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ECONOMICAL RUNGE–KUTTA METHODS FOR NUMERICAL SOLUTION OF STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract.

For the numerical solution of stochastic differential equations an economical Runge–Kutta scheme of second order in the weak sense is proposed. Numerical stability is studied and some examples are presented to support the theoretical results.

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1 Introduction.

In recent years much attention has been devoted to stochastic differential equations (SDEs) due to their application in many fields, including biology, economics and finance. Unfortunately, in many cases analytic solutions are not available, thus numerical methods are needed to approximate them. In this paper we consider an Itô SDE [8]

$$
\begin{aligned}
&dX_t = a(t, X_t)dt + b(t, X_t)dW_t \quad t_0 \le t \le T \\
&X_{t_0} = X_0
\end{aligned} \tag{1.1}
$$

where $W = \{W_t, 0 \le t \le T\}$ denotes a standard Wiener process. The functions a and b are the drift and the diffusion coefficients respectively, and we assume that they are defined and measurable in $[t_0, T] \times R$ and satisfy both Lipschitz and linear growth bound conditions in x. These requirements ensure existence and uniqueness of solution of the SDE (1.1).

Numerical methods for the solution of SDEs are recursive schemes where trajectories of solutions are computed at discrete points. In previous works (see [8], which includes also an extensive bibliography) several numerical methods for

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the solution of (1.1) have been proposed. In [3] the authors gave an overview of methods of Runge–Kutta type for SDEs studied until then. In [2] the numerical solution of SDEs by means of linear multistep formulae has been considered. In [1] and [13] new classes of stochastic Runge–Kutta schemes were derived. They are of second-order accuracy in the weak sense. The methods in [13] has been obtained by generalizing the way to obtain deterministic Runge–Kutta methods from Taylor approximations.

For deterministic equations it is known that the classic Runge–Kutta methods are expensive in terms of function evaluations. Therefore the so-called pseudo Runge–Kutta methods have been proposed (see [5] and the references therein).

In this paper we present an extension of pseudo Runge–Kutta methods for SDEs, in particular of the pseudo Runge–Kutta methods autostarting or of III species [5], which coincides, in the case of second order, with the economical Runge–Kutta method [6].

In order to facilitate the reading of the work, in Section 2 we quote deterministic economical pseudo Runge–Kutta methods and in Section 3 stochastic weak second-order Runge–Kutta methods. In Section 4 we propose the new method. Numerical stability for the proposed method is studied in Section 5, where the domains of stability are obtained and showed in some figures. In the last section numerical examples are given which compare the proposed method to *SME-A* presented in [13].

2 Pseudo Runge–Kutta scheme.

The deterministic pseudo Runge–Kutta method autostarting proposed in [5] for the Cauchy problem $X'(t) = f(t, X)$, $X(t_0) = X_0$ is

$$\begin{cases} X_{n+1} = X_n + h_n \sum_{i=0}^{s} b_i K_{i,n} \quad n = 0, 1, \dots, N-1 \\ X_0 = X(t_0) \end{cases} \tag{2.1}$$

where $b_i \in \mathbf{R}$, s is the number of the stages, $h_n = t_{n+1} - t_n$, and

$$\begin{aligned} K_{i,n} &= f\Big(t_n + c_i h_n, X_n + h_n\Big(\sum_{j=0}^{i-1} b_{ij} K_{j,n} + \sum_{j=0}^{s} \overline{b}_{ij} K_{j,n-1}\Big)\Big) \\ c_i &= \sum_{j=0}^{i-1} b_{ij} + \sum_{j=0}^{s} \overline{b}_{ij} \quad i = 0, \dots, s \quad \sum_{j=0}^{-1} b_{0j} = b_{00}. \end{aligned} \tag{2.2}$$

The order p, that is the smallest natural number s.t.

$$\|x_{n+1} - x(t_{n+1})\| \leq k h_n^p, \quad k \in \mathbf{R}$$

is a function of s and it is known that $s + 1 \leq p \leq 2s$.

For $s = 0$, the simple formula has been derived in [4]:

$$\begin{cases} X_{n+1} = X_n + h_n K_{0,n} \quad n = 0, 1, \dots, N-1 \\ X_0 = X(t_0) \end{cases} \tag{2.3}$$

$$K_{0,n} = f\Big(t_n + \frac{1}{2}h_n, X_n + \frac{1}{2}h_n K_{0,n-1}\Big) \tag{2.4}$$
$$K_{0,-1} = f(t_0, X_0).$$

This method has the same cost and the same interval of stability of Euler's classic method [7], but it has order two. It coincides with the economical second-order Runge–Kutta method proposed in [6], which is obtained, for $s = 2$, from

$$\begin{cases} X_{n+1} = X_n + h_n \sum_{i=2}^{s} b_i K_{i,n} \quad n = 0, 1, \dots, N-1 \\ X_0 = X(t_0) \end{cases} \tag{2.5}$$

$$K_{i,n} = f\Big(t_n + c_i h_n, X_n + h_n\Big(\sum_{j=2}^{i-1} a_{ij} K_{j,n} + a_{i1} K_{s,n-1}\Big)\Big) \tag{2.6}$$
$$K_{s,-1} = K_{1,0} = f(t_0, X_0).$$

where $c_i = \sum_{j=1}^{i-1} a_{ij}$, $i = 2, \dots, s$, and

3 Stochastic second-order Runge–Kutta schemes.

The class of stochastic methods proposed in [13] for the solution of (1.1) have the form

$$X_{n+1} = X_n + (\alpha_1 k_0 + \alpha_2 k_1)\Delta + s_0\Gamma + s_1\Lambda + s_2\Upsilon \tag{3.1}$$

with

$$\begin{aligned} k_0 &= a(t_n, X_n) \\ s_0 &= b(t_n, X_n) \\ k_1 &= a(t_n + \mu_0\Delta, X_n + \lambda_0 k_0\Delta + s_0 L) \\ s_1 &= b(t_n + \rho_0\Delta, X_n + \gamma_0 k_0\Delta + s_0 M) \\ s_2 &= b(t_n + \rho_0\Delta, X_n + \gamma_0 k_0\Delta + s_0 N) \end{aligned} \tag{3.2}$$

where $\Delta = \frac{T-t_0}{N} > 0$ is the equidistant nonrandom step size, $t_n = t_0 + n\Delta$, $n = 0, 1, \dots N$ is the nth step point, lower-case Greek letters are parameters and $\Gamma, \Lambda, \Upsilon$, L, M, N are random variables of mean square order $\frac{1}{2}$.

Analogously with the deterministic case, the technique for obtaining the order conditions consistes in matching the truncated stochastic expansion of the solution about a point with the Itô Taylor approximation of the exact solution [11].

One of the second-order methods of this family is the so-called *Method SME-A*:

$$
\begin{aligned}
X_{n+1} &= X_n + k_1\Delta + \frac{1}{4}(2s_0 + s_1 + s_2)\triangle W_n + \\
&+ \frac{1}{4}(s_2 - s_1)\left(\sqrt{\Delta} - \frac{(\triangle W_n)^2}{\sqrt{\Delta}}\right)
\end{aligned}
\tag{3.3}
$$

with

$$
\begin{aligned}
s_0 &= b(t_n, X_n) \\
k_1 &= a\left(t_n + \frac{1}{2}\Delta, X_n + \frac{1}{2}k_0\Delta + \left(\frac{2-\sqrt{6}}{4}\triangle W_n + \frac{\sqrt{6}}{12}\right)s_0\right) \\
s_1 &= b(t_n + \Delta, X_n + k_0\Delta + \sqrt{\Delta}s_0) \\
s_2 &= b(t_n + \Delta, X_n + k_0\Delta - \sqrt{\Delta}s_0).
\end{aligned}
\tag{3.4}
$$

4 An economical stochastic Runge–Kutta method.

As in the deterministic case, if we save one function call for each step by using information from the previous step, we obtain an economical Runge–Kutta method for SDEs.

THEOREM 4.1. *In accordance with the notation employed in the previous paragraph (see also* [13]*), the following scheme is a second-order Runge–Kutta type method (ESRK in what follows)*

$$
\begin{aligned}
X_{n+1} &= X_n + k_{1,n}\Delta + \frac{1}{4}(2s_0 + s_1 + s_2t)\triangle W_n + \\
&+ \frac{1}{4}(s_2 - s_1)\left(\sqrt{\Delta} - \frac{(\triangle W_n)^2}{\sqrt{\Delta}}\right)
\end{aligned}
\tag{4.1}
$$

$n = 1, 2, \ldots, N,$ *with*

$$
\begin{aligned}
k_{1,0} &= a(t_0, X_0) \\
s_0 &= b(t_n, X_n) \\
k_{1,n} &= a\left(t_n + \frac{1}{2}\Delta, X_n + \frac{1}{2}k_{1,n-1}\Delta + \left(\frac{2-\sqrt{6}}{4}\triangle W_n + \frac{\sqrt{6}}{12}\right)s_0\right) \quad n \geq 1 \\
s_1 &= b(t_n + \Delta, X_n + k_{1,n-1}\Delta + \sqrt{\Delta}s_0) \\
s_2 &= b(t_n + \Delta, X_n + k_{1,n-1}\Delta - \sqrt{\Delta}s_0).
\end{aligned}
\tag{4.2}
$$

PROOF. Using the truncated Taylor expansion of a process $f(t+\Delta, X_t+\triangle W)$ in terms of Δ and $\triangle X = X_{t+\Delta} - X_t$ [8] for the expansion of a and b, it's easy to prove that

$$\Delta^2 k_{1,n-1} \overset{(2)}{\simeq} \Delta^2 a$$

and

$$\Delta\triangle W_n k_{1,n-1} \simeq^{(2)} \Delta\triangle W_n a$$

where the notation $A\simeq^{(2)}B$ means that replacing the variable A by B in a second-order approximation leads to an equivalent approximation.

Thus the ESRK approximation (4.1) and the simplified order two weak Taylor scheme given by

$$\begin{aligned} X_{n+1} = X_n + b\triangle W_n + a\Delta + \frac{1}{2}bb_{01}\big((\triangle W_n)^2 - \Delta\big) \\ + \frac{1}{2}\left(b_{10} + a\ b_{01} + \frac{1}{2}b^2 b_{02} + ba_{01}\right)\Delta\triangle W_n \\ + \frac{1}{2}\left(a_{10} + aa_{01}\frac{1}{2}b^2 a_{02}\right)\Delta^2 \end{aligned}$$

are 2-equivalent. □

5 Stability regions.

In order to study the stability regions for ESRK we consider the Itô test equation

$$dX_t = \lambda X_t dt + \mu X_t dW_t \quad t > t_0\ \lambda, \mu \in \mathbf{C} \tag{5.1}$$

with nonrandom initial conditions $X_{t_0} = x_0 \in \mathbf{R}, x_0 \neq 0$.

DEFINITION 5.1. *We say that a numerical solution* $\{X_n\}_{n\in IN}$ *generated by a scheme with equidistant stepsize applied to test equation* (5.1) *is mean square stable if* $\lim_{n\to\infty} E[|X_n|^2] = 0$.

When we apply the scheme (4.1)–(4.2) to (5.1) we obtain the difference equations

$$\begin{cases} X_{n+1} = P(z_n)X_n + Q(z_n)k_{1,n-1} \\ k_{1,n} = L_n X_n + M k_{1,n-1} \end{cases} \tag{5.2}$$

with $k_{1,-1} = 0$, $z_n = \lambda\Delta + \mu\triangle W_n$ and

$$\begin{cases} P(z_0) = 1 + z_0 + \dfrac{\mu\lambda}{2}\Delta\triangle W_0 + \dfrac{\mu}{2}\big((\triangle W_0)^2 - \Delta\big) \\ P(z_n) = 1 + z_n + \Delta\lambda\mu\left(\dfrac{2-\sqrt{6}}{4}\triangle W_n + \dfrac{\sqrt{6}}{12}\right) + \dfrac{\mu}{2}\big((\triangle W_n)^2 - \Delta\big) \\ Q(z_n) = \dfrac{z_n\Delta}{2} \\ \quad L_n = \lambda + \lambda\mu\left(\dfrac{2-\sqrt{6}}{4}\triangle W_n + \dfrac{\sqrt{6}}{12}\right) \\ \quad M = \dfrac{\lambda\Delta}{2}. \end{cases} \tag{5.3}$$

If we put

$$u_n = [X_n, k_{1,n-1}]^T$$

(5.2) takes the form

$$u_{n+1} = Au_n$$

where

$$A = \begin{bmatrix} P(z_n) & Q(z_n) \\ L_n & M \end{bmatrix}.$$

When we calculate the components of the second moment of u_n, the following one-step difference equation is obtained:

$$Y_{n+1} = \Omega Y_n$$

where

$$Y_n = \begin{bmatrix} Y_n^1 \\ Y_n^2 \\ Y_n^3 \end{bmatrix} = \begin{bmatrix} E\left(u_n^1\right)^2 \\ E\left(u_n^2\right)^2 \\ E\left(u_n^1 u_n^2\right)^2 \end{bmatrix}$$

and Ω is the stability matrix.

Under the pth matrix norm $\|\cdot\|_p$ it is evident that $\lim_{n\to\infty} \|Y_n\| = 0$ if $\|\Omega\|_p < 1$.

The entries of Ω can be determined by direct computation.

THEOREM 5.1. *When method* (4.1)–(4.2) *is applied to the test equation* (5.1), *the stability matrix* Ω *is given by*

$$A = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}$$

where

$$\Omega_{11} = 1 + \left(\mu^2 + 2\lambda + \frac{\lambda\mu}{\sqrt{6}}\right)\Delta + \left(\lambda^2\left(1 + \frac{\mu}{2\sqrt{6}}\right)^2 + \frac{\mu^2}{4}(1 + 4\lambda - 2\sqrt{6}\lambda)\right)\Delta^2 + \left(\frac{2-\sqrt{6}}{4}\right)^2 \lambda^2\mu^2\Delta^3$$

$$\Omega_{12} = \frac{\mu^2}{4}\Delta^3 + \frac{\lambda^2}{4}\Delta^4$$

$$\Omega_{13} = (\lambda + \mu^2)\Delta^2 + \left(\lambda + \frac{2-\sqrt{6}}{4}\mu^2 + \frac{\sqrt{6}}{12}\lambda\mu\right)\lambda\Delta^3$$

$$\Omega_{21} = \lambda^2\left(1 + \frac{\sqrt{6}}{12}\mu\right)^2 + \left(\frac{2-\sqrt{6}}{4}\right)^2\lambda^2\mu^2\Delta$$

$$\Omega_{22} = \frac{\lambda^2}{4}\Delta^2$$

$$\Omega_{23} = \lambda^2\left(1+\frac{\sqrt{6}}{12}\mu\right)\Delta$$

$$\Omega_{31} = \lambda\left(1+\frac{\sqrt{6}}{12}\mu\right) + \frac{\lambda}{12}\left(12\lambda + \left(2\sqrt{6}+\frac{\mu}{2}\right)\lambda\mu + 3(2-\sqrt{6})\mu^2\right) + \left(\frac{2-\sqrt{6}}{4}\right)^2\lambda^2\mu^2\Delta^2$$

$$\Omega_{32} = \frac{\lambda^2}{4}\Delta^3$$

$$\Omega_{33} = \frac{\lambda}{2}\Delta + \lambda\left(\lambda + \frac{\sqrt{6}}{12}\lambda\mu + \frac{2-\sqrt{6}}{8}\mu^2\right)\Delta^2.$$

The scheme is stable in the mean square sense with respect to $\|\cdot\|_\infty$ if

$$\max\{A, B, C\} < 1$$

where

$$A = |\Omega_{11}| + |\Omega_{12}| + |\Omega_{13}|$$
$$B = |\Omega_{21}| + |\Omega_{22}| + |\Omega_{23}|$$
$$C = |\Omega_{31}| + |\Omega_{32}| + |\Omega_{33}|$$

in which Ω_{ij} are as given in Theorem 5.1.

If we restrict attention to $\lambda, \mu \in \mathbf{R}$, the region of mean square stability of the scheme can be obtained for several values of Δ (Figure 5.1).

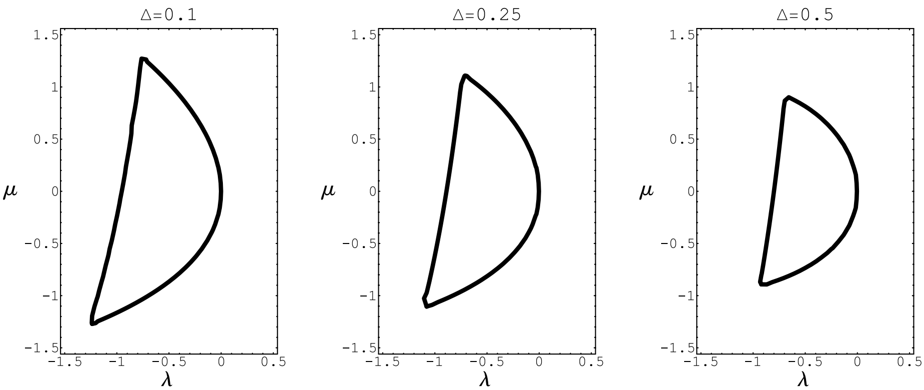

Figure 5.1: Mean square stability regions.

6 Numerical results and conclusions.

In this section, numerical results from the implementation of *ESRK* proposed in this paper are compared to those from the implementation of method

SME-A [13]. In Section 4 we said that in scheme (4.1)–(4.2) at each step we use a function call of the previous step. Therefore the cost of *ESRK* is less than the cost of *SME-A* method.

As test problems linear and nonlinear one-dimensional stochastic differential equations for which the exact solution in terms of the Wiener process is known have been taken. For each of the four examples the approximations are compute and the results are compared with the exact solutions. All the computations were done on a PC with Core 2 processor using Matlab and 5.000 independent simulations were generated for stepsizes $\Delta = 2^{-1}, \ldots, 2^{-5}$. The mean, the standard deviation of the errors and the computational work (the number of function evaluation *nfc*) for each problem are summarized in Tables 6.1–6.4.

EXAMPLE 6.1. Consider the linear SDE [13]

$$\begin{cases} dX_t = \lambda X_t dt + \mu X_t dW_t \\ X_0 = x_0. \end{cases} \tag{6.1}$$

This problem is the Black–Scholes stochastic differential equation used in option pricing. It's easy to estimate the exact value of the first moment $E[X_t] = x_0 e^{\lambda t}$, which was approximated at the point $t = 2$, when $x_0 = 1, \lambda = 1.5, \mu = 0.5$. The obtained results are shown in Table 6.1.

Table 6.1: Error, standard deviation and computational work in the approximation of $E[X_2]$ in (6.1).

	ESRK			SME-A		
Δ	error	st. dev.	nfc	error	st. dev.	nfc
2^{-1}	5.7110	9.2642	90000	7.6151	9.9280	100000
2^{-2}	2.8151	13.3127	170000	4.9142	12.1125	200000
2^{-3}	0.9778	14.0820	330000	2.2497	13.8907	400000
2^{-4}	0.2429	16.6619	650000	1.1344	15.1384	800000
2^{-5}	0.0960	15.4648	1290000	0.5752	15.1021	1600000

For all the considered values of the stepsize we can prove that the scheme is not stable, in particular, for $\Delta = 2^{-1}$, $\|\Omega\|_\infty \approx 4.24$ and for $\Delta = 2^{-5}$, $\|\Omega\|_\infty \approx 2.81$.

If $\lambda = -0.5$, we can see from the first picture of Figure 5.1 that the method is mean square stable. In fact for $\Delta = 2^{-1}$, $\|\Omega\|_\infty \approx 0.71$ and at the point $t = 2$ we have

	ESRK		SME-A	
Δ	error	st. dev.	error	st. dev.
2^{-1}	0.0403	0.2246	0.0710	0.2451

and if $\lambda = -0.5$ and $\mu = 0.1$, for $\Delta = 2^{-1}$, $\|\Omega\|_\infty \approx 0.65$ and at the point $t = 2$ we have

	ESRK		SME-A	
Δ	error	st. dev.	error	st. dev.
2^{-1}	0.0082	0.0447	0.0237	0.0508

EXAMPLE 6.2. Consider now the nonlinear SDE [13, 12]

$$\begin{cases} dX_t = \left(\frac{1}{3}X_t^{\frac{1}{3}} + 6X_t^{\frac{2}{3}}\right)dt + X_t^{\frac{2}{3}}dW_t \\ X_0 = 1. \end{cases} \tag{6.2}$$

The solution is $X_t = (2t + 1 + \frac{W_t}{3})^3$ and the exact value of the first moment is $E[X_t] = 28$ at point $t = 1$. The obtained results are summarized in Table 6.2.

Table 6.2: Error, standard deviation and computational work in the approximation of $E[X_1]$ in (6.2).

	ESRK			SME-A		
Δ	error	st. dev.	nfc	error	st. dev.	nfc
2^{-1}	6.5867	8.4364	50000	9.9445	11.2810	50000
2^{-2}	3.2955	8.2595	90000	6.0137	9.0378	100000
2^{-3}	1.3770	8.8112	170000	3.4208	8.5430	200000
2^{-4}	0.5221	8.8029	330000	1.7712	8.5497	400000
2^{-5}	0.2347	8.9725	650000	0.8990	8.7529	800000

EXAMPLE 6.3. Consider the nonautonomous SDE [12]

$$\begin{cases} dX_t = (t + X_t)dt + t^2 dW_t \\ X_0 = 1. \end{cases} \tag{6.3}$$

Since it is linear, it's easy to see that $E[X_t] = 2e^2 - (1 + t)$, which was approximated at the point $t = 2$. The obtained results are shown in Table 6.3.

Table 6.3: Error and standard deviation in the approximation of $E[X_2]$ in (6.3).

	ESRK			SME-A		
Δ	error	st. dev.	nfc	error	st. dev.	nfc
2^{-1}	1.5226	4.0310	90000	2.7750	4.4232	100000
2^{-2}	0.3161	3.6589	170000	1.3593	3.7129	200000
2^{-3}	0.2567	3.8555	330000	0.9427	3.8241	400000
2^{-4}	0.0724	3.8422	650000	0.3328	3.7739	800000
2^{-5}	0.0440	3.9574	1290000	0.2625	3.9214	1600000

EXAMPLE 6.4. Our fourth test problem is the nonautonomous SDE [12]

$$(6.4)\qquad \begin{cases} dX_t = (tX_t + 10t)dt + bdW_t \\ X_0 = 10 \end{cases}$$

with constant diffusion coefficient $b = 0.1$. The exact value of the first moment is $E[X_t] = 20\sqrt{e} - 10$, which was approximated at $t = 1$. The obtained results are in Table 6.4.

Table 6.4: Error and standard deviation in the approximation of $E[X_2]$ in (6.4).

	ESRK			SME-A		
Δ	error	st. dev.	nfc	error	st. dev.	nfc
2^{-1}	1.5681	1.5701	50000	4.5857	4.5863	50000
2^{-2}	0.5521	0.5628	90000	2.4919	2.4941	100000
2^{-3}	0.1568	0.2060	170000	1.3025	1.3090	200000
2^{-4}	0.0373	0.1438	330000	0.6639	0.6778	400000
2^{-5}	0.0094	0.1386	650000	0.3374	0.3642	800000

Numerical results show that there are practically no numerical differences between the two methods, as far as the error concerns, but ESRK is more economic. Therefore, the idea of pseudo Runge–Kutta and economical Runge–Kutta methods is useful also in the solution of SDEs.

REFERENCES

1. M. I. Abukhaled and E. J. Allen, *A class of second-order Runge–Kutta methods for numerical solution of stochastic differential equations*, Stochast. Anal. Appl., 16(6) (1998), pp. 977–991.
2. L. Brugnano, K. Burrage, and P. M. Burrage, *Adams-type methods for the numerical solution of stochastic ordinary differential equations*, BIT, 40(3) (2000), pp. 451–470.
3. K. Burrage and E. Platen, *Runge–Kutta methods for stochastic differential equations*, Ann. Numer. Math., 1 (1994), pp. 63–78.
4. F. Costabile, *Un metodo con ascisse Gaussiane per l'integrazione numerica dell'equazione differenziale ordinaria*, Rend. Mat. Roma, 4(6) (1973), pp. 1–16.
5. F. Costabile, *A survey of pseudo Runge Kutta methods*, Rend. Mat. Roma, 23(VII) (2003), pp. 217–234.
6. F. Costabile, R. Caira, and M. I. Gualtieri, *Economical Runge Kutta method*, Rend. Mat., 15(VII) (1995), pp. 57–77.
7. E. Hairer, G. Wanner, *Solving Ordinary Differential Equations II*, Springer, Berlin, 1992.
8. P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin, 1992.
9. G. N. Milstein, *A method of second order accuracy integration of stochastic differential equations*, Theory Probab. Appl., 23 (1978), pp. 396–401.
10. Y. Saito and T. Mitsui, *Stability analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal., 33 (1996), pp. 2254–2267.

11. A. Tocino, *Mean-square stability of second-order RungeKutta methods for stochastic differential equations*, J. Comput. Appl. Math., 175 (2005), pp. 355–367.

12. A. Tocino and R. Ardanuy, *Runge–Kutta methods for numerical solution of stochastic differential equations*, J. Comput. Appl. Math., 138(2) (2002), pp. 219–241.

13. A. Tocino and J. Vigo-Aguiar, *Weak second order conditions for stochastic Runge–Kutta methods*, SIAM J. Sci. Comput., 24(2) (2002), pp. 507–523.