|_0 + 2\nu h_0\right)\|e\|_1^2.\end{aligned} \tag{5.13}$$

Applying (5.11)–(5.13) and (5.4) gives

$$0 \leq \nu\left(1 - \frac{4C_1C_3C_7}{\nu^2}\|f\|_0\right)\|e\|_1^2 \leq 0, \tag{5.14}$$

which shows that $e = 0$ by (5.4); i.e., $\tilde{u}_1 = \tilde{u}_2$. Next, applying (5.2) to (5.11) yields that $\tilde{p}_1 = \tilde{p}_2$. Therefore, it follows that (4.6) has a unique solution.

Finally, using the definition of the discrete operator A_h and taking $(v_h, q_h) = (A_h\tilde{u}_h, 0)$ in (4.6), it follows that

$$\nu\|A_h\tilde{u}_h\|_0^2 + b(\tilde{u}_h; \tilde{u}_h, \Gamma_h A_h\tilde{u}_h) = (f, \Gamma_h A_h\tilde{u}_h), \tag{5.15}$$

where, using (2.5), (4.5), and (5.9),

$$\begin{aligned}|(f,\Gamma_h A_h\tilde{u}_h)| &\leq \|f\|_0\|\Gamma_h A_h\tilde{u}_h\|_0 \leq \frac{C_7^2}{\nu}\|f\|_0^2 + \frac{\nu}{4}\|A_h\tilde{u}_h\|_0^2,\\ |b(\tilde{u}_h;\tilde{u}_h,\Gamma_h A_h\tilde{u}_h)| &\leq C_2C_7\|\tilde{u}_h\|_0^{1/2}\|\tilde{u}_h\|_1\|A_h\tilde{u}_h\|_0^{3/2}\\&\quad + \frac{\sqrt{2}}{2}C_2C_7\|\tilde{u}_h\|_1\|\tilde{u}_h\|_0^{1/2}\|A_h\tilde{u}_h\|_0^{3/2}\\&\leq 2C_2C_7\|\tilde{u}_h\|_0^{1/2}\|\tilde{u}_h\|_1\|A_h\tilde{u}_h\|_0^{3/2},\\&\leq \frac{\nu}{2}\|A_h u_h\|_0^2 + \frac{2^{10}}{\nu^9}C_1^8C_2^4C_7^{10}\|f\|_0^6,\end{aligned}$$

which, together with (5.15), completes the proof of (5.5). □

6 Error estimates

In this section we derive error estimates for the finite volume method for the stationary Navier-Stokes equations. We will analyze these estimates for the velocity in the H^1- and L^2-norm and the pressure in the L^2-norm. A superconvergence result is also obtained between the finite element and finite volume solutions. Below the L^2-projection Π_h will also indicate the projection from $(L^2(\Omega))^2$ to $(W_h)^2$, and we will use an assumption that is stronger than (5.4):

$$1 - \frac{2C_1C_3C_7}{\nu^2}\|f\|_0 \geq C_8 > 0. \tag{6.1}$$

Theorem 6.1 *Assume that $h > 0$ satisfies (5.3) and $f \in Y$ and $\nu > 0$ satisfy (6.1). Let $(u, p) \in (X, M)$ and $(\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h)$ be the solution of (2.7) and (4.6), respectively. Then it holds*

$$\|u - \tilde{u}_h\|_1 + \|p - \tilde{p}_h\|_0 \leq Ch(\|u\|_2 + \|p\|_1 + \|f\|_0). \tag{6.2}$$

Moreover, if $(u_h, p_h) \in (X_h, M_h)$ is the solution of (3.13) and $f \in H^1(\Omega)$, then

$$\|u_h - \tilde{u}_h\|_1 + \|p - \tilde{p}_h\|_0 \leq C|\log h|^{1/2}h^2(\|u\|_2 + \|p\|_1 + \|f\|_1). \tag{6.3}$$

Proof Subtracting (4.6) from (3.13), it follows from Lemma 4.2 that

$$\begin{aligned}&\mathcal{C}_h((u_h - \tilde{u}_h, p_h - \tilde{p}_h), (v_h, q_h)) + b(u_h; u_h, v_h) - b(\tilde{u}_h; \tilde{u}_h, \Gamma_h v_h)\\&\quad = (f, v_h - \Gamma_h v_h).\end{aligned} \tag{6.4}$$

Taking $(v_h, q_h) = (e, \eta) = (u_h - \tilde{u}_h, p_h - \tilde{p}_h)$ in (6.4), we see that

$$\mathcal{C}_h((e, \eta), (e, \eta)) + b(e; u_h, e) + b(\tilde{u}_h; e, e) + b(\tilde{u}_h; \tilde{u}_h, e - \Gamma_h e) = (f, e - \Gamma_h e). \tag{6.5}$$

Clearly, from (2.8)–(2.10) and (5.5) it follows that

$$|\mathcal{C}_h((e, \eta), (e, \eta))| \geq \nu\|e\|_1^2, \tag{6.6}$$

$$|b(e; u_h, e)| \leq C_3\|e\|_1^2\|\tilde{u}_h\|_1 \leq \frac{2C_1C_3C_7}{\nu}\|f\|_0\|e\|_1^2, \tag{6.7}$$

$$|b(\tilde{u}_h; e, e)| = 0. \tag{6.8}$$

In addition, setting $\nu_0 = 1 - \frac{2C_1C_3C_7\|f\|_0}{\nu^2}$ and using Lemma 4.1, (3.3), and (3.8), we see that

$$\begin{aligned}|b(\tilde{u}_h; \tilde{u}_h, e - \Gamma_h e)| &= \left|\left(((\tilde{u}_h - \Pi_h\tilde{u}_h) \cdot \nabla)\tilde{u}_h + \frac{1}{2}\operatorname{div}\tilde{u}_h(\tilde{u}_h - \Pi_h\tilde{u}_h), e - \Gamma_h e\right)\right|\\&\leq \left\{\|A_h^{1/2}\tilde{u}_h\|_\infty\|\tilde{u}_h - \Pi_h\tilde{u}_h\|_0 + \frac{1}{2}\|A_h^{1/2}\tilde{u}_h\|_\infty\|\tilde{u}_h - \Pi_h\tilde{u}_h\|_0\right\}\end{aligned}$$

$$
\begin{aligned}
&\quad \times \|e-\Gamma_h e\|_0 \\
&\le C|\log h|^{1/2}h^2\|A_h\tilde{u}_h\|_0\|\tilde{u}_h\|_1\|e\|_1 \\
&\le C|\log h|h^4\|\tilde{u}_h\|_1^2\|A_h\tilde{u}_h\|_0^2+\frac{\nu_0}{4}\|e\|_1^2.
\end{aligned} \tag{6.9}
$$

Similarly, by Lemma 4.1 and (3.8), we have

$$
\begin{aligned}
|(f,e-\Gamma_h e)| = |(f-\Pi_h f, e-\Gamma_h e)| &\le Ch^i\|f\|_i\|e-\Gamma_h e\|_0 \\
&\le Ch^{2(i+1)}\|f\|_i^2+\frac{\nu_0}{4}\|e\|_1^2, \quad i=0,1.
\end{aligned} \tag{6.10}
$$

Combining (5.4) and (6.5)–(6.10) gives

$$
\|e\|_1 \le C\left(|\log h|^{1/2}h^2+h^{i+1}\right)\|f\|_i, \quad i=0,1. \tag{6.11}
$$

In the same argument, it follows from (5.2) and (6.4) that

$$
\|\eta\|_0 \le C\left(|\log h|^{1/2}h^2+h^{i+1}\right)\|f\|_i, \quad i=0,1. \tag{6.12}
$$

Combining (6.11) and (6.12) completes the proof of (6.3).

Finally, using the triangle inequality, (3.14), (6.11), and (6.12), we obtain

$$
\|u-\tilde{u}_h\|_1 \le \|u-u_h\|_1+\|u_h-\tilde{u}_h\|_1 \le Ch(\|u\|_2+\|p\|_1+\|f\|_0), \tag{6.13}
$$

$$
\|p-\tilde{p}_h\|_0 \le \|p-p_h\|_0+\|p_h-\tilde{p}_h\|_0 \le Ch(\|u\|_2+\|p\|_1+\|f\|_0), \tag{6.14}
$$

which completes the proof of (6.2). □

In order to derive an optimal error estimate for the velocity in the L^2-norm, we consider the following dual problem:

$$
a(v,\Phi)+d(v,\Psi)-d(\Phi,q)+b(u;v,\Phi)+b(v;u,\Phi)=(u-\tilde{u}_h,v). \tag{6.15}
$$

Because of convexity of the domain Ω, this problem has a unique solution that satisfies the regularity property [27]

$$
\|\Phi\|_2+\|\Psi\|_1 \le C\|u-\tilde{u}_h\|_0. \tag{6.16}
$$

Below set $(\Phi_h,\Psi_h)=(I_h\Phi,J_h\Psi)\in(X_h,M_h)$, which satisfies, by (3.1),

$$
\|\Phi-\Phi_h\|_0+h(\|\Phi-\Phi_h\|_1+\|\Psi-\Psi_h\|_0) \le Ch^2(\|\Phi\|_2+\|\Psi\|_1). \tag{6.17}
$$

Theorem 6.2 *Let (u,p) and $(\tilde{u}_h,\tilde{p}_h)$ be the solutions of (2.7) and (4.6), respectively. Then, under the assumptions of Theorem 6.1, it holds*

$$
\|u-\tilde{u}_h\|_0 \le Ch^2(\|u\|_2+\|p\|_1+\|f\|_1). \tag{6.18}
$$

Proof Multiplying (2.1) and (2.2) by $\Gamma_h \Phi_h \in \tilde{X}_h$ and $\Psi_h \in M_h$ and integrating over the dual elements $\tilde{K}$ and the primary elements K, respectively, and adding the resulting equations to (4.6) with $(v_h, q_h) = (\Phi_h, \Psi_h)$, we see that

$$\begin{aligned} &A(e, \Gamma_h \Phi_h) + D(\Gamma_h \Phi_h, \eta) + d(e, \Psi_h) + G(\eta, \Psi_h) \\ &\quad + b(e; u, \Gamma_h \Phi_h) + b(u; e, \Gamma_h \Phi_h) - b(e; e, \Gamma_h \Phi_h) = G(p, \Psi_h), \end{aligned} \tag{6.19}$$

where $(e, \eta) = (u - \tilde{u}_h, p - \tilde{p}_h)$. Subtracting (6.19) from (6.15) with $(v, q) = (e, \eta)$ and using (2.1), we obtain

$$\begin{aligned} \|e\|_0^2 &= a(e, \Phi - \Phi_h) + d(e, \Psi - \Psi_h) - d(\Phi - \Phi_h, \eta) - G(\eta, \Psi_h) + G(p, \Psi_h) \\ &\quad + a(e, \Phi_h) - A(e, \Gamma_h \Phi_h) - d(\Phi_h, \eta) - D(\Gamma_h \Phi_h, \eta) \\ &\quad + b(u; e, \Phi - \Gamma_h \Phi_h) + b(e; u, \Phi - \Gamma_h \Phi_h) + b(e; e, \Gamma_h \Phi_h) \\ &= a(e, \Phi - \Phi_h) + d(e, \Psi - \Psi_h) - d(\Phi - \Phi_h, \eta) - G(\eta, \Psi_h) + G(p, \Psi_h) \\ &\quad + b(u; e, \Phi - \Gamma_h \Phi_h) + b(e; u, \Phi - \Gamma_h \Phi_h) + b(e; e, \Gamma_h \Phi_h) \\ &\quad + (f - (u \cdot \nabla) u, \Phi_h - \Gamma_h \Phi_h). \end{aligned} \tag{6.20}$$

Applying (2.4), (2.5), (2.9), (2.10), (3.8), (4.3), (6.16), (6.17), and an inverse inequality, each of terms in (6.20) is estimated as follows:

$$\begin{aligned} &|a(e, \Phi - \Phi_h) + d(e, \Psi - \Psi_h) - d(\Phi - \Phi_h, \eta)| \\ &\quad \le C\,(\|e\|_1 + \|\eta\|_0)\,(\|\Phi - \Phi_h\|_1 + \|\Psi - \Psi_h\|_0) \\ &\quad \le Ch^2\,(\|u\|_2 + \|p\|_1)\,(\|\Phi\|_2 + \|\Psi\|_1) \\ &\quad \le Ch^2\,(\|u\|_2 + \|p\|_1)\,\|e\|_0, \\ &|G(\eta, \Psi_h) - G(p, \Psi_h)| \le Ch\,(\|p - \Pi_h p\|_0 + \|\eta\|_0)\,\|\Psi\|_1 \\ &\qquad\qquad\qquad\qquad\quad \le Ch^2\,(\|u\|_2 + \|p\|_1)\,\|e\|_0, \\ &|b(u; e, \Phi - \Gamma_h \Phi_h) + b(e; u, \Phi - \Gamma_h \Phi_h)| \\ &\quad \le C\|u\|_2\|e\|_1\,(\|\Phi_h - \Gamma_h \Phi_h\|_0 + \|\Phi - \Phi_h\|_0) \\ &\quad \le Ch^2\,(\|u\|_2 + \|p\|_1)\,\|\Phi\|_1 \\ &\quad \le Ch^2\,(\|u\|_2 + \|p\|_1)\,\|e\|_0, \\ &|b(e; e, \Gamma_h \Phi_h)| = |b(e; e, \Gamma_h \Phi_h - \Phi_h) + b(e; e, \Phi_h)| \\ &\qquad\qquad\qquad \le C\left(\|e\|_{0,4}\|e\|_1\|\Gamma_h \Phi_h - \Phi_h\|_{0,4} + \|e\|_1^2\|\Phi_h\|_1\right) \\ &\qquad\qquad\qquad \le Ch^2\,(\|u\|_2 + \|p\|_1)\,\|e\|_0, \\ &|(f - (u \cdot \nabla) u, \Phi_h - \Gamma_h \Phi_h)| \\ &\quad = |([f - \Pi_h f] - [(u \cdot \nabla) u - \Pi_h (u \cdot \nabla) u], \Phi_h - \Gamma_h \Phi_h)| \end{aligned}$$

$$\leq Ch^2(\|f\|_1 + \|\nabla[(u\cdot\nabla)u]\|_0)\|\Phi_h\|_1$$
$$\leq Ch^2(\|f\|_1 + \|u\|_0^{1/2}\|u\|_2^{3/2} + \|u\|_{1,4}^2)\|e\|_0.$$

Combining all these inequalities and (6.20) yields (6.18). □

7 Numerical experiments

In this section we present numerical experiments to check the numerical theory developed in the previous sections. In all the experiments, Ω is the unit square in $\Re^2$, the viscosity $\nu = 1$, the exact solution for the velocity $u = (u_1, u_2)$ and the pressure p is given as follows:

$$p(x) = 10(2x_1 - 1)(2x_2 - 1),$$
$$u_1(x) = 10x_1^2(x_1 - 1)^2 x_2(x_2 - 1)(2x_2 - 1),$$
$$u_2(x) = -10x_1(x_1 - 1)(2x_1 - 1)x_2^2(x_2 - 1)^2,$$

and the right-hand side $f(x)$ is determined by (2.1).

First, we test the convergence rate of the stabilized finite volume method developed for the stationary Navier-Stokes equations. It follows from Table 1 that the numerical results support the theoretical analysis carried out for the stationary Navier-Stokes equations in Sect. 6. The numerical convergence rate for the pressure in the L^2-norm seems slightly better than the theoretical one.

Second, we compare the new stabilized finite element method with the standard Galerkin method with the P_{1b}–P_1 pair, the penalty method, the regular stabilized method, and the multiscale enrichment method with the P_1–P_1 pair for the stationary Navier-Stokes equations on uniform meshes, where P_{1b} indicates the P_1 augmented with the cubic bubbles. These five methods have the common discrete formulation: Find $(\tilde{u}_h, \tilde{p}_h) \in (X_h, M_h)$ such that

$$A(\tilde{u}_h, \Gamma_h v_h) + D(\Gamma_h v_h, \tilde{p}_h) + b(\tilde{u}_h, \tilde{u}_h, \Gamma_h v_h) = (f, \Gamma_h v_h), \tag{7.1}$$
$$d(\tilde{u}_h, q_h) + \Lambda(\tilde{p}_h, q_h) = 0, \tag{7.2}$$

for all $(v_h, q_h) \in (X_h, M_h)$, where

$$\Lambda(p_h, q_h) = \varepsilon(p_h, q_h)/\nu$$

for the penalty method,

$$\Lambda(p_h, q_h) = \delta \sum_K h_K^2(\nabla p_h - f, \nabla q_h)_K$$

for the regular method [26],

$$\Lambda(p_h, q_h) = \delta_1 \sum_K h_K^2(\nabla p_h - f, \nabla q_h)_K + \sum_K \sum_{e\in\partial K} \delta_2 h_e \left\langle \left[\nu \frac{\partial \tilde{u}_h}{\partial n}\right], \left[\nu \frac{\partial v_h}{\partial n}\right]\right\rangle_e$$

Table 1 The rate analysis of the new stabilized finite volume method: P_1–P_1

1/h	CPU	$\frac{\|u-\tilde{u}_h\|_0}{\|u\|_0}$	$\frac{\|u-\tilde{u}_h\|_1}{\|u\|_1}$	$\frac{\|p-\tilde{p}_h\|_0}{\|p\|_0}$	u_{L^2}-rate	u_{H^1}-rate	p_{L^2}-rate
9	0.265	0.3250	0.5394	0.0978			
18	0.797	0.0869	0.2147	0.0293	1.904	1.329	1.742
27	1.890	0.0393	0.1293	0.0145	1.956	1.250	1.738
36	3.422	0.0223	0.0917	0.0088	1.971	1.194	1.728

Table 2 The rate analysis of the penalty finite volume method: P_1–P_1

1/h	CPU	$\frac{\|u-\tilde{u}_h\|_0}{\|u\|_0}$	$\frac{\|u-\tilde{u}_h\|_1}{\|u\|_1}$	$\frac{\|p-\tilde{p}_h\|_0}{\|p\|_0}$	u_{L^2}-rate	u_{H^1}-rate	p_{L^2}-rate
9	0.234	0.1348	0.3849	0.3246			
18	0.828	0.0322	0.1890	0.1840	2.063	1.026	0.819
27	2.000	0.0144	0.1273	0.2037	1.981	0.976	-0.25
36	3.844	0.0079	0.0939	0.1391	2.091	1.058	1.325

Table 3 The rate analysis of the finite volume method: P_{1b}–P_1

1/h	CPU	$\frac{\|u-\tilde{u}_h\|_0}{\|u\|_0}$	$\frac{\|u-\tilde{u}_h\|_1}{\|u\|_1}$	$\frac{\|p-\tilde{p}_h\|_0}{\|p\|_0}$	u_{L^2}-rate	u_{H^1}-rate	p_{L^2}-rate
9	0.265	0.1000	0.7570	0.0528			
18	1.047	0.0241	0.3845	0.0148	2.053	0.977	1.835
27	2.422	0.0106	0.2572	0.0071	2.036	0.992	1.794
36	4.282	0.0059	0.1932	0.0043	2.026	0.995	1.752

for the multiscale enrichment method [2], and $\Lambda(p_h, q_h) = G(p_h, q_h)$ for the new method. Note that $h_e = |e|$ indicates the length of e and $[v]$ denotes the jump of v across e.

It is difficult to obtain optimal values for the stabilization parameters for a given mesh. In practice, these parameters are determined by the trial and error method. The penalty method involves the choice of a stabilization parameter, which must be sufficiently small $1.0e{-}4 > \varepsilon > 0$ to achieve a suboptimal error estimate $O(h/\varepsilon)$ [23]. There is a slight deterioration in the finite element approximation by the regular stabilized method with the selected parameter in [17] for the stationary Navier-Stokes equations. Here we choose $\delta = 1/(80\nu) = 0.0125/\nu$ [24], $\delta_1 = 1/(8\nu) = 0.125/\nu$, and $\delta_2 = 1/(12\nu)$ [2]. As seen from Tables 1–5 on CPU times and error estimates, the new stabilized finite volume method has achieved the best results among the tested methods.

Finally, we check the error between the stabilized finite element solution and the finite volume solution obtained as in Sects. 3 and 6, respectively. Recall that the finite element discretization uses a triangular mesh with the P_1–P_1 pair for the velocity and pressure. The numerical results are presented in Table 6. In particular, it follows from Table 6 that the error between these two solutions has a superconvergence estimate

Table 4 The rate analysis of the regular stabilized finite volume method: P_1–P_1

1/h	CPU	$\frac{\|u-\tilde{u}_h\|_0}{\|u\|_0}$	$\frac{\|u-\tilde{u}_h\|_1}{\|u\|_1}$	$\frac{\|p-\tilde{p}_h\|_0}{\|p\|_0}$	u_{L^2}-rate	u_{H^1}-rate	p_{L^2}-rate
9	0.203	0.0777	0.3101	0.0173			
18	0.812	0.0189	0.1556	0.0053	2.042	0.995	1.693
27	1.922	0.0083	0.1037	0.0028	2.032	1.002	1.634
36	3.532	0.0046	0.0777	0.0018	2.023	1.002	1.568

Table 5 The rate analysis of the multiscale stabilized finite volume method: P_1–P_1

1/h	CPU	$\frac{\|u-\tilde{u}_h\|_0}{\|u\|_0}$	$\frac{\|u-\tilde{u}_h\|_1}{\|u\|_1}$	$\frac{\|p-\tilde{p}_h\|_0}{\|p\|_0}$	u_{L^2}-rate	u_{H^1}-rate	p_{L^2}-rate
9	0.343	0.1154	0.3084	0.0204			
18	1.406	0.0306	0.1553	0.0067	1.914	0.990	1.601
27	3.265	0.0138	0.1035	0.0035	1.976	1.000	1.585
36	5.953	0.0078	0.0776	0.0023	1.990	1.001	1.551

Table 6 Superconvergence between the stabilized finite volume and finite element solutions

1/h	$\frac{\|u-\tilde{u}_h\|_0}{\|u\|_0}$	$\frac{\|u-\tilde{u}_h\|_1}{\|u\|_1}$	$\frac{\|p-\tilde{p}_h\|_0}{\|p\|_0}$	u_{L^2}-rate	u_{H^1}-rate	p_{L^2}-rate
9	0.0273	0.0268	0.0020			
18	0.0073	0.0074	0.0005	1.898	1.858	2.014
27	0.0033	0.0033	0.0002	1.983	1.973	1.998
36	0.0018	0.0019	0.0001	1.993	1.989	1.998

for the velocity and pressure in the H^1-norm and L^2-norm, respectively, which agrees with the theoretical result (6.3).

In conclusion, the numerical experiments presented in this section agree with the stability and accuracy properties of the stabilized finite volume method for the stationary Navier-Stokes equations developed in earlier sections, and this method appears superior to other stabilization methods tested. While only a numerical example is presented, other examples tested give the same conclusion.

8 Conclusions

In this paper, we have provided a theoretical analysis for the stabilized finite volume method based on two local Gauss integrals [22] for the stationary Navier-Stokes equations. The stabilized finite volume method with the P_1–P_1 pair is a highly attractive and usable in scientific computation due to the fact that it is computationally convenient and efficient in a parallel processing and multi-grid context. For the stabilized finite volume solution, we have demonstrated the existence, stability, and optimal error estimate results. Finally, numerical tests have shown that this method is numerically effective for solving the two-dimensional stationary Navier-Stokes equations.

And our future extension will involve the three-dimensional stationary and transient Navier-Stokes equations.

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