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TWO PERTURBATION BOUNDS FOR SINGULAR VALUES AND EIGENVALUES*

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Abstract.

The relative error in $\hat{\alpha} = \alpha(1+\delta)$ as an approximation to α is measured by

$$\delta = \frac{\hat{\alpha} - \alpha}{\alpha}.$$

In terms of this measurement we give a Hoffman–Wielandt type bound of singular values under additive perturbations and a Bauer–Fike type bound of eigenvalues under multiplicative perturbations.

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1 Introduction.

Many problems in science and engineering lead to eigenvalue and singular value problems for matrices. Perturbation bounds of eigenvalues and singular values play an important role in matrix computations. Let α and $\hat{\alpha}$ be two complex numbers. In general, a relative distance

$$\delta = \frac{|\alpha - \hat{\alpha}|}{|\alpha|} \tag{1.1}$$

is used to establish the relative perturbation theory for eigenvalues and singular values (see [1, 2, 3, 6, 5]).

In this note we establish relative perturbation bounds directly in terms of the classical δ-measurement for singular values and eigenvalues. Perturbation bounds have been studied in the context of two different perturbation models: additive perturbations and multiplicative perturbations. For the additive perturbation model we give a Hoffman–Wielandt type perturbation bound of singular

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values, which are derived by using a technique described by Li [4]. For the other model, we use elementary techniques to establish a Bauer–Fike type relative perturbation bound for eigenvalues.

Throughout the paper, $\mathcal{C}^{m\times n}$ denotes the set of $m \times n$ complex matrices; The symbol I stands for the unit matrix. A^* and $\mathrm{eig}(A)$ denote the conjugate transpose and the set of eigenvalues of A, respectively. We use $\|\cdot\|_2$ for the spectral norm and $\|\cdot\|_F$ for Frobenius norm.

2 A Hoffman–Wielandt type bound for singular values.

Let $B, \widehat{B} \in \mathcal{C}^{m\times n}$ $(m \geq n)$ have the singular value decompositions

$$B = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^* \quad \text{and} \quad \widehat{B} = \widehat{U} \begin{pmatrix} \widehat{\Sigma} \\ 0 \end{pmatrix} \widehat{V}^*,$$

respectively, where $U, \widehat{U}$ are $m \times m$ unitary, $V, \widehat{V}$ are $n \times n$ unitary, and

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \quad \widehat{\Sigma} = \mathrm{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_n),$$

and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ and $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \cdots \geq \hat{\sigma}_n \geq 0$.

Using the relative distance (1.1), Demmel and Veselic [1] proved that if B is a nonsingular matrix of order n, then

$$\max_i \left| \frac{\sigma_i - \hat{\sigma}_i}{\sigma_i} \right| \leq \min\left\{ \|B^{-1}(\widehat{B} - B)\|_2, \|(\widehat{B} - B)B^{-1}\|_2 \right\}.$$

No Hoffman–Wielandt type perturbation bound has been proved with respect to the δ-measurement until now. In this section we establish a Hoffman–Wielandt type perturbation bound under the additive perturbation model by using the techniques described by Li [4].

Let $S = (s_{ij}) \in \mathcal{C}^{n\times n}$. S is called a doubly stochastic matrix, if $\sum_{i=1}^n s_{ij} = \sum_{j=1}^n s_{ij} = 1, s_{ij} \geq 0, 1 \leq i, j \leq n$. The following lemma plays an important role in the following proof.

LEMMA 2.1 ([5]). *Let $S = (s_{ij})$ be an $n \times n$ doubly stochastic matrix and let $M = (\theta_{ij}) \in \mathcal{C}^{n\times n}$, then there exists a permutation τ of $\{1, 2, \ldots, n\}$ such that*

$$\sum_{i,j=1}^n |\theta_{ij}| s_{ij} \geq \sum_{i=1}^n |\theta_{i,\tau(i)}|.$$

THEOREM 2.2. *Let B be a nonsingular $n \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. Let $B + E$ be a square matrix with singular values $\hat{\sigma}_1 \geq \hat{\sigma}_2 \geq \cdots \geq \hat{\sigma}_n \geq 0$. Then there exists a permutation τ of $\{1, 2, \ldots, n\}$ such that*

$$\sqrt{\sum_{i=1}^n \left(\frac{\sigma_i - \hat{\sigma}_{\tau(i)}}{\sigma_i} \right)^2} \leq \sqrt{\frac{\|B^{-1}E\|_F^2 + \|EB^{-1}\|_F^2}{2}}. \tag{2.1}$$

PROOF. Let

$$B = U\Sigma V^* \quad \text{and} \quad B + E = \widehat{U}\widehat{\Sigma}\widehat{V}^*$$

be singular value decompositions of B and $B+E$, respectively, where U, V, $\widehat{U}$ and $\widehat{V}$ are $n \times n$ unitary and

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n) \quad \text{and} \quad \widehat{\Sigma} = \operatorname{diag}(\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_n).$$

Note that $B^{-1} = V\Sigma^{-1}U^*$. Therefore we have

$$\begin{aligned} \|B^{-1}E\|_F &= \|B^{-1}(B+E) - I\|_F \\ &= \|V\Sigma^{-1}U^*\widehat{U}\widehat{\Sigma}\widehat{V}^* - I\|_F \\ &= \|\Sigma^{-1}U^*\widehat{U}\widehat{\Sigma} - V^*\widehat{V}\|_F \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \|EB^{-1}\|_F &= \|(B+E)B^{-1} - I\|_F \\ &= \|\widehat{U}\widehat{\Sigma}\widehat{V}^*V\Sigma^{-1}U^* - I\|_F \\ &= \|\widehat{\Sigma}\widehat{V}^*V\Sigma^{-1} - \widehat{U}^*U\|_F \\ &= \|\Sigma^{-1}V^*\widehat{V}\widehat{\Sigma} - U^*\widehat{U}\|_F. \end{aligned} \tag{2.3}$$

Let $W = U^*\widehat{U} = (\omega_{ij})$, $Q = V^*\widehat{V} = (q_{ij})$. From (2.2) and (2.3) , we obtain

$$\begin{aligned} \|B^{-1}E\|_F^2 + \|EB^{-1}\|_F^2 &= \|\Sigma^{-1}W\widehat{\Sigma} - Q\|_F^2 + \|\Sigma^{-1}Q\widehat{\Sigma} - W\|_F^2 \\ &= \sum_{i,j=1}^n \left[\left| \frac{\hat{\sigma}_j}{\sigma_i}\omega_{ij} - q_{ij} \right|^2 + \left| \frac{\hat{\sigma}_j}{\sigma_i} q_{ij} - \omega_{ij} \right|^2 \right] \\ &= \sum_{i,j=1}^n \left[\left(\frac{\hat{\sigma}_j^2}{\sigma_i^2} + 1 \right) \left(|\omega_{ij}|^2 + |q_{ij}|^2 \right) - 4\frac{\hat{\sigma}_j}{\sigma_i} Re(\omega_{ij} q_{ij}) \right] \\ &\geq \sum_{i,j=1}^n \left[\left(\frac{\hat{\sigma}_j^2}{\sigma_i^2} + 1 \right) \left(|\omega_{ij}|^2 + |q_{ij}|^2 \right) - 2\frac{\hat{\sigma}_j}{\sigma_i} \left(|\omega_{ij}|^2 + |q_{ij}|^2 \right) \right] \\ &= 2\sum_{i,j=1}^n \left[\left(\frac{\hat{\sigma}_j}{\sigma_i} - 1 \right)^2 \frac{|\omega_{ij}|^2 + |q_{ij}|^2}{2} \right], \end{aligned} \tag{2.4}$$

where $Re(a)$ denotes the real part of the complex number a. Since W and Q are $n \times n$ unitary,

$$H = \left(\frac{|\omega_{ij}|^2 + |q_{ij}|^2}{2} \right)$$

is a doubly stochastic matrix. Hence applying Lemma 2.1 to (2.4), we know that there exists a permutation τ of $\{1, 2, \ldots, n\}$ such that

$$\|B^{-1}E\|_F^2 + \|EB^{-1}\|_F^2 \geq 2\sum_{i=1}^n \left(\frac{\hat{\sigma}_{\tau(i)}}{\sigma_i} - 1 \right)^2,$$

which yields the desired bound (2.1). □

REMARK 2.1. Generally, it is natural to order the σ_i's and $\hat{\sigma}_j$'s in Theorem 2.1 descendingly. Thus we expect that τ will be the identity permutation. But this may not be true without further assumptions. Li [6] showed that the following equality isn't true.

$$\sum_{i=1}^{n}\left(\frac{\sigma_i-\hat{\sigma}_i}{\sigma_i}\right)^2=\min_{\tau}\sum_{i=1}^{n}\left(\frac{\sigma_i-\hat{\sigma}_{\tau(i)}}{\sigma_i}\right)^2.$$

3 A Bauer–Fike type bound for eigenvalues.

In this section we establish a Bauer–Fike type perturbation bound of eigenvalues under the multiplicative perturbation model with respect to the δ-measurement.

Let A be $n\times n$ diagonalizable matrix, that is, there exists a nonsingular matrix X such that

$$A=X\Lambda X^{-1},\quad \Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n), \tag{3.1}$$

and λ_i are the eigenvalues of A. Let $\widehat{A}=D_1AD_2$ be a multiplicative perturbed matrix of A, where D_1 and D_2 are $n\times n$ nonsingular matrices, typically close to the identity matrices. For any $\hat{\lambda}\in\mathrm{eig}(\widehat{A})$, Eisenstat and Ipsen used the Bauer–Fike theorem with residual bound to obtain the following a relative bound (see Theorem 6.1 in [2], Theorem 5.1 in [3])

$$\min_{1\le i\le n}|\lambda_i-\hat{\lambda}|\le|\hat{\lambda}|\kappa(X)\big\|I-D_1^{-1}D_2^{-1}\big\|_2, \tag{3.2}$$

where $\kappa(X)=\|X\|_2\|X^{-1}\|_2$ is the condition number of an eigenvector matrix of A.

Next we use elementary techniques to give a Bauer–Fike type relative bound, which slightly improves the one (3.2).

THEOREM 3.1. *Let $A,\widehat{A}=D_1AD_2\in\mathcal{C}^{n\times n}$, where D_1,D_2 are nonsingular, and let A be diagonalizable and have the spectral decomposition* (3.1). *Then for each $\hat{\lambda}\in\mathrm{eig}(\widehat{A})$,*

$$\min_{1\le i\le n}|\lambda_i-\hat{\lambda}|\le|\hat{\lambda}|\big\|X^{-1}\big(I-D_1^{-1}D_2^{-1}\big)X\big\|_2. \tag{3.3}$$

PROOF. For any $\hat{\lambda}\in\mathrm{eig}(\hat{A})$, if also $\hat{\lambda}\in\mathrm{eig}(A)$, then (3.3) holds. Assume $\hat{\lambda}\notin\mathrm{eig}(A)$. Then

$$\begin{aligned}\widehat{A}-\hat{\lambda}I&=D_1AD_2-\hat{\lambda}I\\&=D_1\big(A-\hat{\lambda}D_1^{-1}D_2^{-1}\big)D_2\\&=D_1X\big(\Lambda-\hat{\lambda}X^{-1}D_1^{-1}D_2^{-1}X\big)X^{-1}D_2\\&=D_1X\big[(\Lambda-\hat{\lambda}I)+\hat{\lambda}X^{-1}\big(I-D_1^{-1}D_2^{-1}\big)X\big]X^{-1}D_2\\&=D_1X(\Lambda-\hat{\lambda}I)\big[I+(\Lambda-\hat{\lambda}I)^{-1}\hat{\lambda}X^{-1}\big(I-D_1^{-1}D_2^{-1}\big)X\big]X^{-1}D_2.\end{aligned} \tag{3.4}$$

Note that $\widehat{A} - \hat{\lambda} I$ is singular. Therefore from (3.4), we get

$$\left\|(\Lambda - \hat{\lambda} I)^{-1} \hat{\lambda} X^{-1}\left(I - D_1^{-1} D_2^{-1}\right) X\right\|_2 \geq 1,$$

which implies the desired bound (3.3). □

REMARK 3.1. Since

$$\left\|X^{-1}\left(I - D_1^{-1} D_2^{-1}\right) X\right\|_2 \leq \|X\|_2 \|X^{-1}\|_2 \left\|I - D_1^{-1} D_2^{-1}\right\|_2,$$

it is obvious that the bound (3.3) slightly improves the one (3.2).

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