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A NOTE ON THE EULER–MARUYAMA SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH A DISCONTINUOUS MONOTONE DRIFT COEFFICIENT*

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Abstract.

It is shown that the Euler–Maruyama scheme applied to a stochastic differential equation with a discontinuous monotone drift coefficient, such as a Heaviside function, and additive noise converges strongly to a solution of the stochastic differential equation with the same initial condition. The proof uses upper and lower solutions of the stochastic differential equations and the Euler–Maruyama scheme.

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1 Introduction.

In [5] we proved the existence of strong solutions for d-dimensional autonomous Itô stochastic differential equations

$$dX_t = f(X_t)dt + g(X_t)dW_t, \quad t \in [0,T] \tag{1.1}$$

for which the drift coefficient is a monotone increasing function, but not necessarily continuous, and the diffussion coefficient is Lipschitz continuous. By an increasing function we mean that $f(x) \leq f(y)$ whenever $x \leq y$, where the inequalities are interpreted componentwise. A motivating example is the scalar SDE

$$dX_t = H(X_t)dt + dW_t, \tag{1.2}$$

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where $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function, which is defined by

$$H(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Such equations arise, for example, when one considers the effects of background noise on switching systems or other discontinuous ordinary differential equations.

In this paper we show that the Euler–Maruyama scheme applied to a stochastic differential equation such as (1.2) can be used to obtain numerical approximations which converge strongly to a solution with the same initial value. Specifically, we consider the numerical approximation of such stochastic differential equations with additive noise, i.e. of the form

$$dX_t = f(X_t)dt + AdW_t, \quad t \in [0,T] \tag{1.3}$$

for which the drift coefficient is a monotone increasing function, which is continuous from below but not necessarily continuous, and A is a $d \times k$ matrix and W_t a k-dimensional Wiener process.

Gyöngy and Krylov [4] have investigated such problems with a discontinuous drift coefficient and a more general diffusion term than ours. In particular, they proved that the Euler–Maruyama approximations converge in probability to the unique solution (see Theorem 2.8). Later, under the same conditions as in [4] (see Theorem 2.6) plus a monotonicity condition on the drift coefficient, Gyöngy [3] extended the results in [4] to obtain the almost sure convergence of the Euler–Maruyama approximations. However, the monotonicity condition in [3] does not apply to our model problem (1.3), so our goal here is to give a different set of hypotheses which ensures the strong convergence of the Euler–Maruyama scheme. Nevertheless, Theorem 2.8 of [4] will play a crucial role in our work. Moreover, both the existence proof and the numerical results make extensive use of upper and lower solutions.

2 Existence and uniqueness theory.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\{\mathcal{F}_t\}_{t\geq 0}$ be the smallest filtration generated by the k-dimensional Wiener process W_t in the SDE (1.1).

DEFINITION 2.1. *A strong solution of the SDE* (1.1) *on an interval* $[0,T]$ *is a stochastic process* X_t *which is* $\mathcal{F}_t$*-measurable for each* $t \in [0,T]$ *with* $\mathbb{E}\|X_t\|^2 < \infty$ *for all* $t \in [0,T]$ *such that*

$$X_t = X_0 + \int_0^t f(X_s)ds + \int_0^t g(X_s)dW_s, \quad t \in [0,T], \text{ w.p.1.} \tag{2.1}$$

DEFINITION 2.2. *A* $\mathcal{F}_t$*-measurable stochastic process* U_t *is an upper solution of the SDE* (1.1) *on the interval* $[0,T]$ *if the inequality (interpreted component wise)*

$$U_t \geq U_0 + \int_0^t f(U_s)ds + \int_0^t g(U_s)dW_s, \quad t \in [0,T], \tag{2.2}$$

holds with probability 1. *If the process satisfies the reversed inequality (i.e. with* $\leq$*), then we say that it is a lower solution.*

Upper and lower solutions of stochastic differential equations have been considered previously under other names in [1], where conditions ensuring their existence were given in [1], see also [2]. We proved the following theorem in [5].

THEOREM 2.1. *Suppose that* $f : \mathbb{R}^d \to \mathbb{R}^d$ *and* $g : \mathbb{R}^d \to \mathbb{R}^{d\times k}$ *both satisfy a linear growth bound,*

$$\|f(x)\| + \|g(x)\| \leq K + L\|x\|, \quad x \in \mathbb{R}^d, \tag{2.3}$$

and that

- f *is increasing (but not necessarily continuous), i.e.,* $f(x) \leq f(y)$ *whenever* $x \leq y$ *(where the inequalities are interpreted componentwise),*
- g *is Lipschitz continuous.*

In addition, suppose that the SDE (1.1) *has mean-square continuous upper and lower solutions* U_t *and* L_t *on* $[0, T]$ *with* $\int_0^T \mathbb{E}\|f(L_t)\|^2 dt < \infty$, $\int_0^T \mathbb{E}\|f(U_t)\|^2 dt < \infty$ *and* $L_t \leq U_t$ *for* $t \in [0, T]$, *w.p.1. Finally, suppose that* X_0 *is* $\mathcal{F}_0$*-measurable with* $\mathbb{E}\|X_0\|^2 < \infty$ *and* $L_0 \leq X_0 \leq U_0$, *w.p.1.*

Then the SDE (1.1) *has at least one mean-square continuous strong solution* X_t *with initial value* X_0. *Moreover,* $L_t \leq X_t \leq U_t$ *for* $t \in [0, T]$, *w.p.1.*

Theorem 2.1 applies in particular for the scalar SDE (1.2) with the Heaviside drift coefficient $f(x) = H(x)$ and diffusion coefficient $g(x) \equiv 1$. First we note that $H(x)$ is an increasing function and then that

$$X_0 + \int_0^t dW_s \leq X_0 + \int_0^t H(X_s)ds + \int_0^t dW_s \leq X_0 + \int_0^t 1ds + \int_0^t dW_s$$

for any sample path continuous, non-anticipative stochastic process X_t, so

$$L_t := X_0 + W_t, \quad U_t := X_0 + t + W_t$$

are lower and upper solutions, respectively, for the Heaviside SDE (1.2). Thus, the Heaviside SDE (1.2) has at least one mean-square continuous strong solution X_t^* taking values between those of these lower and upper solutions, specifically with

$$X_0 + W_t \leq X_t^* \leq X_0 + t + W_t, \quad t \in [0, T], \text{ w.p.1.}$$

3 The Euler–Maruyama scheme.

The Euler–Maruyama scheme with constant step size Δ for the SDE (1.3) is

$$X_{n+1}^{(\Delta)} = X_n^{(\Delta)} + f\big(X_n^{(\Delta)}\big)\Delta + A\Delta W_n, \tag{3.1}$$

for $n = 0, 1, \dots, N_\Delta - 1$, where $N_\Delta := T/\Delta$ and the the components of ΔW_n are $N(0, \Delta)$ distributed and independent, and the ΔW_n for different n are also independent.

Our aim is to prove the following theorem.

THEOREM 3.1. *Suppose that the assumptions of Theorem 2.1 hold and, in addition, that the drift coefficient f is continuous from below. Then there exists a mean-square continuous strong solution X_t of the SDE* (1.3) *with the same initial value and with $L_t \le X_t \le U_t$ for $t \in [0, T]$ such that the solutions of the Euler scheme converge strongly (i.e. mean-square) to this solution, i.e.,*

$$\lim_{\Delta \to 0} \sup_{n=0,1,\dots,N_\Delta} \mathbb{E}\left\|X_n^{(\Delta)} - X_{t_n}\right\|^2 = 0.$$

The usual theorems on the strong convergence of the Euler–Maruyama scheme as in [7] do not apply to the SDE (1.3) because of the lack of regularity of the drift coefficient. Our proof below applies the Euler–Maruyama scheme to more regular equations and then takes the limit. It makes extensive use of upper and lower numerical solutions.

3.1 Upper and lower numerical solutions.

Let the step size $\Delta > 0$ be fixed. A lower solution of the Euler–Maruyama scheme applied to the SDE (1.3) with constant step size Δ is a finite sequence of $\mathcal{F}_{t_n}$-measurable random variables $L_n^{(\Delta)}$ with $\mathbb{E}\|L_n^{(\Delta)}\|^2 < \infty$ which satisfies

$$L_{n+1}^{(\Delta)} \le L_n^{(\Delta)} + f\big(L_n^{(\Delta)}\big)\Delta + A\Delta W_n, \tag{3.2}$$

for $n = 0, 1, \dots, N_\Delta - 1$, where $N_\Delta := T/\Delta$. An upper solution $U_n^{(\Delta)}$ is defined analogously with $\le$ replaced by $\ge$.

If the SDE has a lower solution L_t, then Euler–Maruyama scheme has a lower solution $L_n^{(\Delta)}$ defined by

$$L_{n+1}^{(\Delta)} = \min\left\{L_{t_n}, L_n^{(\Delta)} + f(L_n^{(\Delta)})\Delta + [A\Delta W_n]^-\right\}, \quad n = 0, 1, \dots, N_\Delta - 1, \tag{3.3}$$

with $L_0^{(\Delta)} \le L_0$, where (this differs in sign from convention!)

$$[x]^- := \min\{0, x\} = \begin{cases} 0 & \text{if } x \ge 0 \\ x & \text{if } x \le 0 \end{cases}.$$

By construction it follows that $L_n^{(\Delta)} \le L_{t_n}$ for each n.

Similarly, if the SDE has a upper solution U_t, then Euler–Maruyama scheme has a upper solution $U_n^{(\Delta)}$ defined by

$$U_{n+1}^{(\Delta)} = \max\left\{U_{t_n}, U_n^{(\Delta)} + f(U_n^{(\Delta)})\Delta + [A\Delta W_n]^+\right\}, \quad n = 0, 1, \dots, N_\Delta - 1, \tag{3.4}$$

with $U_0^{(\Delta)} \geq U_0$, where

$$[x]^+ := \max\{0, x\} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

By construction it follows that $U_n^{(\Delta)} \geq U_{t_n}$ for each n.

4 Successively iterated stochastic differentials.

To prove Theorem 3.1 we shall apply the Euler–Maruyama scheme to successively iterated stochastic differentials and take their convergence.

Recall that as well as being an increasing function with a linear growth bound, the drift f is continuous from below, i.e., left continuous in the scalar case. In addition, we assume that a lower solution L_t and an upper solutions U_t of the SDE (1.3) are known with $L_t \leq U_t$ for $t \in [0, T]$.

We consider the following successive interations.

$$X_t^{(j+1)} = X_0 + \int_0^t f\big(X_s^{(j)}\big)ds + AW_t, \quad t \in [0, T], \tag{4.1}$$

for $j = 0, 1, 2, \ldots$ with $X_t^{(0)} \equiv L_t$ and some $\mathcal{F}_0$-measurable initial value X_0 with $\mathbb{E}\|X_0\|^2 < \infty$ and $L_0 \leq X_0 \leq U_0$. Equation (4.1) is not so much a stochastic differential equations as a stochastic differential

$$dX_t^{(j+1)} = f\big(X_t^{(j)}\big)dt + AdW_t, \quad t \in [0, T],$$

i.e., the right hand side is a known function which does not involve the unknown "solution" $X_t^{(j+1)}$. It has a unique strong solution $X_t^{(j+1)}$ (in the sense of Theorem 2.1) for each $j = 0, 1, 2, \ldots$ by Theorem 2.2 in [5].

In particular, $X_t^{(j)}$ is a lower solution and U_t an upper solution with $X_t^{(j)} \leq X_t^{(j+1)} \leq U_t$ for all $t \in [0, T]$. To see this note that

$$X_t^{(j+1)} - X_t^{(j)} = \int_0^t \big(f\big(X_s^{(j)}\big) - f\big(X_s^{(j-1)}\big)\big)ds, \quad t \in [0, T],$$

for $j = 1, 2, \ldots$ Thus, if $X_t^{(j)} \geq X_t^{(j-1)}$ for all $t \in [0, T]$, then $f(X_t^{(j)}) \geq f(X_t^{(j-1)})$ for all $t \in [0, T]$ and it follows that $X_t^{(j+1)} \geq X_t^{(j)}$ for all $t \in [0, T]$. Now for $j = 1$, we have $X_t^{(1)} \geq X_t^{(0)} \equiv L_t$ since L_t is a lower solution of the SDE (1.3) and

$$X_t^{(1)} - L_t \geq \int_0^t (f(L_s) - f(L_s))\, ds \equiv 0, \quad t \in [0, T].$$

The sequence of functions $\{X_t^{(j)}(\omega)\}$ is monotonically increasing and bounded above by $U_t(\omega)$ on the interval $[0, T]$, so converges pointwise for each t and ω.

We need to show that the limits form a stochastic process, in particular, satisfies the required measurability and integrability conditions.

Since $X_t^{(j)} \in [L_t, U_t]$ for all $t > 0$, then $\mathbb{E}\|X_t^{(j)}\|^2$ is bounded for any fixed t. Thus, $X_t^{(j)}$ converges weakly in $L^2(\Omega, \mathbb{R}^d)$ to some $\bar{X}_t$ for any t, i.e., $\mathbb{E}\langle X_t^{(j)}, g\rangle \to \mathbb{E}\langle \bar{X}_t, g\rangle$ for any $g \in L^2(\Omega, \mathbb{R}^d)$. Choosing g to be a constant function equal to any one of the unit vectors in $\mathbb{R}^d$, it follows (componentwise) that $\mathbb{E}X_t^{(j)} \to \mathbb{E}\bar{X}_t$.

We now want to show that $X_t^{(j)} \leq \bar{X}_t$, w.p.1., for every $j \in \mathbb{N}$ and for every $t > 0$. Suppose not. Then there exists some $j \in \mathbb{N}$ and $A \subseteq \Omega$ with $P(A) > 0$ such that $X_t^{(j)} > \bar{X}_t$, a.s., on A for some $t > 0$. Hence, $\mathbb{E}\{X_t^{(j)}\mathbb{I}_A\} > \mathbb{E}\{\bar{X}_t\mathbb{I}_A\}$ where $\mathbb{I}_A$ is the indicator function of A. Moreover, we know that $X_t^{(j+1)} \geq X_t^{(j)}$, a.s., so

$$\mathbb{E}\big\{X_t^{(j+1)}\mathbb{I}_A\big\} \geq \mathbb{E}\big\{X_t^{(j)}\mathbb{I}_A\big\} > \mathbb{E}\{\bar{X}_t\mathbb{I}_A\}.$$

Now, since $\mathbb{I}_A \in L^2(\Omega, \mathbb{R})$ we have that $\mathbb{E}\{X_t^{(j+1)}\mathbb{I}_A\} \to \mathbb{E}\{\bar{X}_t\mathbb{I}_A\}$, and thus have a contradiction.

Now using the Markov inequality, we obtain

$$P\big(\bar{X}_t - X_t^{(j)} \geq \varepsilon\big) \leq \frac{\mathbb{E}\{\bar{X}_t - X_t^{(j)}\}}{\varepsilon},$$

for every $\varepsilon > 0$, so $X_t^{(j)} \to \bar{X}_t$ in probability and hence from the boundedness (see [12, Theorem 17.4, p. 146]) we have $X_t^{(j)} \to \bar{X}_t$ in $L^2(\Omega, \mathbb{R}^d)$. Finally, since the sequence $X_t^{(j)}$ is monotone, it converges a.s, to $\bar{X}_t$ for each t and $\bar{X}_t$ is $\mathcal{F}_t$ measurable.

There thus exists a unique limit $\bar{X}_t(\omega)$ with $X_t^{(j)}(\omega) \leq \bar{X}_t(\omega) \leq U_t(\omega)$ for $t \in [0, T]$. In particular, by the continuity from below,

$$\lim_{j\to\infty} f\big(X_t^{(j)}(\omega)\big) = f(\bar{X}_t(\omega)).$$

Taking the limit in the integral equation (4.1), using the Lebesgue Dominated Convergence theorem, we obtain

$$\bar{X}_t(\omega) = X_0 + \int_0^t f(\bar{X}_s(\omega))ds + AW_t(\omega), \quad t \in [0, T],$$

i.e., $\bar{X}_t$ is a strong solution of the SDE (1.3).

5 Successively iterated Euler–Maruyama solutions.

Let the step size $\Delta > 0$ be fixed and let $L_n^{(\Delta)}$ and $U_n^{(\Delta)}$ be lower and upper solutions of the Euler–Maruyama scheme with $L_0^{(\Delta)} \leq X_0 \leq U_0^{(\Delta)}$.

We consider an Euler–Maruyama analogue of the successively iterated differentials (4.1), namely

(5.1)
$$X_{n+1}^{(j+1),\Delta} = X_n^{(j+1),\Delta} + f\big(X_n^{(j),\Delta}\big)\Delta + A\Delta W_n, \quad n = 0, 1, \dots, N_\Delta := T/\Delta,$$

for $j = 0, 1, 2, \dots$ with $X_n^{(0),\Delta} \equiv L_n^\Delta$. Then $X_n^{(j+1),\Delta}(\omega) \geq X_n^{(j),\Delta}(\omega)$ for each $n = 0, 1, \dots, N_\Delta$ and all $j = 0, 1, 2, \dots$ To see this first note that pathwise

$$X_{n+1}^{(j+1),\Delta} - X_{n+1}^{(j),\Delta} = X_n^{(j+1),\Delta} - X_n^{(j),\Delta} + \big[f\big(X_n^{(j),\Delta}\big) - f\big(X_n^{(j-1),\Delta}\big)\big]\Delta.$$

But for $j = 1$ we have $X_n^{(0),\Delta} = L_n^{(\Delta)}$, and

$$X_{n+1}^{(1),\Delta} - L_{n+1}^\Delta \geq X_n^{(1),\Delta} - L_n^\Delta + \big[f\big(L_n^\Delta\big) - f\big(L_n^\Delta\big)\big]\Delta$$

with $X_0^{(1),\Delta} - L_0^\Delta = X_0 - L_0^\Delta \geq 0$. Thus, $X_{n+1}^{(1),\Delta} \geq L_{n+1}^\Delta$ for every $n \in \mathbb{N}$. By induction and exploiting the monotonicity of f we have the desired result.

The vector of iterations $(X_0^{(j),\Delta}(\omega), \dots, X_{N_\Delta}^{(j),\Delta}(\omega))$ is componentwise monotonically increasing and bounded from above by the upper numerical solution vector $(U_0^\Delta(\omega), \dots, U_{N_\Delta}^\Delta(\omega))$. Thus it converges componentwise to the limit

$$\big(\bar{X}_0^\Delta(\omega), \cdots, \bar{X}_{N_\Delta}^\Delta(\omega)\big),$$

which is also bounded above componentwise by the upper numerical solution vector. As before we can prove that $X_0^{(j),\Delta} \to \bar{X}_0^\Delta$ in $L^2(\Omega, \mathbb{R}^d)$ as well as a.s.

Since the convergence is from below, we have

$$\lim_{j\to\infty} f\big(X_n^{(j),\Delta}(\omega)\big) = f\big(\bar{X}_n^\Delta(\omega)\big)$$

for each $n = 0, 1, \dots, N_\Delta$. Thus in the limit we have pathwise

$$\bar{X}_{n+1}^\Delta = \bar{X}_n^\Delta + f\left(\bar{X}_n^\Delta\right)\Delta + A\Delta W_n$$

for $n = 0, 1, \dots, N_\Delta$, i.e. $\bar{X}_n^\Delta$ satisfies the Euler–Maruyama scheme applied to the SDE (1.3).

6 Convergence.

Finally, we need to show that the Euler–Maruyama solution $\bar{X}_n^\Delta$ converges strongly to the solution $\bar{X}_t$ of the SDE (1.3).

First we note that since the successive iterations $X_t^{(j)}$ converge almost surely to $\bar{X}_t$ and are bounded from above by a mean-square integrable expression, they also converge in the mean-square sense, i.e.,

$$\mathbb{E}\big(\big\|X_t^{(j)} - \bar{X}_t\big\|^2\big) \to 0 \quad \text{as } j \to \infty$$

and similarly for the Euler iterations, i.e.,

$$\mathbb{E}\left(\left\|X_{t_n}^{(j),\Delta} - \bar{X}_{t_n}^{\Delta}\right\|^2\right) \to 0 \quad \text{as } j \to \infty.$$

Thus,

$$\mathbb{E}\left(\left\|\bar{X}_{t_n}^{\Delta} - \bar{X}_{t_n}\right\|^2\right) \le \mathbb{E}\left(\left\|X_{t_n}^{(j),\Delta} - X_{t_n}^{(j)}\right\|^2\right) + \varepsilon,$$

for every $\varepsilon > 0$ and j large enough.

In addition, for each j the Euler–Maruyama solution $X_n^{(j),\Delta}$ converges in probability (see [4]) to the successive iteration $X_t^{(j)}$. Moreover, since the $X_n^{(j),\Delta}$ are bounded, the convergence here is also in $L^2(\Omega, \mathbb{R}^d)$, i.e.,

$$\mathbb{E}\left(\left\|X_{t_n}^{(j),\Delta} - X_{t_n}^{(j)}\right\|^2\right) \to 0 \quad \text{as } \Delta \to 0.$$

Taking now the limit as $\Delta \to 0$ in the above inequality we have that

$$\lim_{\Delta \to 0} \mathbb{E}\left(\left\|\bar{X}_{t_n}^{\Delta} - \bar{X}_{t_n}\right\|^2\right) \le \varepsilon.$$

Since, $\varepsilon > 0$ was arbitrary chosen we have the desired result. This completes the proof of Theorem 3.1.

REMARK 6.1. The result remains true if the drift coefficient is continuous from above rather than from below. The essential change in the proof is to start with $X_t^{(0)} = U_t$, i.e., the upper solution, and to construct a decreasing rather than increasing sequence of iterations.

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