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Pathwise Taylor schemes for random ordinary differential equations

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Abstract Random ordinary differential equations (RODEs) are ordinary differential equations which contain a stochastic process in their vector fields. They can be analyzed pathwise using deterministic calculus, but since the driving stochastic process is usually only Hölder continuous in time, the vector field is not differentiable in the time variable. Traditional numerical schemes for ordinary differential equations thus do not achieve their usual order of convergence when applied to RODEs. Nevertheless, deterministic calculus can still be used to derive higher order numerical schemes for RODEs by means of a new kind of integral Taylor expansion. The theory is developed systematically here, applied to illustrative examples involving Brownian motion and fractional Brownian motion as the driving processes and compared with other numerical schemes for RODEs in the literature.

Keywords Random ordinary differential equations · Integral Taylor expansion · One-step numerical scheme · Pathwise convergence · Brownian motion · Fractional Brownian motion

Mathematics Subject Classification (2000) 65C30 · 65L05 · 65L20

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1 Introduction

Taylor expansions are a very basic tool in numerical analysis and other areas of mathematics which require approximations. In particular, they allow us to derive one step numerical schemes for ordinary differential equations (ODEs) of arbitrary high order, although in practice such Taylor schemes are rarely implemented but are used instead as a theoretical comparison for determining the convergence orders of other schemes that have been derived by more heuristic methods. On the other hand, in view of the less robust nature of the Ito stochastic integral, stochastic Taylor schemes are the essential starting point for the derivation of consistent higher order numerics schemes for stochastic differential equations (SDEs). Other types of schemes for SDEs, such as derivative-free schemes, are then obtained by modifying the corresponding stochastic Taylor schemes, see e.g. Kloeden and Platen [14].

Random ordinary differential equations (RODEs), which are ordinary differential equations containing a stochastic process in their vector field functions, have been used for many years in a wide range of applications, see e.g. [1, 2, 16, 17] and the papers cited therein. They are also very useful for investigating pathwise properties of SDEs since, by the results of Doss and Sussmann and their generalizations, see e.g. [9, 19], every SDE can be transformed into a RODE and vice versa. In particular, RODEs are nonautonomous ODEs to which deterministic calculus can be applied pathwise. Typically, however, the driving stochastic process has at most Hölder continuous sample paths, so the solutions sample paths are certainly continuously differentiable but with the derivative sample paths being at most Hölder continuous in time. Equivalently, the resulting vector field after insertion of the driving stochastic process is at most Hölder continuous in time, no matter how smooth the vector field is in its original variables. Consequently, although classical numerical schemes for ODEs can be used pathwise for RODEs, they rarely attain their traditional order.

Grüne and Kloeden [8] showed how averaging schemes can be used to regain the traditional order for RODEs with a special nonlinear affine structure, i.e. affine in the noise and possibly nonlinear in the state variables. (See [7] for a generalization of this idea to numerical schemes for affine nonlinear control systems.) More recently, Jentzen and Kloeden [12] (see also [11]) used an integral equation expansions to derive numerical schemes of arbitrary high order for general RODEs, but these integrals equations were of an implicit nature and the schemes obtained often contained more terms than were needed for the required order.

In this paper we generalize the Wagner-Platen multi-index notation that was used in [14] to formulate succinctly Taylor expansions and Taylor schemes for SDEs. The main difference is that matrix valued multi-indices are required to handle the possibly different Hölder exponents of the different components of the driving noise processes. As in the SDE case the coefficient functions are obtained by iterated application of differential operators to the vector field of the RODE, but instead of iterated integrals of the components of the noise processes single integrals of suitable powers of increments of them are used. This approach is simpler and more direct than that in [12] as well as more amenable to further development.

The paper is structured as follows. A brief introduction to RODEs is given in Sect. 2. In Sect. 3 background material is presented and the notation needed for formulating integral Taylor expansions for RODEs is developed, specifically, the types

of driving noise processes, matrix-valued multi-indices, iterated integrals as well iterated total differential operators. The basic idea underlying integral Taylor expansions for RODEs is then sketched in Sect. 4. General Taylor schemes for RODEs are defined in Sect. 5, where the essential RODE-Taylor schemes which involve the minimal number of terms needed for a desired order are also considered. Various examples of RODE-Taylor schemes for Brownian and fractional Brownian motion are presented in Sects. 6 and 7 some other numerical schemes for RODEs from the literature are recalled. Numerical simulations for two test examples are given in Sect. 9 after some remarks in Sect. 8 on approximating the integrals in the schemes by suitably chosen Riemann sums. Most proofs are given in Sect. 10, in particular the proof of the main theorem of the paper on the convergence order of the RODE-Taylor schemes.

2 Random ordinary differential equations

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a complete probability space an let $(\zeta_t)_{t\geq 0}$ be a $\mathbb{R}^m$-valued stochastic process with continuous sample paths. In addition, let $f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function.

A random ordinary differential equation (RODE) in $\mathbb{R}^d$, i.e.,

$$\frac{dx}{dt} = f(\zeta_t(\omega), x), \tag{2.1}$$

is a nonautonomous ordinary differential equation (ODE)

$$\frac{dx}{dt} = F_\omega(t, x) := f(\zeta_t(\omega), x) \tag{2.2}$$

for almost every realization $\omega \in \Omega$. For convenience, we will assume that this holds for all $\omega \in \Omega$, by restricting Ω to a subset of full probability, if necessary.

Also for convenience, we will assume that f is infinitely often continuously differentiable in its variables, although k-times continuously differentiable with k sufficiently large would suffice. In particular, f is then locally Lipschitz in x, so the initial value problem

$$\dot{x}_t(\omega) = f(\zeta_t(\omega), x_t(\omega)), \qquad x_0(\omega) = X_0(\omega), \tag{2.3}$$

where the initial value X_0 is a $\mathbb{R}^d$-valued random variable, has a unique pathwise solution $x_t(\omega)$ for every $\omega \in \Omega$, which will be assumed to exist on the finite time interval $[0, T]$ under consideration. Sufficient conditions that guarantee the existence and uniqueness of such solutions can be found in [2], see also [1].

The solution of (2.3) is a stochastic process $(x_t)_{t\in[0,T]}$, which is non-anticipative if the driving process ζ_t is non-anticipative. Its sample paths $t \to x_t(\omega)$ are continuously differentiable but need not be further differentiable, since the vector field $F_\omega(t, x)$ of the nonautonomous ODE (2.2) is usually only continuous but not differentiable in t, no matter how smooth the function f is in its variables.

3 Preliminaries and notation

Let $\mathbb{N}$ be the set of natural numbers and let $\mathbb{N}_0$ denote the set of nonnegative integers.

We will use the norms

$$\|v\|_1 = |v_1| + \cdots + |v_l|, \qquad \|v\|_2 = \sqrt{v_1^2 + \cdots + v_l^2}$$

for a vector $v = (v_1, \ldots, v_l) \in \mathbb{R}^l$ and for a function $g : [0, T] \to \mathbb{R}$ we will use the supremum norm

$$|g|_\infty := \sup_{s \in [0,T]} |g(s)|$$

and the Hölder norm

$$|g|_\eta := |g|_\infty + \sup_{r \neq s \in [0,T]} \frac{|g(r) - g(s)|}{|r - s|^\eta}$$

for $\eta > 0$ (where these exist). For a vector valued function $g = (g_1, \ldots, g_l) : [0, T] \to \mathbb{R}^l$ we use the supremum norm

$$|g|_\infty := \sup_{s \in [0,T]} \|g(s)\|_2$$

and we define the Hölder norm

$$|g|_\eta := \max_{j=1,\ldots,l} |g_j|_{\eta_j}$$

for a vector $\eta = (\eta_1, \ldots, \eta_l) \in (0, 1]^l$.

3.1 Regularity of the driving stochastic process

We assume that the driving stochastic process $\zeta_t = (\zeta_t^1, \ldots, \zeta_t^m)$ has continuous sample paths, so the maximum can be used in the supremum norms $|\zeta^j|_\infty$, $j = 1, \ldots, m$, and $|\zeta|_\infty$, which are in fact random variables. The Hölder norms $|\zeta^j|_{\eta_j}$, $j = 1, \ldots, m$, and $|\zeta|_\eta$ defined pathwise as above, are also random variables.

The following standing assumption about the Hölder continuity of the driving stochastic process will allow us to exploit better the different strengths Hölder continuity of the different component processes.

Assumption 3.1 There is a vector $\theta = (\theta_1, \ldots, \theta_m) \in (0, 1]^m$ such that

$$|\zeta|_\eta < \infty$$

holds pathwise for all $0 < \eta < \theta$, i.e. with $0 < \eta_1 < \theta_1, \ldots, 0 < \eta_m < \theta_m$.

Finally, let $\vartheta := \min(\theta_1, \ldots, \theta_m)$ be the minimum of $\theta_1, \ldots, \theta_m$.

3.2 Multi-index notation

We now introduce a multi-index notation that uses matrix-valued indices rather than vector-valued indices as in Kloeden and Platen [14].

For a non empty set A and $l, k \in \mathbb{N}$ we mean by $A^{l\times k} := (A^l)^k$ the set of all $l \times k$ matrices with entries in A respectively the set of all k-tuples of l-tuples of A, i.e.

$$A^{l\times k} := \left\{ \left(\left(a_{1,1}, \ldots, a_{1,l}\right), \ldots, \left(a_{k,1}, \ldots, a_{k,l}\right)\right) \middle| a_{i,j} \in A, 1 \le i \le k, 1 \le j \le l \right\}.$$

In particular, we consider $m \times i$-matrices $\mathbb{N}_0^{m\times i}$ of nonnegative integers with $i \ge 1$. We present an element $\alpha = (\alpha_1, \ldots, \alpha_i) \in \mathbb{N}_0^{m\times i}$ with $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,m}) \in \mathbb{N}_0^m$ for $j = 1, \ldots, i$ as

$$\alpha = (\alpha_1, \ldots, \alpha_i) = \begin{pmatrix} \alpha_{1,1} & \ldots & \alpha_{i,1} \\ \vdots & & \vdots \\ \alpha_{1,m} & \ldots & \alpha_{i,m} \end{pmatrix}.$$

For such an $\alpha \in \mathbb{N}_0^{m\times i}$, we define $i(\alpha) := i$. In addition, we write $\mathbb{N}_0^{m\times 0} = \mathbb{N}_0^0 := \{\emptyset\}$, where $\emptyset$ is the empty set symbol, which will be used to denote the "empty" index.

Finally, we denote the set of all such matrix-valued multi-indices by

$$\mathscr{A}_m := \bigcup_{i=0}^{\infty} \mathbb{N}_0^{m\times i}.$$

If $\alpha \in \mathscr{A}_m \setminus \{\emptyset\}$, so $i(\alpha) \ge 1$, we define

$$\alpha! := \alpha_{1,1}! \cdots \alpha_{1,m}! \cdots \alpha_{i,1}! \cdots \alpha_{i,m}!$$

and

$$|\alpha| := \alpha_{1,1} + \cdots + \alpha_{1,m} + \cdots + \alpha_{i,1} + \cdots + \alpha_{i,m}.$$

Since α^T is an $i \times m$-matrix for $\alpha \in \mathscr{A}_m$ with $i(\alpha) \ge 1$ and $\theta = (\theta_1, \ldots, \theta_m)$ in Assumption 3.1 is an m-dimensional vector, we have

$$\alpha^T\theta = \begin{pmatrix} \alpha_{1,1}\theta_1 + \cdots + \alpha_{1,m}\theta_m \\ \vdots \\ \alpha_{i,1}\theta_1 + \cdots + \alpha_{i,m}\theta_m \end{pmatrix} \in \mathbb{R}^m_{\ge 0}$$

and

$$\|\alpha^T\theta\|_1 = \theta_1(\alpha_{1,1} + \cdots + \alpha_{i,1}) + \cdots + \theta_m(\alpha_{1,m} + \cdots + \alpha_{i,m}).$$

Finally, for $\alpha = \emptyset$ we define

$$i(\emptyset) := 0, \qquad \emptyset! := 1, \qquad |\emptyset| := 0, \qquad \|\emptyset^T\theta\|_1 := 0.$$

3.3 Iterated integrals

We use the following abbreviations for iterated integrals.

For a function $g : [0, \infty) \to \mathbb{R}^m$ and for $0 \leq t_0 \leq s_0 < \infty$ we write

$$\Delta g_{t_0,s_0} := g(s_0) - g(t_0), \qquad \Delta_{t_0,s_0} := s_0 - t_0.$$

Given such a function and a matrix index $\alpha = (\alpha_1, \ldots, \alpha_i) \in \mathscr{A}_m$ with $i(\alpha) = i \geq 1$, we define

$$\left(\Delta g_{t_0,s_0}\right)^{\alpha_k} := \prod_{j=1}^{m} \left(\Delta g^j_{t_0,s_0}\right)^{\alpha_{k,j}}, \quad k = 1, \ldots, i,$$

where $\Delta g_{t_0,s_0} = (\Delta g^1_{t_0,s_0}, \ldots, \Delta g^m_{t_0,s_0})$ and which we will only use for the driving stochastic process ζ_t in iterated integrals, which we define pathwise by

$$I_{\alpha,t_0,s_0} := \int_{t_0}^{s_0} \ldots \int_{t_0}^{s_{i(\alpha)-1}} \left[\left(\Delta\zeta_{t_0,s_1}\right)^{\alpha_1} \cdots \left(\Delta\zeta_{t_0,s_{i(\alpha)}}\right)^{\alpha_{i(\alpha)}}\right] ds_{i(\alpha)} \ldots ds_1$$

for $i(\alpha) \geq 1$ and by $I_{\emptyset,t_0,s_0} := 1$. Obviously, I_{α,t_0,s_0} is a random variable.

3.4 Function spaces

Let $\mathscr{F}_0 = C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ be the vector space over $\mathbb{R}$ of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ and for $i \in \mathbb{N}$ let $\mathscr{F}_i = C^\infty(\mathbb{R}^{m\times i} \times \mathbb{R}^d, \mathbb{R}^d)$ be the vector space over $\mathbb{R}$ of all smooth functions from $\mathbb{R}^{m\times i} \times \mathbb{R}^d$ to $\mathbb{R}^d$. For example, the vector field f of the RODE (2.1) is in $\mathscr{F}_1$. Let

$$\mathscr{F} := \bigcup_{i=0}^{\infty} \mathscr{F}_i.$$

We will write the components of the variables $(w, y) \in \mathbb{R}^{m\times i} \times \mathbb{R}^d$ of a function in $\mathscr{F}_i$ with $i \geq 1$ as

$$w = (w_1, \ldots, w_i) = \begin{pmatrix} w_{1,1} & \ldots & w_{i,1} \\ \vdots & & \vdots \\ w_{1,m} & \ldots & w_{i,m} \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix}.$$

For a matrix multi-index $\alpha \in \mathscr{A}_m$ with $i = i(\alpha) \geq 1$ and a function $H \in \mathscr{F}_i$, we define the α-derivative of H with respect to $w \in \mathbb{R}^{m\times i}$ as

$$\partial^\alpha H := \left(\partial_{w_{1,1}}\right)^{\alpha_{1,1}} \cdots \left(\partial_{w_{1,m}}\right)^{\alpha_{1,m}} \cdots \left(\partial_{w_{i,1}}\right)^{\alpha_{i,1}} \cdots \left(\partial_{w_{i,m}}\right)^{\alpha_{i,m}} H$$

and we define $\partial^\emptyset H = H$ for $H \in \mathscr{F}_0$.

Obviously, $\partial^\alpha H \in \mathscr{F}_i$ when $H \in \mathscr{F}_i$ for any $i = i(\alpha) \geq 0$ and $\alpha \in \mathscr{A}_m$.

3.5 Iterated differential operators

Let $(x_t)_{t\in[0,T]}$ be the stochastic process, which is the pathwise solution of the initial value problem (2.3) for the RODE (2.1) with the driving stochastic process $(\zeta_t)_{t\geq 0}$. We want to define iterated differential operators of functions of this solution process and thus require appropriate total derivatives.

For $i \in \mathbb{N}_0$ we define the linear differential operator $L_i : \mathscr{F}_i \to \mathscr{F}_{i+1}$ by

$$L_i H(w_1, \ldots, w_{i+1}, y) := \underbrace{\partial_y H(w_1, \ldots, w_i, y)}_{d\times d\text{-matrix}} \cdot \underbrace{f(w_{i+1}, y)}_{\text{vector in } \mathbb{R}^d}$$

for $w_1, \ldots, w_{i+1} \in \mathbb{R}^m$, $y \in \mathbb{R}^d$ and $H \in \mathscr{F}_i$. (For $i = 0$, omit the $w_1, \ldots, w_i$ variables.) The right side here is the product of a $d \times d$-matrix and a d-dimensional vector.

We have an analogue of the fundamental theorem of calculus for these differential operators and functions in $\mathscr{F}$.

Lemma 3.1 *Let $H \in \mathscr{F}_i$ for some $i \in \mathbb{N}_0$ and $0 \leq t_0 \leq s_0 \leq T$. Then,*

$$\begin{aligned} &H(w_1, \ldots, w_i, x_{s_0}(\omega)) \\ &\quad = H(w_1, \ldots, w_i, x_{t_0}(\omega)) + \int_{t_0}^{s_0} (L_i H)(w_1, \ldots, w_i, \zeta_t(\omega), x_t(\omega))\, dt \end{aligned} \tag{3.1}$$

for all $w_1, \ldots, w_i \in \mathbb{R}^m$ and each $\omega \in \Omega$. (For $i = 0$, omit the $w_1, \ldots, w_i$ variables.)

Proof Fix $\omega \in \Omega$ and consider the function $g : [0, T] \to \mathbb{R}^d$ given by

$$g(t) := H(w_1, \ldots, w_i, x_t(\omega)), \quad t \in [0, T],$$

which is continuously differentiable with the derivative

$$\begin{aligned} g'(t) &= \partial_y H(w_1, \ldots, w_i, x_t(\omega)) \cdot f(\zeta_t(\omega), x_t(\omega)) \\ &= (L_i H)(w_1, \ldots, w_i, \zeta_t(\omega), x_t(\omega)). \end{aligned}$$

The assertion thus follows by applying the fundamental theorem of calculus to the function g. □

More generally, we define the differential operator $L : \mathscr{F} \to \mathscr{F}$ by

$$LH := L_i H \quad \text{if } H \in \mathscr{F}_i \text{ for some } i \in \mathbb{N}_0.$$

If $H \in \mathscr{F}_i$ for some $i \in \mathbb{N}_0$, then $L(H) \in F_{i+1}$ and $L(L(H)) \in F_{i+2}$, and so on. Thus, we can iterate the L-operator:

$$L^k := \underbrace{L \circ L \circ \cdots \circ L}_{k \text{ times}} : \mathscr{F}_i \to \mathscr{F}_{i+k}$$

as

$$\mathscr{F}_i \xrightarrow{L} \mathscr{F}_{i+1} \xrightarrow{L} \cdots \xrightarrow{L} \mathscr{F}_{i+k}.$$

4 Integral equation expansions

We will now derive an integral equation expansion of the solution of the RODE (2.3) using iterated integrals and the above differential operators. This will be the key to developing the RODE-Taylor schemes in the next section. The underlying idea is to use Lemma 3.1 iteratively. A similar method has already been used in a number of other contexts [7, 10, 14, 18].

Instead of a general $H \in \mathscr{F}_0$, we restrict attention to the identity function $id : \mathbb{R}^d \to \mathbb{R}^d$, i.e.

$$id(y) = y \quad \text{for all } y \in \mathbb{R}^d.$$

This is obviously in $\mathscr{F}_0$, so we can apply Lemma 3.1 with $H = id$ to obtain

$$x_{s_0} = x_{t_0} + \int_{t_0}^{s_0} (L\, id)\left(\zeta_{s_1}, x_{s_1}\right) ds_1,$$

which is just an integral version of the RODE (2.1) since $L\, id = L_0\, id = f$.

Then, Lemma 3.1 applied to the integrand $L\, id \in \mathscr{F}_1$ over the interval $[t_0, s_1]$ yields

$$(L\, id)(\zeta_{s_1}, x_{s_1}) = (L\, id)(\zeta_{s_1}, x_{t_0}) + \int_{t_0}^{s_1} (L^2 id)\left(\zeta_{s_1}, \zeta_{s_2}, x_{s_2}\right) ds_2.$$

Inserting this into the previous equation, we have

$$x_{s_0} = x_{t_0} + \int_{t_0}^{s_0} (L\, id)\left(\zeta_{s_1}, x_{t_0}\right) ds_1 + \int_{t_0}^{s_0} \int_{t_0}^{s_1} (L^2 id)\left(\zeta_{s_1}, \zeta_{s_2}, x_{s_2}\right) ds_2\, ds_1.$$

Iterating this idea, we obtain an *integral equation expansion* of the solution of the RODE (2.3):

$$\begin{aligned} x_{s_0} = x_{t_0} &+ \sum_{j=1}^{k} \int_{t_0}^{s_0} \cdots \int_{t_0}^{s_{j-1}} (L^j\, id)\left(\zeta_{s_1}, \ldots, \zeta_{s_j}, x_{t_0}\right) ds_j \ldots ds_1 \\ &+ \int_{t_0}^{s_0} \cdots \int_{t_0}^{s_k} (L^{k+1}\, id)\left(\zeta_{s_1}, \ldots, \zeta_{s_{k+1}}, x_{s_{k+1}}\right) ds_{k+1} \ldots ds_1, \end{aligned} \tag{4.1}$$

for each $k \in \mathbb{N}_0$. Here $t_0 \le s_0$ and $t_0, s_0 \in [0, T]$.

To illustrate this integral equation expansion above we give two examples of that expansion. For $k = 0$ the integral equation expansion (4.1) reduces to

$$x_{s_0} = x_{t_0} + \int_{t_0}^{s_0} f(\zeta_{s_1}, x_{s_1})\, ds_1,$$

i.e., the original RODE (2.1), whereas for $k = 2$ we have

$$\begin{aligned} x_{s_0} = x_{t_0} &+ \int_{t_0}^{s_0} f(\zeta_{s_1}, x_{t_0})\, ds_1 + \int_{t_0}^{s_0}\int_{t_0}^{s_1} f_y(\zeta_{s_1}, x_{t_0}) f(\zeta_{s_2}, x_{t_0})\, ds_2\, ds_1 \\ &+ \int_{t_0}^{s_0}\int_{t_0}^{s_1}\int_{t_0}^{s_2} f_{yy}(\zeta_{s_1}, x_{s_3}) \big(f(\zeta_{s_2}, x_{s_3}),\, f(\zeta_{s_3}, x_{s_3})\big)\, ds_3\, ds_2\, ds_1 \\ &+ \int_{t_0}^{s_0}\int_{t_0}^{s_1}\int_{t_0}^{s_2} f_y(\zeta_{s_1}, x_{s_3}) \big(f_y(\zeta_{s_2}, x_{s_3}) \big(f(\zeta_{s_3}, x_{s_3})\big)\big)\, ds_3\, ds_2\, ds_1, \end{aligned} \tag{4.2}$$

where f_y and f_{yy} are abbreviations for the partial derivatives $\partial_y f$ and $\partial_y^2 f$. Note that for example the expression

$$f_y(\zeta_{s_1}, x_{s_3}) \big(f_y(\zeta_{s_2}, x_{s_3}) \big(f(\zeta_{s_3}, x_{s_3})\big)\big)$$

can be written as

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \partial_{y_i} f(\zeta_{s_1}, x_{s_3}) \cdot [\partial_{y_j} f(\zeta_{s_2}, x_{s_3})]^i \cdot [f(\zeta_{s_3}, x_{s_3})]^j,$$

where $[\cdot]^k$ for $k \in \mathbb{N}$ denotes the projection to the k-th component.

5 RODE-Taylor schemes

The integral equation expansions (4.1) of the solution of the RODE (2.3) can be simplified further by using the Assumption 3.1 that the driving stochastic process $(\zeta_t)_{t\geq 0}$ is Hölder-continuous.

Since $L^i\, id : \mathbb{R}^{m\times i} \times \mathbb{R}^d \to \mathbb{R}^d$ is in $\mathscr{F}_i$, we can approximate it by a Taylor expansion in its first $m \times i$ variables. We will use these in the integral equation expansion (4.1) to construct a temporally discretized approximation of the solution of the RODE. The resulting numerical schemes will be called RODE-Taylor schemes.

We will use the abbreviation

$$f_\alpha = \frac{1}{\alpha!}\left(\partial^\alpha L^{i(\alpha)}\, id\right) \in \mathscr{F}_{i(\alpha)}$$

for a matrix multi-index $\alpha \in \mathscr{A}_m$ and for a vector $v = (v_1, \ldots, v_l) \in \mathbb{R}^l$ for some $l \in \mathbb{N}$ we denote by $v^{\times k}$ for $k \in \mathbb{N}$ the $l \times k$ matrix

$$v^{\times k} := \underbrace{(v, \ldots, v)}_{k \text{ times}} = \begin{pmatrix} v_1 & \ldots & v_1 \\ \vdots & & \vdots \\ v_l & \ldots & v_l \end{pmatrix}.$$

Moreover, we will consider specific subsets of matrix multi-indices of the form

$$\mathscr{A}_m^\gamma := \big\{\alpha \in \mathscr{A}_m \,\big|\, i(\alpha) + \|\alpha^T \theta\|_1 < \gamma + 1\big\},$$

where $\gamma > 0$.

For simplicity, we consider an equidistant partition $(t_n^h)_{n=0,1,\ldots,N_h}$ of $[0,T]$ with constant stepsize $h \in (0,1]$, $N_h := \lceil \frac{T}{h} \rceil$, $t_n^h := nh$ for $n = 0,1,\ldots,N_h - 1$ and $t_{N_h}^h := T$. Here $\lceil a \rceil$ for a real number $a \geq 0$ is the smallest positive number $b \in \mathbb{N}_0$ such that $a \leq b$. For $\gamma > 0$ we define the function $\Phi_\gamma : \mathbb{R}^d \times [0,T] \times [0,1] \times \Omega \to \mathbb{R}^d$ by

$$\Phi_\gamma(x,t,h,\omega) := \sum_{\alpha \in \mathscr{A}_m^\gamma} f_\alpha(\zeta_t(\omega)^{\times i(\alpha)}, x) \cdot I_{\alpha,t,t+h}(\omega).$$

Finally, we define the *γ-RODE-Taylor scheme* ω-wise by

$$X_{n+1}^h(\omega) = \Phi_\gamma(X_n^h(\omega), t_n^h, \Delta_{t_n^h, t_{n+1}^h}, \omega), \quad n = 0,1,\ldots,N_h - 1, \tag{5.1}$$

with the initial value $X_0^h(\omega) = X_0(\omega)$ for each $\omega \in \Omega$. The γ-RODE-Taylor scheme generates a discrete time stochastic process $(X_n^h)_{n=0,1,\ldots,N_h}$. To facilitate error estimates, we will extend it to a continuous time stochastic process $(\bar{x}_t^h)_{t\in[0,T]}$ which we define by $\bar{x}_{t_n^h}^h = X_n$ for $n = 0,1,\ldots,N_h$ and

$$\bar{x}_t^h = \Phi_\gamma(\bar{x}_{t_n^h}^h, t_n^h, \Delta_{t_n^h,t}) \quad \text{for } t \in (t_n^h, t_{n+1}^h),\ n = 0,1,\ldots,N_h - 1.$$

The process $(\bar{x}_t^h)_{t\in[0,T]}$ obviously has continuous sample paths and it is also non-anticipative if the stochastic process ζ_t is non-anticipative.

The following pathwise global discretization error estimate will be proved in Sect. 10.

Theorem 5.1 (Discretization error of the RODE-Taylor schemes) *Let $\gamma > 0$ and let $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_m) \in [0,1]^m$ with $\|\varepsilon\|_1 = \sum_{i=1}^m \varepsilon_i < \gamma$. Then,*

$$\sup_{0\leq t\leq T} \|x_t(\omega) - \bar{x}_t^h(\omega)\|_2 \leq D_\gamma^\varepsilon(\omega) \cdot h^{(\gamma - \|\varepsilon\|_1)}$$

for all $0 < h \leq h_0(\omega)$ and all $\omega \in \Omega$ with the nonnegative random variables

$$D_\gamma^\varepsilon = (C_\gamma^\varepsilon T) e^{L_\gamma T}, \qquad h_0 = \left(D_\gamma^\varepsilon\right)^{1/(\|\varepsilon\|_1 - \gamma)} \wedge 1,$$

where

$$C_\gamma^\varepsilon := \sum_{\substack{\alpha \in \mathscr{A}_m \setminus \mathscr{A}_m^\gamma \\ i(\alpha) \leq \lceil \gamma + 1 \rceil \\ |\alpha| \leq \lceil \frac{\gamma}{\vartheta} \rceil}} \frac{1}{i(\alpha)!} \left(|\zeta|_{\theta - \varepsilon/\lceil \frac{\gamma}{\vartheta} \rceil} \right)^{|\alpha|} \sup_{\substack{\|w_1\|_2,\ldots,\|w_{i(\alpha)}\|_2 \leq |\zeta|_\infty \\ \|y\|_2 \leq |x|_\infty}} \|f_\alpha(w_1,\ldots,w_{i(\alpha)},y)\|_2,$$

and

$$L_\gamma := \sum_{\alpha \in \mathscr{A}_m^\gamma \setminus \{\emptyset\}} (2|\zeta|_\infty)^{|\alpha|} \left(\sup_{0\leq t\leq T} \sup_{\|y - x_t\| \leq 1} \|\partial_y f_\alpha(\zeta_t^{\times i(\alpha)}, y)\|_2 \right).$$

Suppose that the stochastic process $(\zeta_t)_{t\geq 0}$, $\theta = (\theta_1, \ldots, \theta_m)$ for ζ_t and the desired order $\gamma > 0$ are given. Then $\varepsilon \in [0,1]^m$ should be chosen so, that $\|\varepsilon\|_1 < \gamma$ and that for $i = 1, \ldots, n$ we have $\varepsilon_i = 0$ iff $|\zeta^i|_{\theta_i} < \infty$. In this case, we have $D_\gamma^\varepsilon < \infty$.

5.1 The essential RODE-Taylor schemes

Although we defined the γ-RODE-Taylor scheme for every $\gamma > 0$ only some $\gamma > 0$ are important. Consider the set

$$\mathscr{V}_\theta := \mathbb{N}_0 + \theta_1\mathbb{N}_0 + \cdots + \theta_m\mathbb{N}_0 = \left\{k + l_1\theta_1 + \cdots + l_m\theta_m \middle| k, l_1, \ldots, l_m \in \mathbb{N}_0\right\}.$$

It is not hard to see that $\mathscr{V}_\theta$ coincides with a sequence

$$0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots .$$

From Lemma 5.1 below we conclude that for $\gamma \in (\gamma_i, \gamma_{i+1}]$ for some $i \in \mathbb{N}_0$ the γ-RODE-Taylor scheme is the same as the γ_{i+1}-RODE-Taylor scheme. Hence we need only consider the γ_i-RODE-Taylor schemes for $i \in \mathbb{N}$.

Define

$$[\gamma] = [\gamma]_\theta := \min_{\substack{\gamma_i \geq \gamma \\ \gamma_i \in \mathscr{V}_\theta}} \gamma_i$$

for $\gamma > 0$. Clearly, $\gamma \leq [\gamma]$ and $[\gamma] \in \mathscr{V}_\theta$.

Lemma 5.1 *Let $\gamma, \tilde{\gamma} > 0$. Then*

$$\mathscr{A}_m^\gamma = \mathscr{A}_m^{\tilde{\gamma}} \quad \textit{if and only if} \quad [\gamma] = [\tilde{\gamma}].$$

In particular, $\mathscr{A}_m^\gamma = \mathscr{A}_m^{[\gamma]}$ and $\Phi_\gamma = \Phi_{[\gamma]}$.

Proof We can assume without loss of generality that $\tilde{\gamma} \geq \gamma$.

First we prove "$\Leftarrow$" that direction. Of course, $\mathscr{A}_m^\gamma \subset \mathscr{A}_m^{\tilde{\gamma}}$. We will suppose that there is an $\alpha \in \mathscr{A}_m^{\tilde{\gamma}} \setminus \mathscr{A}_m^\gamma$ and obtain

$$\gamma + 1 \leq i(\alpha) + \|\alpha^T\theta\|_1 < \tilde{\gamma} + 1.$$

Obviously, $i(\alpha) \geq 1$, so

$$\gamma \leq \underbrace{(i(\alpha) - 1) + \|\alpha^T\theta\|_1}_{\in \mathscr{V}_\theta} < \tilde{\gamma},$$

which implies that

$$[\gamma] \leq \underbrace{(i(\alpha) - 1) + \|\alpha^T\theta\|_1}_{\in \mathscr{V}_\theta} < \tilde{\gamma} \leq [\tilde{\gamma}] = [\gamma]$$

and we obtain a contradiction.

Now we prove the "⇒" direction. We assume that $[\gamma] < [\tilde{\gamma}]$. Then there are $a, b_1, \ldots, b_m \in \mathbb{N}_0$ such that

$$\gamma \leq [\gamma] = a + b_1\theta_1 + \cdots + b_m\theta_m < [\tilde{\gamma}]$$

and, furthermore,

$$[\gamma] < \tilde{\gamma} \leq [\tilde{\gamma}].$$

Now define

$$\alpha = \big((\alpha_{1,1}, \ldots, \alpha_{1,m}), \ldots, (\alpha_{i,1}, \ldots, \alpha_{i,m})\big) \in \mathbb{N}^{m \times i}$$

with $i = a + 1 \in \mathbb{N}$ and

$$\begin{aligned} &\alpha_{1,1} = b_1, \qquad \alpha_{1,2} = b_2, \quad \ldots, \quad \alpha_{1,m} = b_m, \\ &\alpha_{2,1} = \cdots = \alpha_{2,m} = 0, \quad \ldots, \quad \alpha_{i,1} = \cdots = \alpha_{i,m} = 0. \end{aligned}$$

Finally, we have

$$\gamma + 1 \leq \underbrace{i(\alpha) + \|\alpha^T\theta\|_1}_{=1+a+b_1\theta_1+\cdots+b_m\theta_m=1+[\gamma]} < \tilde{\gamma} + 1,$$

so α is in $\mathcal{A}_m^{\tilde{\gamma}}$ but not in $\mathcal{A}_m^{\gamma}$, which is a contradiction. □

6 Examples of the RODE-Taylor schemes

We present three representative examples of the RODE-Taylor schemes. In particular, given $0 \leq t_0 < t_0 + h \leq T$ and $y_0 \in \mathbb{R}^d$, we determine the function $\Phi_\gamma(y_0, t_0, h)$ with stepsize $h \in (0, 1]$ of the γ-RODE-Taylor scheme for different γ, θ, d and m.

6.1 The case: $\gamma = 1.5$, $\theta = (\frac{1}{2}, \frac{3}{4})$, $d = m = 2$

Both the RODE and the driving stochastic process $\zeta_t = (\zeta_t^1, \zeta_t^2)$ are 2-dimensional. In view of the choice of θ, the first component of the noise process could be a Brownian motion and the second a fractional Brownian motion with Hurst Index $H = \frac{3}{4}$.

Since $\theta = (\frac{1}{2}, \frac{3}{4})$ we have

$$\mathbb{N}_0 + \frac{1}{2}\mathbb{N}_0 + \frac{3}{4}\mathbb{N}_0 = \left\{0, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}, \ldots\right\}.$$

A representative choice is $\gamma = \gamma_5 = 1.5$ with the multi-index set

$$\mathcal{A}_2^{1.5} = \left\{\alpha \in \mathcal{A}_2 \;\middle|\; i(\alpha) + \left\|\alpha^T \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix}\right\|_1 \leq \frac{9}{4}\right\},$$

that is,

$$\mathscr{A}_2^{1.5} = \left\{\emptyset, \binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}, \binom{2}{0}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

For these multi-indices in $\mathscr{A}_2^{1.5}$ we have

$$f_{\emptyset}(y) = y, \qquad f_{\binom{i}{j}}(w, y) = \frac{\partial_{w_2}^j \partial_{w_1}^i f(w, y)}{i!j!},$$

$$f_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)}(w, y) = \sum_{j=1}^{2} \partial_{y_j} f(w, y)[f(w, y)]^j$$

and

$$I_{\binom{i}{j}, t_0, t_0+h} = \int_{t_0}^{t_0+h} \left(\Delta\zeta_{t_0,t}^1\right)^i \left(\Delta\zeta_{t_0,t}^2\right)^j dt, \qquad I_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), t_0, t_0+h} = \frac{h^2}{2}.$$

Hence, the 1.5-RODE-Taylor scheme in this case is

$$\begin{aligned} \Phi_{1.5}(y_0, t_0, h) = y_0 &+ hf + \partial_{w_1} f \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^1 \, dt + \partial_{w_2} f \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^2 \, dt \\ &+ \partial_{w_1, w_2} f \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^1 \Delta\zeta_{t_0,t}^2 \, dt + \frac{\partial_{w_1}^2 f}{2} \int_{t_0}^{t_0+h} \left(\Delta\zeta_{t_0,t}^1\right)^2 dt \\ &+ \left(\partial_{y_1} f[f]^1 + \partial_{y_2} f[f]^2\right) \frac{h^2}{2}, \end{aligned} \tag{6.1}$$

where the coefficient functions are evaluated at (ζ_{t_0}, y_0).

6.2 The case: $\gamma = 3, \theta = (\frac{1}{2}), d = m = 1$

Here the RODE and the driving stochastic process ζ_t are 1-dimensional and the noise process could be a Brownian motion. We consider the Taylor scheme for $\gamma = \gamma_6 = 3.0 \in \mathbb{N}_0 + \frac{1}{2}\mathbb{N}_0$. The multi-index set here is

$$\mathscr{A}_1^3 = \left\{ i(\alpha) + \frac{1}{2}\|\alpha\|_1 \leq \frac{7}{2} \right\},$$

that is

$$\mathscr{A}_1^3 = \left\{ \begin{array}{c} \emptyset, (0), (1), (2), (3), (4), (5) \\ (0,0), (1,0), (0,1), (1,1), (2,0), (0,2) \\ (2,1), (1,2), (3,0), (0,3) \\ (0,0,0), (1,0,0), (0,1,0), (0,0,1) \end{array} \right\}.$$

The 3.0-RODE-Taylor scheme in this case is thus

$$\begin{aligned}\Phi_{3.0}(y_0,t_0,h) = y_0 + hf + \sum_{i=1}^{5}\frac{1}{i!}\partial_w^i f\, I_{(i)} + \frac{1}{2}f_y f h^2 + f_{w,y} f\, I_{(1,0)} \\ + f_y f_w\, I_{(0,1)} + f_{w,y} f_w\, I_{(1,1)} + \frac{1}{2}f_{w,w,y} f\, I_{(2,0)} + \frac{1}{2}f_y f_{w,w}\, I_{(0,2)} \\ + \frac{1}{2}f_{w,w,y} f_w\, I_{(2,1)} + \frac{1}{2}f_{w,y} f_{w,w}\, I_{(1,2)} + \frac{1}{6}(\partial_w^3 f_y) f\, I_{(3,0)} \\ + \frac{1}{6}f_y(\partial_w^3 f)\, I_{(0,3)} + \frac{1}{6}f_y^2 f h^3 + \frac{1}{6}f_{y,y} f^2 h^3 \\ + \left(f_{w,y} f_y f + f_{w,y,y} f^2\right) I_{(1,0,0)} + \left(f_y f_{w,y} f + f_{y,y} f_w f\right)\, I_{(0,1,0)} \\ + \left(f_y^2 f_w + f_{y,y} f f_w\right) I_{(0,0,1)}, \qquad (6.2)\end{aligned}$$

where $f_{\underbrace{w,\ldots,w}_{l\text{ times}},\underbrace{y,\ldots,y}_{k\text{ times}}}$ is just $\partial_w^l \partial_y^k f(\zeta_{t_0}, y_0)$ and the integrals over the interval $[t_0, t_0+h]$ are given by

$$I_{(i)} = \int_{t_0}^{t_0+h} (\Delta\zeta_{t_0,t})^i\, dt \quad \text{for } i = 1,\ldots,6,$$

$$I_{(1,0)} = \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}\Delta_{t_0,t}\, dt, \qquad I_{(0,1)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} \Delta\zeta_{t_0,t}\, dt\, ds,$$

$$I_{(1,1)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} \Delta\zeta_{t_0,s}\Delta\zeta_{t_0,t}\, dt\, ds, \qquad I_{(2,0)} = \int_{t_0}^{t_0+h} (\Delta\zeta_{t_0,t})^2\Delta_{t_0,t}\, dt,$$

$$I_{(0,2)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} (\Delta\zeta_{t_0,t})^2\, dt\, ds, \qquad I_{(2,1)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} (\Delta\zeta_{t_0,s})^2\Delta\zeta_{t_0,t}\, dt\, ds,$$

$$I_{(1,2)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} \Delta\zeta_{t_0,s}(\Delta\zeta_{t_0,t})^2\, dt\, ds, \qquad I_{(3,0)} = \int_{t_0}^{t_0+h} (\Delta\zeta_{t_0,t})^3\Delta_{t_0,t}\, dt,$$

$$I_{(0,3)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} (\Delta\zeta_{t_0,t})^3\, dt\, ds, \qquad I_{(1,0,0)} = \frac{1}{2}\int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}(\Delta_{t_0,t})^2\, dt,$$

$$I_{(0,1,0)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s} \Delta\zeta_{t_0,t}\Delta_{t_0,t}\, dt\, ds, \qquad I_{(0,0,1)} = \int_{t_0}^{t_0+h}\int_{t_0}^{s}\int_{t_0}^{t} \Delta\zeta_{t_0,v}\, dv\, dt\, ds.$$

6.3 The case: $\gamma = 0.6, 1.6, 2.1$, $\theta = (0.3)$, $d = m = 1$

In this scalar case we consider $\gamma = 0.6$, $\gamma = 1.6$ and $\gamma = 2.1$. Since

$$\mathbb{N}_0 + \theta_1\mathbb{N}_0 = \{0, 0.3, 0.6, 0.9, 1, 1.2, 1.3, 1.5, 1.6, 1.8, 1.9, 2.0, 2.1, 2.2, 2.3, \ldots\},$$

we have $\gamma_2 = 0.6$, $\gamma_8 = 1.6$ and $\gamma_{12} = 2.1$. The multi-index sets are

$$\mathscr{A}_1^{0.6} = \{\emptyset, (0), (1)\},$$
$$\mathscr{A}_1^{1.6} = \{\emptyset, (0), (1), (2), (3), (4), (5), (0,0), (1,0), (0,1)\}$$

and

$$\mathscr{A}_1^{2.1} = \left\{ \begin{array}{c} \emptyset, (0), (1), (2), (3), (4), (5), (6) \\ (0,0), (1,0), (0,1), (1,1), (2,0), (0,2) \\ (3,0), (0,3), (2,1), (1,2), (0,0,0) \end{array} \right\}.$$

Therefore,

$$\Phi_{0.6}(y_0, t_0, h) = y_0 + hf + f_w \int_{t_0}^{t} \Delta\zeta_{t_0,u}\, du$$

is the 0.6-RODE-Taylor scheme,

$$\Phi_{1.6}(y_0, t_0, h) = y_0 + hf + \sum_{i=1}^{5} \frac{1}{i!} \partial_w^i f\, I_{(i)} + f_y f \frac{h^2}{2} + f_{w,y} f\, I_{(1,0)} + f_y f_w I_{(0,1)}$$

for the 1.6-RODE-Taylor scheme, and

$$\begin{aligned} \Phi_{2.1}(y_0, t_0, h) = \Phi_{1.6}(y_0, t_0, h) &+ \frac{1}{6!} \partial_w^6 f\, I_{(6)} + f_{w,y} f_w\, I_{(1,1)} + \frac{1}{2} f_{w,w,y} f\, I_{(2,0)} \\ &+ \frac{1}{2} f_y f_{w,w}\, I_{(0,2)} + \frac{1}{2} f_{w,w,y} f_w\, I_{(2,1)} + \frac{1}{2} f_{w,y} f_{w,w}\, I_{(1,2)} \\ &+ \frac{1}{6} \partial_w^3 f_y f\, I_{(3,0)} + \frac{1}{6} f_y \partial_w^3 f\, I_{(0,3)} + \frac{1}{6} f_y^2 f\, h^3 + \frac{1}{6} f_{y,y} f^2 h^3 \end{aligned} \tag{6.3}$$

for the 2.1-RODE-Taylor scheme.

7 Other numerical schemes for RODEs

For numerical comparison later, we recall briefly other numerical schemes for RODEs from the literature. These are also one-step schemes so we just give their functions $\Phi(y_0, t_0, h)$.

7.1 The local linearization scheme for RODEs

The local linearization method (LL) for RODEs was proposed by Carbonell et al. [3], where its discretization error was analyzed. In our context the LL scheme has the form

$$\Phi_{LL}(y_0, t_0, h) = y_0 + \int_{t_0}^{t_0+h} e^{f_y \cdot (t_0+h-t)}\, dt \cdot f + \sum_{j=1}^{m} \int_{t_0}^{t_0+h} e^{f_y \cdot (t_0+h-t)} \Delta\zeta_{t_0,t}^{j}\, dt \cdot f_{w_j}$$

respectively

$$\Phi_{LL}(y_0, t_0, h) = y_0 + \int_0^h e^{f_y \cdot t}\, dt \cdot f + \sum_{j=1}^{m} \int_{t_0}^{t_0+h} e^{f_y \cdot (t_0+h-t)}\, \Delta\zeta_{t_0,t}^{j}\, dt \cdot f_{w_j} \quad (7.1)$$

if we omit the argument (ζ_{t_0}, x_{t_0}) of the functions f_y, f and f_{w_j} for $j = 1, \ldots, m$. Note that $f_y(\zeta_{t_0}, x_{t_0})$ is a $d \times d$-matrix, so $e^{f_y(\zeta_{t_0}, x_{t_0}) \cdot (t_0+h-t)}$ is the matrix exponential function, and the $f_{w_j}(\zeta_{t_0}, x_{t_0})$ are vectors.

In [3] it was assumed that all of the components of the driving noise were of the same type, i.e., $\theta = (\vartheta, \ldots, \vartheta) \in (0, 1]^m$, and that the vector field f was global Lipschitz continuous in the state variable uniformly in the noise variables. Under these assumptions, it was shown in [3] that the LL scheme converges with order 2ϑ. It is thus comparable with a 2ϑ-RODE-Taylor scheme, which is given by

$$\Phi_{2\vartheta}(y_0, t_0, h) = y_0 + hf + \sum_{j=1}^{m} f_{w_j} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt$$

when $\vartheta \leq \frac{1}{2}$ and

$$\Phi_{2\vartheta}(y_0, t_0, h) = y_0 + hf + \sum_{j=1}^{m} f_{w_j} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt + \frac{1}{2} f_y f\, h^2$$

when $\vartheta > \frac{1}{2}$. In the second case, we obtain

$$\begin{aligned}
\Phi_{2\vartheta}(y_0, t_0, h) &= y_0 + hf + \sum_{j=1}^{m} f_{w_j} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt + \frac{1}{2} f_y f\, h^2 \\
&= y_0 + \left(h \cdot I + \frac{h^2 f_y}{2} \right) f + \sum_{j=1}^{m} f_{w_j} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt \\
&= y_0 + (f_y)^{-1} \left(hf_y + \frac{1}{2}(hf_y)^2 \right) f + \sum_{j=1}^{m} f_{w_j} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt
\end{aligned}$$

and therefore

$$\begin{aligned}
\Phi_{2\vartheta}(y_0, t_0, h) &\approx y_0 + (f_y)^{-1} \left(e^{hf_y} - I \right) f + \sum_{j=1}^{m} f_{w_j} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt \\
&= y_0 + \int_0^h e^{f_y \cdot t} dt \cdot f + \sum_{j=1}^{m} \int_{t_0}^{t_0+h} \Delta\zeta_{t_0,t}^{j}\, dt \cdot f_{w_j} \\
&\approx \Phi_{LL}(y_0, t_0, h),
\end{aligned}$$

if f_y is invertible. The coefficients and integrals of the RODE-Taylor scheme are easier to compute when the noise process is scalar or low dimensional and the state

space dimensional is high (although they still require the same number of evaluations of the driving stochastic process ζ_t, see Sect. 8), since in the LL scheme (7.1) one then has to calculate a large matrix exponential $e^{f_y \cdot (t_0+h-t)}$ for many $t \in [t_0, t_0+h]$ in approximations of the integrals

$$\int_{t_0}^{t_0+h} e^{f_y \cdot (t_0+h-t)} \Delta\zeta_{t_0,t}^j \, dt, \quad j = 1, \ldots, m.$$

7.2 The averaged Euler scheme

Grüne and Kloeden [8] introduced the averaged Euler scheme for RODEs with the vector field f special separable structure (hence $m = d+1$)

$$f(w_1, \ldots, w_d, w_{d+1}, x) = \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} + w_{d+1} \cdot H(x)$$

for a smooth function $H : \mathbb{R}^d \to \mathbb{R}^d$. The averaged Euler scheme for such RODEs is given by

$$\Phi_{AE}(y_0, t_0, h) = y_0 + \int_{t_0}^{t_0+h} \begin{pmatrix} \zeta_t^1 \\ \vdots \\ \zeta_t^d \end{pmatrix} dt + H(y_0) \int_{t_0}^{t_0+h} \zeta_t^{d+1} \, dt$$

and has order 1. It is easy to see that the averaged Euler scheme is exactly the 1-RODE-Taylor scheme for these RODEs.

7.3 The numerical schemes in [12]

In [12] Jentzen and Kloeden introduced numerical schemes for RODEs that converge with arbitrary high order. The schemes there are very similar to the RODE-Taylor schemes in this paper. For example, in the situation $d = m = 1$ and $\theta = (\frac{1}{2})$ the schemes there are the RODE-Taylor schemes of order $\gamma = 0.5, 1, 1.5, 2.0$ and 2.5. However, for higher order they contain not only the terms of the RODE-Taylor schemes in this paper, but also more terms than are needed for the required order. An example of that situation is $\gamma = 3.0$. A tedious calculation gives the 3.0-scheme of [12] as

$$\Phi(y_0, t_0, h) = \Phi_{3.0}(y_0, t_0, h) + \frac{f_{yy} f_w^2}{2} \int_{t_0}^{t_0+h} \left(\int_{t_0}^{s} \Delta\zeta_u \, du \right)^2 ds,$$

where $\Phi_{3.0}(y_0, t_0, h)$ is the 3.0-RODE-Taylor scheme of this paper (see Sect. 6.2). The reason for this phenomenon is the recursive technique used in [12].

7.4 A random Euler scheme

In [13] Jentzen and Neuenkirch analyzed a numerical scheme for RODEs that converges independent of ϑ with the order $\frac{1}{2}$ to the exact solution of the RODE. This scheme is called random Euler scheme there and is given by

$$\Phi(y_0, t_0, h) = x_0 + h \cdot f\left(\zeta_{t_0+R\cdot h}, x_0\right),$$

where R is a on $[0, 1]$ uniform distributed random variable. Hence, not only the equation, but also the numerical scheme there is random. With this approach the stochastic process $(\zeta_t)_{t\geq 0}$ may even be discontinuous, in particular processes with jumps are allowed.

8 Approximation of the integrals

We recall that the γ-RODE-Taylor scheme was defined by

$$\Phi_\gamma(x, t, h) := \sum_{\alpha\in\mathscr{A}_m^\gamma} f_\alpha(\zeta_t^{\times i(\alpha)}, x)\cdot I_{\alpha,t,t+h}.$$

To implement such a scheme, we have to approximate the integrals

$$I_{\alpha,t_0,s_0} = \int_{t_0}^{s_0}\cdots\int_{t_0}^{s_{i(\alpha)-1}} \left(\Delta\zeta_{t_0,s_1}\right)^{\alpha_1}\cdots\left(\Delta\zeta_{t_0,s_{i(\alpha)}}\right)^{\alpha_{i(\alpha)}} ds_{i(\alpha)}\ldots ds_1$$

for appropriate time intervals $[t_0, s_0]\subset[0, T]$ and multi-indices $\alpha\in\mathscr{A}_m^\gamma$.

We will use Riemann sums because the sample paths of the driving stochastic process $\zeta_t(\omega)$ are only Hölder continuous and not differentiable, so there are, in general, no better quadrature formulas of higher order.

Write $i = i(\alpha)$ and let $n\in\mathbb{N}$. This natural number will be the discretization accuracy for the integrals. We approximate the above integrals as follows:

$$\begin{aligned}
I_{\alpha,t_0,s_0} &= \int_{t_0}^{s_0}\int_{t_0}^{s_1}\cdots\int_{t_0}^{s_{i-1}} \left(\left(\Delta\zeta_{t_0,s_1}\right)^{\alpha_1}\cdots\left(\Delta\zeta_{t_0,s_i}\right)^{\alpha_i}\right) ds_i\ldots ds_2\, ds_1\\
&= \int_{t_0}^{s_0}\left(\Delta\zeta_{t_0,s_1}\right)^{\alpha_1}\left(\cdots\left(\int_{t_0}^{s_{i-1}}\left(\Delta\zeta_{t_0,s_i}\right)^{\alpha_i} ds_i\right)\cdots\right) ds_1\\
&\approx \left(\frac{\Delta_{t_0,s_0}}{n}\right)^i \sum_{j_1=1}^{n-1}\left(\Delta\zeta_{t_0,\tau_{j_1}}\right)^{\alpha_1}\left(\cdots\left(\sum_{j_i=1}^{j_{i-1}-1}\left(\Delta\zeta_{t_0,\tau_{j_i}}\right)^{\alpha_i}\right)\cdots\right)\\
&=: Q_{\alpha,t_0,s_0,n}
\end{aligned}$$

with equidistant evaluation points $\tau_j = t_0 + \frac{j}{n}\Delta_{t_0,s_0}$ for $j = 0, 1, \ldots, n$.

Note that there are not more than $m\cdot i\cdot n = O(n)$ calculation steps needed to evaluate the expression $Q_{\alpha,t_0,s_0,n}$ with the following strategy:

– Simulate one random realization of $\Delta\zeta_{t_0,\tau_j} = \zeta_{\tau_j} - \zeta_{t_0}$ for $j = 0, 1, \ldots, n$.
– Calculate $F_i(k) = \sum_{j=1}^{k} (\Delta\zeta_{t_0,\tau_j})^{\alpha_i}$ for each $k = 0, \ldots, n-1$.
– Calculate $F_{i-1}(k) = \sum_{j=1}^{k} (\Delta\zeta_{t_0,\tau_j})^{\alpha_{i-1}} F_i(j-1)$ for every $k = 0, \ldots, n-1$.
– ...
– Calculate $F_1(k) = \sum_{j=1}^{k} (\Delta\zeta_{t_0,\tau_j})^{\alpha_1} F_2(j-1)$ for every $k = 0, \ldots, n-1$.
– Finally $Q_{\alpha,t_0,s_0,n} = (\frac{\Delta_{t_0,s_0}}{n})^i \cdot F_1(n-1)$.

Of course, we can write

$$Q_{\alpha,t_0,s_0,n} = \left(\frac{\Delta_{t_0,s_0}}{n}\right)^i \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{j_1-1} \cdots \sum_{j_i=1}^{j_{i-1}-1} \left((\Delta\zeta_{t_0,\tau_{j_1}})^{\alpha_1} \cdots (\Delta\zeta_{t_0,\tau_{j_i}})^{\alpha_i}\right).$$

It can be shown that, if one chooses

$$n(h,\gamma) = \left\lceil h^{(1-\frac{\gamma-\|\varepsilon\|_1}{(\vartheta-\delta)})} \right\rceil$$

in the expressions $Q_{\alpha,t_0,s_0,n(h,\gamma)}$ for a small δ with $0 < \delta < \vartheta$, then the numerical scheme

$$\tilde{\Phi}_\gamma(x_0,t_0,h) := \sum_{\alpha\in\mathscr{A}_m^\gamma} f_\alpha \cdot Q_{\alpha,t_0,t_0+h,n(h,\gamma)}$$

also has the desired order $\gamma - \|\varepsilon\|_1$ for any $\gamma > 0$.

The number of evaluations of the stochastic process ζ_t does not change for different RODE-Taylor schemes for the same accuracy. Why use a high order RODE-Taylor scheme then? Larger step sizes can be used to achieve a given desired accuracy, so fewer evaluations of the vector field f and its derivatives (but not fewer evaluations of the stochastic process) are required. This is particularly advantageous when the spatial dimension is high and the noise dimension is small (see [8]). If one descretizes a parabolic partial differential equation, that is disturbed by a finite dimensional additive noise process, in space, than one ends up with a high dimensional SDE respectively RODE but with low dimensional noise. See for example Grecksch and Kloeden [6] for those considerations and see also Da Prato and Zabczyk [4] and Prévot and Röckner [15] for examples of stochastic partial differential equations with additive noise.

9 Numerical results

The above theoretical results are illustrated here with two numerical examples of the RODE-Taylor schemes presented in Sects. 6.2 and 6.3 as well as the LL scheme of Sect. 7.1.

In each figure the error of the approximation of the solution in the interval $[0, 1]$ is calculated and shown in the log-log-plot.

In Fig. 1 for the first example the three orderlines correspond to the orders 0.5, 2.0 and 3.0, while in Fig. 2 for the second example the three order lines correspond

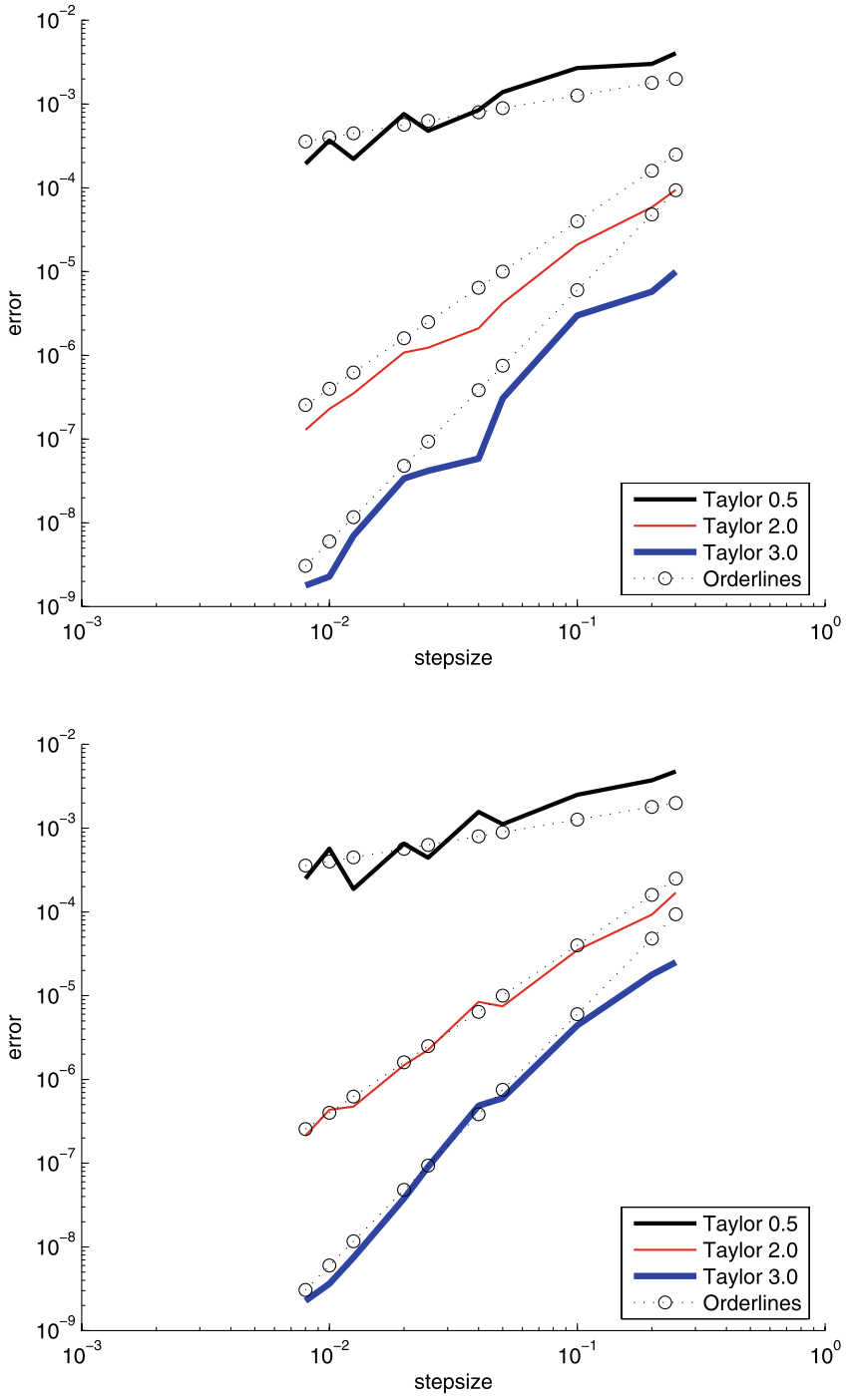

Fig. 1 Example (9.1)/(9.2): pathwise maximum error vs. stepsize for two random realizations

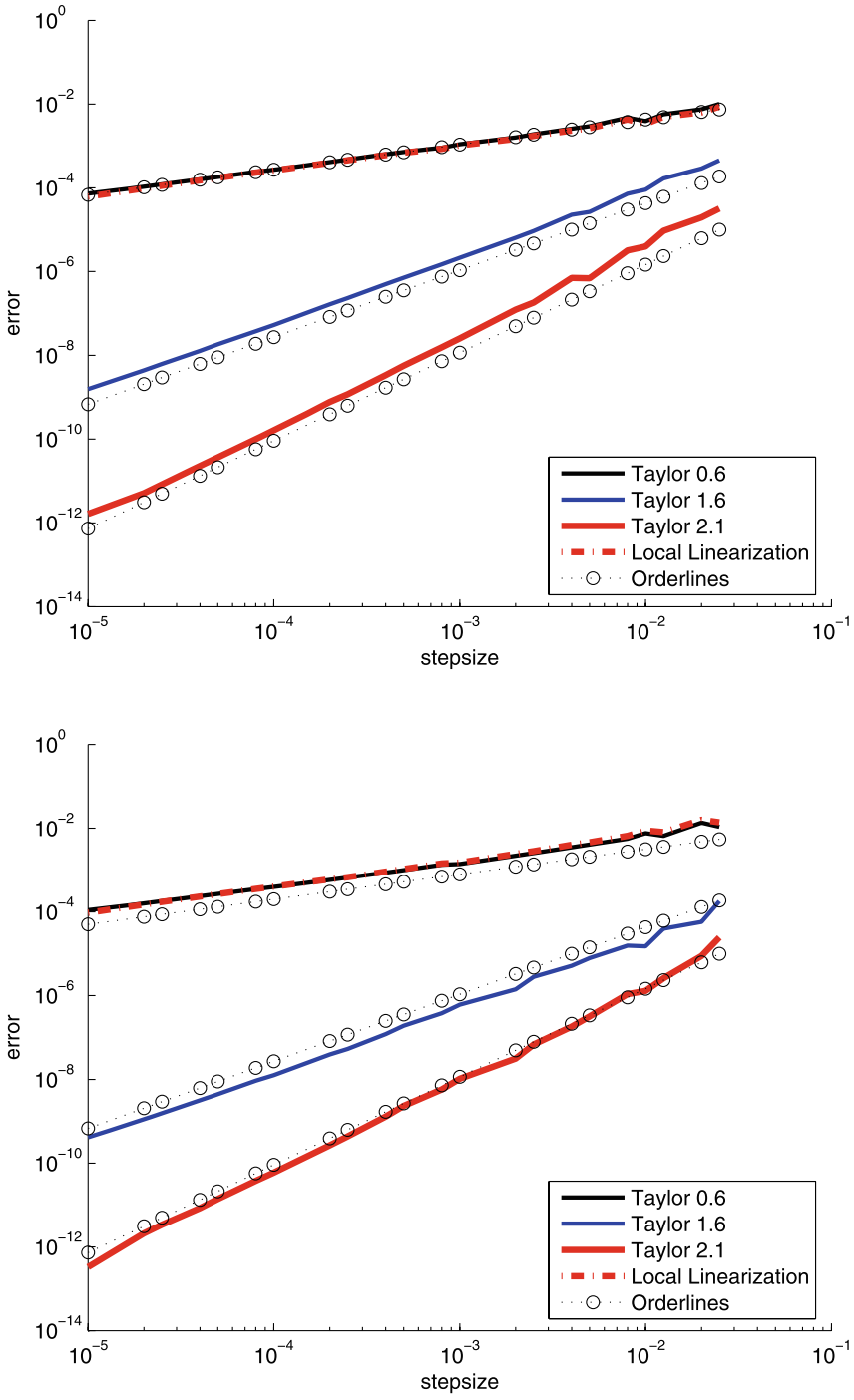

Fig. 2 Example (9.3)/(9.4): pathwise maximum error vs. stepsize for two random realizations

to the orders 0.6, 1.6 and 2.1. All examples of the RODE-Taylor schemes converge with their predicted order and the local linearization scheme in the second example also converges with its predicted order 0.6.

9.1 First example

We consider the scalar ODE (see [12])

$$\dot{x}_t = f(\zeta_t, x_t), \qquad f(w,x) = -\frac{1}{11}(w-1)^2\left(x-\frac{1}{2}\right)^2, \qquad x_0 = 1, \tag{9.1}$$

with the noise process ζ_t given by

$$\zeta_t = \frac{1}{|W_t| + \frac{1}{2}} + \frac{1}{11}\int_0^t \sqrt{\left|W_s + \frac{1}{2}\right|}\, ds + |V_t|, \qquad \theta = (\theta_1) = \left(\frac{1}{2}\right),$$

where W_t and V_t denote two independent standard Brownian motions.

This RODE with the initial value $x_0 = 1$ has the explicit pathwise solution

$$x_t(\omega) = \frac{1}{2} + \frac{1}{2 + \frac{1}{11}\int_0^t (\zeta_s(\omega) - 1)^2\, ds} \tag{9.2}$$

for each $\omega \in \Omega$ on $[0, 1]$.

We simulate the 0.5-, 2.0- and the 3.0-RODE-Taylor scheme (see Fig. 1). Note that the 0.5-RODE-Taylor scheme here is just the classical Euler scheme.

9.2 Second example

Now we consider the linear scalar RODE (see [8])

$$\dot{x}_t = f(\zeta_t, x_t), \qquad x_0 = X_0 \tag{9.3}$$

with

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad f(w,x) = \sin(w) \cdot x$$

and where $(\zeta_t)_{t\in[0,1]}$ is a fractional Brownian motion with Hurst Index $H = 0.3$, so $\theta = (0.3)$ and where X_0 is a random variable, uniformly distributed on the interval $[\frac{1}{2}, \frac{3}{2}]$ and independent of $(\zeta_t)_{t\in[0,1]}$. Thus this RODE has the unique pathwise solution

$$x_t(\omega) = e^{\int_0^t \sin(\zeta_s(\omega))\, ds} \cdot X_0(\omega) \tag{9.4}$$

for each $\omega \in \Omega$ on $[0, 1]$.

Here we simulate paths of the 0.6-, 1.6-, 2.1-RODE-Taylor scheme and the local linearization method in Fig. 2.

10 Proof of Theorem 5.1

We prove Theorem 5.1 with the following lemmata.

Lemma 10.1 *Let $l \in \mathbb{N}_0$, $i \in \mathbb{N}$. Then*

$$\begin{aligned}
&(L^i id)\left(\zeta_{s_1}, \dots, \zeta_{s_i}, x_{t_0}\right) \\
&\quad = \sum_{\substack{|\alpha| \le l \\ \alpha \in \mathbb{N}_0^{m \times i}}} f_\alpha(\zeta_{t_0}^{\times i}, x_{t_0}) \cdot (\Delta\zeta_{t_0,s_1})^{\alpha_1} \dots (\Delta\zeta_{t_0,s_i})^{\alpha_i} \\
&\qquad + \sum_{\substack{|\alpha| = l+1 \\ \alpha \in \mathbb{N}_0^{m \times i}}} \left[\int_0^1 f_\alpha(\zeta_{t_0} + u\Delta\zeta_{t_0,s_1}, \dots, \zeta_{t_0} + u\Delta\zeta_{t_0,s_i}, x_{t_0}) \cdot (1-u)^l \, du \right] \\
&\qquad \times (l+1) \cdot (\Delta\zeta_{t_0,s_1})^{\alpha_1} \cdots (\Delta\zeta_{t_0,s_i})^{\alpha_i}
\end{aligned}$$

holds for all $t_0, s_1, \dots, s_i \in [0, T]$.

Proof Fix $\omega \in \Omega$. Then we simply apply a Taylor expansion of order l to the function $g : [0, 1] \to \mathbb{R}^d$ given by

$$g(u) := (L^i id)(\zeta_{t_0}(\omega) + u\Delta\zeta_{t_0,s_1}(\omega), \dots, \zeta_{t_0}(\omega) + u\Delta\zeta_{t_0,s_i}(\omega), x_{t_0}(\omega))$$

and obtain

$$g(1) = g(0) + g'(0) + \cdots + \frac{g^{(l)}(0)}{l!} + \int_0^1 g^{(l+1)}(u) \, \frac{(1-u)^l}{l!} \, du,$$

which is the assertion. □

We need an estimate for the integrand in I_{α,t_0,s_0}.

Lemma 10.2 *Let $[t_0, s_0] \subset [0, T]$, $t_0 \le s_0$ and $\eta = (\eta_1, \dots, \eta_m) \in (0, 1]^m$. Then,*

$$|(\Delta\zeta_{t_0,s_1})^{\alpha_1} \cdots (\Delta\zeta_{t_0,s_{i(\alpha)}})^{\alpha_{i(\alpha)}}| \le \left(|\zeta|_\eta\right)^{|\alpha|} \cdot \left(\Delta_{t_0,s_0}\right)^{\|\alpha^T \eta\|_1}$$

for all $s_1, \dots, s_{i(\alpha)} \in [t_0, s_0]$ and $\alpha \in \mathscr{A}_m$ with $i(\alpha) \ge 1$.

Proof Let j be in $\{1, \dots, i(\alpha)\}$. Then,

$$\begin{aligned}
\left|(\Delta\zeta_{t_0,s_j})^{\alpha_j}\right| &= \left|\left(\Delta\zeta^1_{t_0,s_1}\right)^{\alpha_{j,1}}\right| \cdots \left|\left(\Delta\zeta^m_{t_0,s_j}\right)^{\alpha_{j,m}}\right| \\
&\le \left(|\zeta^1|_{\eta_1} \left(\Delta_{t_0,s_1}\right)^{\eta_1}\right)^{\alpha_{j,1}} \cdots \left(|\zeta^m|_{\eta_m} \left(\Delta_{t_0,s_j}\right)^{\eta_m}\right)^{\alpha_{j,m}} \\
&\le \left(|\zeta|_\eta\right)^{\alpha_{j,1}+\cdots+\alpha_{j,m}} \left(\Delta_{t_0,s_0}\right)^{\alpha_{j,1}\eta_1+\cdots+\alpha_{j,m}\eta_m}.
\end{aligned}$$

We also have

$$\left|(\Delta\zeta_{t_0,s_1})^{\alpha_1}\cdots(\Delta\zeta_{t_0,s_{i(\alpha)}})^{\alpha_{i(\alpha)}}\right| = |(\Delta\zeta_{t_0,s_1})^{\alpha_1}|\cdots|(\Delta\zeta_{t_0,s_{i(\alpha)}})^{\alpha_{i(\alpha)}}|$$

and the assertion follows. □

As is usual for numerical schemes for deterministic ODEs, we first study the local discretization error and than analyze the global discretization error.

Lemma 10.3 *Let $\gamma > 0$ and let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in [0,1]^m$ with $\|\varepsilon\|_1 < \gamma$. Then,*

$$\left\|x_{s_0} - \Phi_\gamma(x_{t_0}, t_0, \Delta_{t_0,s_0})\right\|_2 \le C_\gamma^\varepsilon \cdot \left(\Delta_{t_0,s_0}\right)^{(\gamma+1-\|\varepsilon\|_1)}$$

for all $t_0, s_0 \in [0,T]$ with $0 \le s_0 - t_0 \le 1$.

Proof We use the integral equation expansion (4.1)

$$\begin{aligned} x_{s_0} = x_{t_0} &+ \sum_{i=1}^{\lceil\gamma\rceil} \int_{t_0}^{s_0} \cdots \int_{t_0}^{s_{i-1}} (L^i\, id)\left(\zeta_{s_1}, \ldots, \zeta_{s_i}, x_{t_0}\right) ds_i \ldots ds_1 \\ &+ \int_{t_0}^{s_0} \cdots \int_{t_0}^{s_{\lceil\gamma\rceil}} (L^{\lceil\gamma\rceil+1}\, id)\left(\zeta_{s_1}, \ldots, \zeta_{s_{\lceil\gamma\rceil+1}}, x_{s_{\lceil\gamma\rceil+1}}\right) ds_{\lceil\gamma\rceil+1} \ldots ds_1 \end{aligned}$$

with $k = \lceil\gamma\rceil \in \mathbb{N}_0$. Then we use Lemma 10.1 to determine the Taylor expansion of the integrands $(L^i\, id)(\zeta_{s_1}, \ldots, \zeta_{s_i}, x_{t_0})$ in the first $m \cdot i$-variables of order

$$l_i = \left\lceil \frac{\gamma + 1 - i}{\vartheta} \right\rceil - 1, \quad i = 1, \ldots, \lceil\gamma\rceil$$

and obtain

$$x_{s_0} = x_{t_0} + \sum_{i=1}^{\lceil\gamma\rceil} \sum_{\substack{|\alpha|\le l_i \\ \alpha\in\mathbb{N}_0^{m\times i}}} f_\alpha(\zeta_{t_0}^{\times i}, x_{t_0}) \cdot I_{\alpha,t_0,s_0} + E_1 + E_2 \tag{10.1}$$

with

$$E_1 = \int_{t_0}^{s_0} \cdots \int_{t_0}^{s_{\lceil\gamma\rceil}} (L^{\lceil\gamma\rceil+1}\, id)\left(\zeta_{s_1}, \ldots, \zeta_{s_{\lceil\gamma\rceil+1}}, x_{s_{\lceil\gamma\rceil+1}}\right) ds_{\lceil\gamma\rceil+1} \ldots ds_1$$

and

$$\begin{aligned} E_2 = \sum_{i=1}^{\lceil\gamma\rceil} \sum_{\substack{|\alpha|=l_i+1 \\ \alpha\in\mathbb{N}_0^{m\times i}}} &\int_{t_0}^{s_0} \cdots \int_{t_0}^{s_{i-1}} (\Delta\zeta_{t_0,s_1})^{\alpha_1} \cdots (\Delta\zeta_{t_0,s_i})^{\alpha_i} \cdot (l_i+1) \\ &\times \left(\int_0^1 f_\alpha(\zeta_{t_0} + u\Delta\zeta_{t_0,s_1}, \ldots, \zeta_{t_0} + u\Delta\zeta_{t_0,s_i}, x_{t_0}) \cdot (1-u)^{l_i}\, du \right) ds_i \ldots ds_1. \end{aligned}$$

Now, since $|\alpha|\vartheta \leq \|\alpha^T\theta\|_1$, we have

$$\mathscr{A}_m^\gamma = \left\{\alpha \in \mathscr{A}_m \,\middle|\, i(\alpha) + \|\alpha^T\theta\|_1 < \gamma + 1\right\} \subset \left\{\alpha \in \mathscr{A}_m \,\middle|\, |\alpha| \leq l_{i(\alpha)}\right\}.$$

Hence, we can write equation (10.1) in the form

$$x_{s_0} = x_{t_0} + \sum_{i=1}^{\lceil\gamma\rceil} \sum_{\substack{\alpha\in\mathscr{A}_m^\gamma \\ i(\alpha)=i}} f_\alpha(\zeta_{t_0}^{\times i}, x_{t_0}) \cdot I_{\alpha,t_0,s_0} + E_1 + E_2 + E_3 \tag{10.2}$$

with

$$E_3 = \sum_{i=1}^{\lceil\gamma\rceil} \sum_{\substack{|\alpha|\leq l_i \\ \alpha\in\mathbb{N}_0^{m\times i}\setminus\mathscr{A}_m^\gamma}} f_\alpha(\zeta_{t_0}^{\times i}, x_{t_0}) \cdot I_{\alpha,t_0,s_0}.$$

Equation (10.2) is just

$$x_{s_0} = \Phi_\gamma(x_{t_0}, t_0, \Delta_{t_0,s_0}) + E_1 + E_2 + E_3, \tag{10.3}$$

so we need to estimate E_1, E_2 and E_3 and complete the proof. We have

$$\|E_1\|_2 \leq \left(\sup_{\substack{s_1,\ldots,s_{\lceil\gamma\rceil+1} \\ \in[0,T]}} \left\|(L^{\lceil\gamma\rceil+1} id)(\zeta_{s_1}, \ldots, \zeta_{s_{\lceil\gamma\rceil+1}}, x_{s_{\lceil\gamma\rceil+1}})\right\|_2 \right) \frac{\left(\Delta_{t_0,s_0}\right)^{(\lceil\gamma\rceil+1)}}{(\lceil\gamma\rceil+1)!},$$

$$\|E_2\|_2 \leq \sum_{i=1}^{\lceil\gamma\rceil} \sum_{\substack{|\alpha|=l_i+1 \\ \alpha\in\mathbb{N}_0^{m\times i}}} \left(\frac{\left(|\zeta|_\eta\right)^{|\alpha|}}{i!} \left(\Delta_{t_0,s_0}\right)^{(i+\|\alpha^T\eta\|_1)} R_\alpha \right)$$

and

$$\|E_3\|_2 \leq \sum_{i=1}^{\lceil\gamma\rceil} \sum_{\substack{|\alpha|\leq l_i \\ \alpha\in\mathbb{N}_0^{m\times i}\setminus\mathscr{A}_m^\gamma}} \left(\frac{\left(|\zeta|_\eta\right)^{|\alpha|}}{i!} \left(\Delta_{t_0,s_0}\right)^{(i+\|\alpha^T\eta\|_1)} R_\alpha \right)$$

for every $\eta \in (0,1]^m$, where R_α is the random variable

$$R_\alpha = \sup_{\substack{\|w_1\|_2,\ldots,\|w_i\|_2\leq|\zeta|_\infty \\ \|y\|_2\leq|x|_\infty}} \|f_\alpha(w_1,\ldots,w_i,y)\|_2$$

for $\alpha \in \mathscr{A}_m$ with $i = i(\alpha) \geq 1$. Hence, we have

$$\|x_{s_0} - \Phi_\gamma(x_{t_0}, t_0, \Delta_{t_0,s_0})\|_2 \leq \sum_{\substack{i=1,\ldots,\lceil\gamma\rceil+1 \\ |\alpha|\leq\lceil\frac{\gamma}{\vartheta}\rceil \\ \alpha\in\mathbb{N}_0^{m\times i}\setminus\mathscr{A}_m^\gamma}} \left(\frac{\left(|\zeta|_\eta\right)^{|\alpha|}}{i!} \left(\Delta_{t_0,s_0}\right)^{(i+\|\alpha^T\eta\|_1)} R_\alpha \right)$$

for every $\eta \in (0,1]^m$. This, in particular, implies that

$$\|x_{s_0} - \Phi_\gamma(x_{t_0}, t_0, \Delta_{t_0,s_0})\|_2 \leq \sum_{\substack{\alpha \in \mathscr{A}_m \setminus \mathscr{A}_m^\gamma \\ i(\alpha) \leq \lceil \gamma \rceil + 1 \\ |\alpha| \leq \lceil \frac{\gamma}{\vartheta} \rceil}} \left(\frac{\left(|\zeta|_\eta\right)^{|\alpha|}}{i(\alpha)!} \left(\Delta_{t_0,s_0}\right)^{(i(\alpha) + \|\alpha^T \eta\|_1)} R_\alpha \right)$$

$$\leq \sum_{\substack{\alpha \in \mathscr{A}_m \setminus \mathscr{A}_m^\gamma \\ i(\alpha) \leq \lceil \gamma \rceil + 1 \\ |\alpha| \leq \lceil \frac{\gamma}{\vartheta} \rceil}} \left(\frac{\left(|\zeta|_\eta\right)^{|\alpha|}}{i(\alpha)!} R_\alpha \right) \left(\Delta_{t_0,s_0}\right)^{(\gamma + 1 - \|\varepsilon\|_1)}$$

with $\eta = \theta - \varepsilon / \lceil \frac{\gamma}{\vartheta} \rceil$, since, for $i = 1, \ldots, \lceil \gamma \rceil + 1$ and $\alpha \in \mathbb{N}_0^{m \times i} \setminus \mathscr{A}_m^\gamma$ with $|\alpha| \leq \lceil \frac{\gamma}{\vartheta} \rceil$, we have

$$i + \|\alpha^T \eta\|_1 = i + \|\alpha^T \theta\|_1 - \left\| \alpha^T \frac{\varepsilon}{\lceil \frac{\gamma}{\vartheta} \rceil} \right\|_1 \geq \gamma + 1 - \left\| \alpha^T \frac{\varepsilon}{\lceil \frac{\gamma}{\vartheta} \rceil} \right\|_1$$

$$\geq \gamma + 1 - |\alpha| \frac{\|\varepsilon\|_1}{\lceil \frac{\gamma}{\vartheta} \rceil} \geq \gamma + 1 - \|\varepsilon\|_1.$$

This completes the proof. □

For the global discretization error, we need a Lipschitz estimate for the function Φ_γ.

Lemma 10.4 *Let $[t_0, s_0] \subset [0, T]$ with $0 \leq h = s_0 - t_0 \leq 1$ and let $\gamma > 0$. Then, for each $\omega_0 \in \Omega$ we have*

$$\left\| (\Phi_\gamma(y_1, t_0, h, \omega_0) - y_1) - (\Phi_\gamma(y_2, t_0, h, \omega_0) - y_2) \right\|_2 \leq h\, L_\gamma(\omega_0)\, \|y_1 - y_2\|_2$$

for every $y_1, y_2 \in \mathbb{R}^d$ with $\|x_{t_0}(\omega_0) - y_1\|_2 \leq 1$, $\|x_{t_0}(\omega_0) - y_2\|_2 \leq 1$.

Proof We have

$$\Phi_\gamma(y_j, t_0, h, \omega_0) - y_j = \sum_{\alpha \in \mathscr{A}_m^\gamma \setminus \{\emptyset\}} f_\alpha(\zeta_{t_0}^{\times i(\alpha)}(\omega_0), y_j) \cdot I_{\alpha,t_0,s_0}(\omega_0)$$

for $j = 1$ and 2. Moreover,

$$\left| I_{\alpha,t_0,s_0} \right| \leq \frac{1}{i(\alpha)!} h^{i(\alpha)} \left(2|\zeta|_\infty\right)^{|\alpha|} \leq h \left(2|\zeta|_\infty\right)^{|\alpha|}$$

for $\alpha \in \mathscr{A}_m$ with $i(\alpha) \geq 1$ and, by the fundamental theorem of calculus,

$$f_\alpha(\zeta_{t_0}^{\times i(\alpha)}, y_2) - f_\alpha(\zeta_{t_0}^{\times i(\alpha)}, y_1) = \int_0^1 \left(\partial_y f_\alpha\right)(\zeta_{t_0}^{\times i(\alpha)}, y_1 + u(y_2 - y_1)) \cdot (y_2 - y_1)\, du.$$

Combining the results, we obtain

$$
\begin{aligned}
&\left\| (\Phi_\gamma(y_1, t_0, h, \omega_0) - y_1) - (\Phi_\gamma(y_2, t_0, h, \omega_0) - y_2) \right\|_2 \\
&\quad \le \sum_{\alpha \in \mathscr{A}_m^\gamma \setminus \{\emptyset\}} \left\| f_\alpha(\zeta_{t_0}^{\times i(\alpha)}(\omega_0), y_1) - f_\alpha(\zeta_{t_0}^{\times i(\alpha)}(\omega_0), y_2) \right\|_2 |I_{\alpha, t_0, s_0}(\omega_0)| \\
&\quad \le h \left(\sum_{\alpha \in \mathscr{A}_m^\gamma \setminus \{\emptyset\}} (2 |\zeta|_\infty(\omega_0))^{|\alpha|} \left\| f_\alpha(\zeta_{t_0}^{\times i(\alpha)}(\omega_0), y_2) - f_\alpha(\zeta_{t_0}^{\times i(\alpha)}(\omega_0), y_1) \right\|_2 \right) \\
&\quad \le h \left(\sum_{\alpha \in \mathscr{A}_m^\gamma \setminus \{\emptyset\}} (2 |\zeta|_\infty(\omega_0))^{|\alpha|} \right. \\
&\qquad \left. \times \int_0^1 \left\| (\partial_y f_\alpha) \left(\zeta_{t_0}^{\times i(\alpha)}(\omega_0), y_1 + u(y_2 - y_1) \right) \right\|_2 du \right) \|y_2 - y_1\|_2 \\
&\quad \le h \cdot L_\gamma(\omega_0) \cdot \|y_1 - y_2\|_2,
\end{aligned}
$$

which is the assertion of the Lemma. □

Finally, the required global discretization error follows by a standard technique (with the Gronwall lemma) for deterministic ODEs from Lemmas 10.3 and 10.4 (see for example [5]).

This completes the proof of Theorem 5.1.

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