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THE LOCAL LINEARIZATION METHOD FOR NUMERICAL INTEGRATION OF RANDOM DIFFERENTIAL EQUATIONS ⋆

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Abstract.

A Local Linearization (LL) method for the numerical integration of Random Differential Equations (RDE) is introduced. The classical LL approach is adapted to this type of equations, which are defined by random vector fields that are typically at most Hölder continuous with respect to the time argument. The order of strong convergence of the method is studied. It turns out that the LL method improves the order of convergence of conventional numerical methods that have been applied to RDEs. Additionally, the performance of the LL method is illustrated by means of numerical simulations, which show that it behaves well even in those equations with complicated noisy dynamics where conventional methods fail.

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1 Introduction.

During the last four decades the use of Random Differential Equations (RDE) has became a useful tool for modeling many physical, biological and engineering phenomena [24, 19, 16, 18, 15, 22]. Recently, a renovated interest in the study of RDEs has been motivated by the development of the theory of random dynamical systems (see [1] and references therein). The main reason is the fact that the dynamics of random systems is better understood in the framework of deterministic systems than in the framework of stochastic integration theory. For instance, RDEs have been recently used for the analysis of the bifurcation behavior of random nonlinear systems [7, 8].

Since in most common cases no explicit solution of the equations are known, the use of numerical methods in the treatment of RDEs has become an important

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need [17, 3, 2, 13, 21]. Essentially, a RDE is a non autonomous Ordinary Differential Equation (ODE) coupled with a stochastic process, which usually is used to model the noisy perturbations of deterministic systems. Thus, in principle, a RDE can be integrated by applying conventional numerical methods for ODEs. For instance, in a recent paper [5] the authors applied the classical Euler and Heun schemes for the integration of RDEs and introduced "averaged" versions of these schemes that retained their standard order of convergence. However, the application of these averaged methods is not only restricted to the particular class of separable RDEs but it also requires a finer partition for the noise term that, typically, increase the computational effort of these numerical algorithms.

In this paper, an alternative numerical integrator based on the Local Linearization approach is introduced. That approach has been successfully applied in the framework of ODEs [9] and Stochastic Differential Equations (SDEs) [11] to construct efficient and stable numerical schemes. A key step of the LL approach is the piece-wise linear approximation (by the first-order Taylor expansion) of the vector field that define the differential equations. Because the vector field of RDEs is typically at most Hölder continuous with respect to the temporal variable, the use of the differential version of the Taylor expansion and so the application of the conventional LL methods for nonautonomous ODEs is not possible. Therefore, for this class of equations, the LL approach must be reconsidered.

The goal of this work is justly to study the viability of the LL approach for the numerical integration of RDEs. The plan of the paper is the following. In Section 2, the LL method is derived. In Section 3, the convergence of the method is studied and, in the last section, the performance of LL scheme is evaluated and compared with other numerical integrators by mean of simulations.

2 Local Linearization method.

Let $(\Omega, \mathcal{F}, P)$ be the underlying complete probability space and $\{\mathcal{F}_t, t \geq t_0\}$ be an increasing right continuous family of complete sub σ-algebras of $\mathcal{F}$ and $\mathbf{f} : \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}^d$ be a twice continuously differentiable function. Consider the RDE

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \xi(t)), \quad t \in [t_0, T], \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{2.1}$$

where ξ is a k-dimensional $\mathcal{F}_t$-adapted and separable finite continuous process. Suppose that conditions for the existence and uniqueness of an almost surely continuous solution are assumed (see Theorem 3.1 in [6]). Let $(\tau)_h$ be a time partition given by

$$(\tau)_h = \{t_0 < t_1 < \cdots < t_n < \cdots\},$$

where

$$\sup_n (t_{n+1} - t_n) \leq h < 1,$$

and define

$$n_t := \max\{n = 0, 1, 2, \ldots : t_n \leq t\}.$$

Suppose that a realization of ξ is given and that $\mathbf{y}_{t_n} \in \mathbb{R}^d$ is a point closed to $\mathbf{x}(t_n)$. Consider the first-order Taylor expansion of the function $\mathbf{f}$ at the point $(\mathbf{y}_{t_n}, \xi(t_n))$,

$$\mathbf{f}(\mathbf{u},\zeta) \approx \mathbf{f}(\mathbf{y}_{t_n},\xi(t_n)) + \mathbf{f}'_{\mathbf{x}}(\mathbf{y}_{t_n},\xi(t_n)))(\mathbf{u}-\mathbf{y}_{t_n}) + \mathbf{f}'_{\xi}(\mathbf{y}_{t_n},\xi(t_n))(\zeta - \xi(t_n)),$$

for all $\mathbf{u} \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}^k$, where $\mathbf{f}'_{\mathbf{x}}$ and $\mathbf{f}'_{\xi}$ denote the derivatives of $\mathbf{f}$ respecting to $\mathbf{x}$ and ξ, respectively. Thus, using this, the solution of Equation (2.1) can be locally approximated by the solution of the linear equation

$$\begin{aligned} \dot{\mathbf{y}}(t) &= \mathbf{A}(\mathbf{y}(t_n))\mathbf{y}(t) + \mathbf{a}(\mathbf{y}(t_n),t), \quad t \in [t_n, t_{n+1}], \\ \mathbf{y}(t_n) &= \mathbf{y}_{t_n}, \end{aligned} \tag{2.2}$$

where

$$\mathbf{A}(\mathbf{y}(t_n)) = \mathbf{f}'_{\mathbf{x}}(\mathbf{y}(t_n),\xi(t_n))$$

and

$$\mathbf{a}(\mathbf{y}(t_n),t) = \mathbf{f}(\mathbf{y}(t_n),\xi(t_n)) - \mathbf{f}'_{\mathbf{x}}(\mathbf{y}(t_n),\xi(t_n))\mathbf{y}(t_n) + \mathbf{f}'_{\xi}(\mathbf{y}(t_n),\xi(t_n))(\xi(t)-\xi(t_n)).$$

The solution of Equation (2.2) is given by

$$\mathbf{y}(t_n+s) = \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))s}\left(\mathbf{y}(t_n) + \int_{t_n}^{t_n+s} \mathrm{e}^{-\mathbf{A}(\mathbf{y}(t_n))(u-t_n)}\mathbf{a}(\mathbf{y}(t_n),u)\,\mathrm{d}u\right),$$

which, by means of the integral identity

$$\int_0^h \exp(-\mathbf{A}u)\,\mathrm{d}u\mathbf{A} = -(\exp(-\mathbf{A}h) - \mathbf{I})$$

can be rewritten as

$$\begin{aligned} \mathbf{y}(t_n+s) &= \mathbf{y}(t_n) + \int_0^s \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))(s-u)}\mathbf{f}(\mathbf{y}(t_n),\xi(t_n))\,\mathrm{d}u + \\ &\quad + \int_0^s \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))(s-u)}\mathbf{f}'_{\xi}(\mathbf{y}(t_n),\xi(t_n))(\xi(t_n+u)-\xi(t_n))\,\mathrm{d}u, \end{aligned} \tag{2.3}$$

for all $t_n + s \in [t_n, t_{n+1}]$. Therefore, numerical integrators for Equation (2.1) might be obtained by choosing suitable approximations to the second integral in the expression above.

For instance, a natural approximation to the term $\xi(t_n+u) - \xi(t_n)$ is given by the following linear spline interpolation [23]

$$\xi(t_n+u) - \xi(t_n) = \frac{\Delta\xi(t_n)}{h_n}u,$$

where $h_n = t_{n+1} - t_n$, $\Delta\xi(t_n) = \xi(t_{n+1}) - \xi(t_n)$, $n = 0, 1, \ldots$. Then, substituting this in (2.3) it is obtained the Local Linear Approximation

$$\mathbf{y}(t_n+s) = \mathbf{y}(t_n) + \int_0^s \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))(s-u)}\mathbf{f}(\mathbf{y}(t_n),\xi(t_n))\,\mathrm{d}u + \tag{2.4}$$

$$
\begin{aligned}
&+\int_0^s \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))(s-u)}\mathbf{f}'_\xi(\mathbf{y}(t_n),\xi(t_n))\frac{\Delta\xi(t_n)}{h_n}u\,\mathrm{d}u \\
= &\ \mathbf{y}(t_n)+\int_0^s \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))(s-u)}\mathbf{f}(\mathbf{y}(t_n),\xi(t_n))\,\mathrm{d}u \\
&+\int_0^s\int_0^u \mathrm{e}^{\mathbf{A}(\mathbf{y}(t_n))(s-u)}\mathbf{f}'_\xi(\mathbf{y}(t_n),\xi(t_n))\frac{\Delta\xi(t_n)}{h_n}\,\mathrm{d}s\,\mathrm{d}u,
\end{aligned}
$$

which, according to [10], can be rewritten as

$$\mathbf{y}(t)=\mathbf{y}(t_{n_t})+\mathbf{g}(\mathbf{y}(t_{n_t}),\xi(t_{n_t});\Delta t),\quad t\geq t_0, \tag{2.5}$$

where the vector $\mathbf{g}(\mathbf{y}(t_{n_t}),\xi(t_n);\Delta t)$ is defined by the block matrix

$$
\begin{pmatrix}
\mathbf{F}(\mathbf{y}(t_{n_t}),\xi(t_{n_t});\Delta t) & \mathbf{f}_1(\mathbf{y}(t_{n_t}),\xi(t_{n_t});\Delta t) & \mathbf{g}(\mathbf{y}(t_{n_t}),\xi(t_{n_t});\Delta t)\\
\mathbf{0} & 1 & \mathbf{f}_2(\mathbf{y}(t_{n_t}),\xi(t_{n_t});\Delta t)\\
\mathbf{0} & 0 & 1
\end{pmatrix} = \mathrm{e}^{\Delta t\mathbf{C}}
$$

with $\Delta t = t - t_{n_t}$ and

$$
\mathbf{C} = \begin{pmatrix}
\mathbf{f}'_{\mathbf{x}}(\mathbf{y}(t_{n_t}),\xi(t_{n_t})) & \mathbf{f}'_\xi(\mathbf{y}(t_{n_t}),\xi(t_{n_t}))\frac{\Delta\xi(t_n)}{h_n} & \mathbf{f}(\mathbf{y}(t_{n_t}),\xi(t_{n_t}))\\
\mathbf{0} & 0 & 1\\
\mathbf{0} & 0 & 0
\end{pmatrix}
\in \mathbb{R}^{(d+2)\times(d+2)}.
$$

Now, taking $t = t_{n+1}$ in (2.5) it is obtained the following LL scheme:

$$\mathbf{y}_{t_{n+1}} = \mathbf{y}_{t_n} + \mathbf{g}(\mathbf{y}_{t_n},\xi(t_n);h_n). \tag{2.6}$$

It is clear that, for a given realization of ξ, the LL Approximation (2.5) is a continuous function that coincides with the above LL scheme at each point of the time partition $(\tau)_h$.

It should be also noted that the LL scheme (2.6) is computational feasible and its numerical implementation is reduced to the use of a convenient algorithm to compute matrix exponentials, e.g., those based on rational Padé approximations [4], the Schur decomposition [4] or Krylov subspace methods [14]. The selection of one of them will mainly depend on the size and structure of the matrix $\mathbf{C}$. For instance, for many low-dimensional system of equations it is enough to use the algorithm developed in [20], which takes advantage of the special structure of the matrix $\mathbf{C}$. Whereas, for large systems of equations, the Krylov subspace methods are strongly recommended.

3 Convergence.

In this section, a study of the uniform error in the LL Approximation is presented. It is shown that the order of convergence depends on the moduli of continuity of the stochastic process involved in the equation.

Suppose that there exits separable almost surely finite stochastic processes L, K_0 and K_1 such that for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,

$$\|\mathbf{f}(\mathbf{u}, \xi(t)) - \mathbf{f}(\mathbf{v}, \xi(t))\| \leq L(t)\|\mathbf{u} - \mathbf{v}\| \tag{3.1}$$

and

$$\|\mathbf{f}(\mathbf{u}, \xi(t))\| \leq K_0(t)(1 + \|\mathbf{u}\|), \tag{3.2}$$
$$\left\|\mathbf{f}'_{\mathbf{x}}(\mathbf{u}, \xi(t))\right\| + \left\|\mathbf{f}'_{\xi}(\mathbf{u}, \xi(t))\right\| \leq K_1(t). \tag{3.3}$$

Assume also that for all $\mathbf{u} \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}^k$ the following condition

$$\left\|f''_{\mathbf{x}\mathbf{x}}(\mathbf{u}, \zeta)\right\| + \left\|f''_{\mathbf{x}\xi}(\mathbf{u}, \zeta)\right\| + \left\|f''_{\xi\xi}(\mathbf{u}, \zeta)\right\| \leq K_2 \tag{3.4}$$

holds for some positive constant K_2. Finally, let

$$\omega(h) := \sup_{|t-s|\leq h} \|\xi(t) - \xi(s)\|$$

be the moduli of continuity of ξ.

Suppose that a realization of the process ξ is given. Then, consider the corresponding realization of the solution process $\mathbf{x}$ and its respective LL approximation $\mathbf{y}$. Theorem 3.2 below states the order of convergence of $\mathbf{y}$ to $\mathbf{x}$. The following lemma shall be very useful for the proof of that theorem.

LEMMA 3.1. *There exists positive constants C_1, C_2 and C_3 such that the inequalities*

$$\sup_{t_0 \leq t \leq T} \|\mathbf{y}(t)\| \leq C_1$$

and

$$\|\mathbf{y}(t) - \mathbf{y}(t_{n_t})\| \leq (C_2 + C_3\omega(h))h$$

hold for all $t \in [t_0, T]$.

PROOF. From Equation (2.2) and conditions (3.2), (3.3) it is obtained that

$$\begin{aligned}
\sup_{t_0 \leq s \leq t} \|\mathbf{y}(s)\| &\leq \|\mathbf{y}_0\| + \sup_{t_0 \leq s \leq t} \sum_{n=0}^{n_s - 1} \int_{t_{n_s}}^{s} (K_1(t_{n_u})\|\mathbf{y}(u) - \mathbf{y}(t_{n_u})\| + \\
&\quad + K_0(t_{n_u})(1 + \|\mathbf{y}(t_{n_u})\|) + K_1(t_{n_u})\|\xi(u) - \xi(t_{n_u})\|)\, \mathrm{d}u \\
&\leq \|\mathbf{y}_0\| + \int_{t_0}^{t} ((K_0(t_{n_u}) + 2K_1(t_{n_u})) \sup_{t_0 \leq s \leq u} \|\mathbf{y}(s)\| + K_0(t_{n_u}) + \\
&\quad + K_1(t_{n_u})\|\xi(u) - \xi(t_{n_u})\|)\, \mathrm{d}u,
\end{aligned}$$

which yields to

$$\sup_{t_0 \leq s \leq t} \|\mathbf{y}(s)\| \leq \|\mathbf{y}_0\| + \int_{t_0}^{t} ((\widetilde{K}_0 + 2\widetilde{K}_1) \sup_{t_0 \leq s \leq u} \|\mathbf{y}(s)\|\, \mathrm{d}u + (\widetilde{K}_0 + \widetilde{K}_1\omega(h))(t - t_0),$$

where

$$\widetilde{K}_i = \sup_{t_0 \leq s \leq T} K_i(s) < \infty, \quad i = 0, 1.$$

Then, from the Gronwall inequality follows that

$$\begin{aligned}
(3.5) \qquad & \sup_{t_0 \leq s \leq t} \|\mathbf{y}(s)\| \\
& \leq (\|\mathbf{y}_0\| + (\widetilde{K}_0 + \widetilde{K}_1 \omega(h))(t - t_0)) \mathrm{e}^{(\widetilde{K}_0 + 2\widetilde{K}_1)(t - t_0)} \\
& \leq \Big(\|\mathbf{y}_0\| + \Big(\widetilde{K}_0 + 2\widetilde{K}_1 \sup_{t_0 \leq s \leq t} \|\xi(s)\|\Big)(t - t_0)\Big) \mathrm{e}^{(\widetilde{K}_0 + 2\widetilde{K}_1)(t - t_0)},
\end{aligned}$$

which for $t = T$ gives the first assertion of the lemma.

On the other hand,

$$\begin{aligned}
\|\mathbf{y}(t) - \mathbf{y}(t_{n_t})\| \leq & \int_{t_{n_t}}^{t} (K_1(t_{n_t}) \|\mathbf{y}(u) - \mathbf{y}(t_{n_t})\| + K_0(t_{n_t})(1 + \|\mathbf{y}(t_{n_u})\|) + \\
& + K_1(t_{n_t}) \|\xi(u) - \xi(t_{n_u})\|) \,\mathrm{d}u,
\end{aligned}$$

which, by (3.5), yields to

$$\begin{aligned}
& \|\mathbf{y}(t) - \mathbf{y}(t_{n_t})\| \\
& \leq \int_{t_{n_t}}^{t} (\widetilde{K}_1 + \widetilde{K}_0) \|\mathbf{y}(u) - \mathbf{y}(t_{n_t})\| \,\mathrm{d}u + (\widetilde{K}_0(\widetilde{C}_1 + \widetilde{C}_2 \omega(h)) + \widetilde{K}_1 \omega(h))h,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{C}_1 &= 1 + (\|\mathbf{y}_0\| + \widetilde{K}_0)(T - t_0) \mathrm{e}^{(\widetilde{K}_0 + 2\widetilde{K}_1)(T - t_0)}, \\
\widetilde{C}_2 &= \widetilde{K}_1 (T - t_0) \mathrm{e}^{(\widetilde{K}_0 + 2\widetilde{K}_1)(T - t_0)}.
\end{aligned}$$

Hence, the Gronwall inequality implies that

$$\begin{aligned}
\|\mathbf{y}(t) - \mathbf{y}(t_{n_t})\| &\leq (\widetilde{K}_0 \widetilde{C}_1 + (\widetilde{K}_1 + \widetilde{K}_0 \widetilde{C}_2) \omega(h)) \mathrm{e}^{(\widetilde{K}_0 + \widetilde{K}_1)h} h \\
&\leq (C_2 + C_3 \omega(h))h,
\end{aligned}$$

where

$$\begin{aligned}
C_2 &= \widetilde{K}_0 \widetilde{C}_1 \mathrm{e}^{(\widetilde{K}_0 + \widetilde{K}_1)}, \\
C_3 &= (\widetilde{K}_1 + \widetilde{K}_0 \widetilde{C}_2) \mathrm{e}^{(\widetilde{K}_0 + \widetilde{K}_1)}.
\end{aligned}$$

This concludes the second statement of the lemma. □

THEOREM 3.2. *If*

$$\|\mathbf{x}_0 - \mathbf{y}_0\| \leq D_1 h^{\min(2, 2\gamma)} \quad \textit{and} \quad \omega(h) \leq D_2 h^{\gamma}$$

for some positive constants $D_1, D_2, \gamma > 0$ *then*

$$\sup_{t_0 \leq t \leq T} \|\mathbf{x}(t) - \mathbf{y}(t)\| \leq C_T(\xi) h^{\min(2, 2\gamma)},$$

where $C_T(\xi)$ *is a positive constant.*

PROOF. The following expressions follow respectively from Equations (2.1) and (2.2)

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}(t_{n_t}) + \int_{t_{n_t}}^{t} \mathbf{f}(\mathbf{x}(u), \xi(u))\,\mathrm{d}u,\\ \mathbf{y}(t) &= \mathbf{y}(t_{n_t}) + \int_{t_{n_t}}^{t} (\mathbf{A}(t_{n_t})\mathbf{y}(u) + \mathbf{a}(t_{n_t}, u))\,\mathrm{d}u\end{aligned}$$

which used in a recursive way yield to

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}(t_0) + \sum_{n=0}^{n_t-1} \int_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{x}(u), \xi(u))\,\mathrm{d}u + \int_{t_{n_t}}^{t} \mathbf{f}(\mathbf{x}(u), \xi(u))\,\mathrm{d}u,\\ \mathbf{y}(t) &= \mathbf{y}(t_0) + \sum_{n=0}^{n_t-1} \int_{t_n}^{t_{n+1}} (\mathbf{A}(t_n)\mathbf{y}(u) + \mathbf{a}(t_n, u))\,\mathrm{d}u +\\ &\quad + \int_{t_{n_t}}^{t} (\mathbf{A}(t_{n_t})\mathbf{y}(u) + \mathbf{a}(t_{n_t}, u))\,\mathrm{d}u.\end{aligned}$$

From these equalities it is obtained

$$e(t) \le \|\mathbf{x}_0 - \mathbf{y}_0\| + P(t) + Q(t), \tag{3.6}$$

where

$$e(t) = \sup_{t_0 \le s \le t} \|\mathbf{x}(s) - \mathbf{y}(s)\|$$

and

$$\begin{aligned}P(t) &= \sup_{t_0 \le s \le t} \Bigg\| \sum_{n=0}^{n_s-1} \int_{t_n}^{t_{n+1}} (\mathbf{f}(\mathbf{x}(u), \xi(u)) - \mathbf{f}(\mathbf{y}(u), \xi(u)))\,\mathrm{d}u +\\ &\quad + \int_{t_{n_s}}^{s} (\mathbf{f}(\mathbf{x}(u), \xi(u)) - \mathbf{f}(\mathbf{y}(u), \xi(u)))\,\mathrm{d}u \Bigg\|,\end{aligned}$$

$$\begin{aligned}Q(t) &= \sup_{t_0 \le s \le t} \Bigg\| \sum_{n=0}^{n_s-1} \int_{t_n}^{t_{n+1}} (\mathbf{f}(\mathbf{y}(u), \xi(u)) - \mathbf{A}(t_{n_u})\mathbf{y}(u) - \mathbf{a}(t_{n_u}, u))\,\mathrm{d}u +\\ &\quad + \int_{t_{n_s}}^{s} (\mathbf{f}(\mathbf{y}(u), \xi(u)) - \mathbf{A}(t_{n_t})\mathbf{y}(u) - \mathbf{a}(t_{n_t}, u))\,\mathrm{d}u \Bigg\|.\end{aligned}$$

From (3.1) it is obtained

$$P(t) \le \sup_{t_0 \le s \le t} \int_{t_0}^{s} L(u)\|\mathbf{x}(u) - \mathbf{y}(u)\|\,\mathrm{d}u \le \widetilde{L} \int_{t_0}^{t} \sup_{t_0 \le s \le u} \|\mathbf{x}(s) - \mathbf{y}(s)\|\,\mathrm{d}u, \tag{3.7}$$

where

$$\widetilde{L} = \sup_{t_0 \le s \le T} L(s) < \infty.$$

On the other hand, by applying the Taylor formulae with Lagrange rest to the function $\mathbf{f}$ it is obtained

$$\begin{aligned}
&\|\mathbf{f}(\mathbf{y}(u),\xi(u)) - \mathbf{A}(t_{n_u})\mathbf{y}(u) - \mathbf{a}(t_{n_u},u)\| \\
&\quad \le \frac{1}{2}\sup_{\theta\in[0,1]}\{\|f''_{\mathbf{xx}}(\mathbf{y}^\theta,\xi^\theta)\|\|\mathbf{y}(u)-\mathbf{y}(t_{n_u})\|^2 + \\
&\qquad + 2\|f''_{\mathbf{x}\xi}(\mathbf{y}^\theta,\xi^\theta)\|\|\mathbf{y}(u)-\mathbf{y}(t_{n_u})\|\|\xi(u)-\xi(t_{n_u})\| + \\
&\qquad + \|f''_{\xi\xi}(\mathbf{y}^\theta,\xi^\theta)\|\|\xi(u)-\xi(t_{n_u})\|^2\},
\end{aligned}$$

where

$$\mathbf{y}^\theta = \mathbf{y}(t_{n_u}) + \theta(\mathbf{y}(u)-\mathbf{y}(t_{n_u})), \qquad \xi^\theta = \xi(t_{n_u}) + \theta(\xi(u)-\xi(t_{n_u})), \quad \theta\in[0,1].$$

Moreover, by condition (3.4) follows that

$$\|\mathbf{f}(\mathbf{y}(u),\xi(u)) - \mathbf{A}(t_{n_u})\mathbf{y}(u) - \mathbf{a}(t_{n_u},u)\| \le K_2(\|\mathbf{y}(u)-\mathbf{y}(t_{n_u})\|^2 + \|\xi(u)-\xi(t_{n_u})\|^2).$$

Hence, by Lemma 3.1,

$$Q(t) \le K_2 \int_{t_0}^{t} (2C_2^2h^2 + 2C_3^2\omega(h)^2h^2 + \omega(h)^2)\,\mathrm{d}u. \tag{3.8}$$

Then, by combining (3.7) and (3.8) in (3.6) it is obtained from the Gronwall inequality that

$$e(t) \le \|\mathbf{x}_0 - \mathbf{y}_0\| + K_2(2C_2^2h^2 + 2C_3^2\omega(h)^2h^2 + \omega(h)^2)(t-t_0)\mathrm{e}^{\widetilde{L}(t-t_0)}. \tag{3.9}$$

Finally,

$$e(T) \le C_T(\xi)h^{\min(2,2\gamma)},$$

where

$$C_T(\xi) = D_1 + K_2(2C_2^2 + 2C_3^2D_2^2 + D_2^2)(T-t_0)\mathrm{e}^{\widetilde{L}(T-t_0)}. \qquad \square$$

Remark 3.1. It is worth to emphasize that for any stochastic process ξ the LL method converges twice faster than the Euler method. In addition, for $\gamma \ge 0.5$ and any process ξ with moduli of continuity $\omega(h) = \mathrm{O}(h^\gamma)$ the LL method converges faster than the averaged Euler method without additional computational effort. Note also that for the particular case of a deterministic ξ, the theorem above provides the order of convergence of the LL method for non autonomous ODEs (see [11]).

As it was mentioned at the beginning of this section, Theorem 3.2 holds for a given realization of the processes $\mathbf{x}$ and $\mathbf{y}$. Thus, the constant $C_T(\xi)$ that appears in (3.9) is actually a realization of a finite random variable that depend on the process ξ.

The next corollary gives an estimate of the order of strong convergence of the LL approximation $\mathbf{y}$ to the process $\mathbf{x}$.

Corollary 3.3. *If*

$$E(\|\mathbf{x}_0 - \mathbf{y}_0\|) \leq D_1 h^{\min(2,2\gamma)}$$

and the stochastic processes ω, K_0, K_1 and L satisfy either

(i) $E(\omega(h)^2) \leq D_2 h^{2\gamma}$ *and K_0, K_1, L are positive constants,* *or*

(ii) $E(\omega(h)^4) \leq D_2 h^{4\gamma}$ *and for all $t \geq t_0$ there exists the respective moment generating functions of the random variables* $\widetilde{L} = \sup\limits_{t_0 \leq s \leq T} L(s)$, $\widetilde{K}_i = \sup\limits_{t_0 \leq s \leq T} K_i(s)$, $i = 0, 1$,

then

$$E\Big(\sup_{t_0 \leq t \leq T} \|\mathbf{x}(t) - \mathbf{y}(t)\|\Big) \leq C_T h^{\min(2,2\gamma)},$$

for some positive constant C_T.

Proof. This follows by taking expectations in the expression (3.9). Under condition (i), the values of C_2 and C_3 do not depend on ξ and the result is trivial. On the other hand, if condition (ii) holds, the corollary follows by using the Cauchy–Schwarz inequality and expressing the expectations of the powers of C_2 and C_3 in terms of the moment generating functions $\phi_{\widetilde{L}}(t) = E(\mathrm{e}^{\widetilde{L}t})$ and $\phi_{\widetilde{K}_i}(t) = E(\mathrm{e}^{\widetilde{K}_i t})$, $i = 0, 1$. □

4 Numerical experiments.

In this section the performance of the LL method is illustrated by means of three test examples. The first one belongs to the class of separable RDE considered in [5]. Thus, a comparison among the Euler scheme, the averaged Euler scheme and the LL scheme is achieved. For the second example, a simulation study is carried out to estimate the order of strong convergency of the LL scheme, and so, to corroborate the theoretical estimated obtained in the previous section. In these two examples the dynamics behavior of the random equations is very similar to that of their deterministic counterpart. Therefore, in the last example a comparison between the Euler scheme and the LL scheme is carried out for a random equation with a more complicated noisy dynamics.

For all examples, the matrix exponential that appears in the LL scheme (2.6) is computed by the rational Padé approximation with the 'scaling and squaring' procedure (see Algorithm 11.3.1 in [4] for details).

Example 1. Consider the RDE

$$\begin{aligned}\dot{x}_1(t) &= -x_2(1 + B(t)),\\ \dot{x}_2(t) &= x_2(1 + B(t)),\\ x_1(0) &= x_2(0) = 0.2,\end{aligned}$$

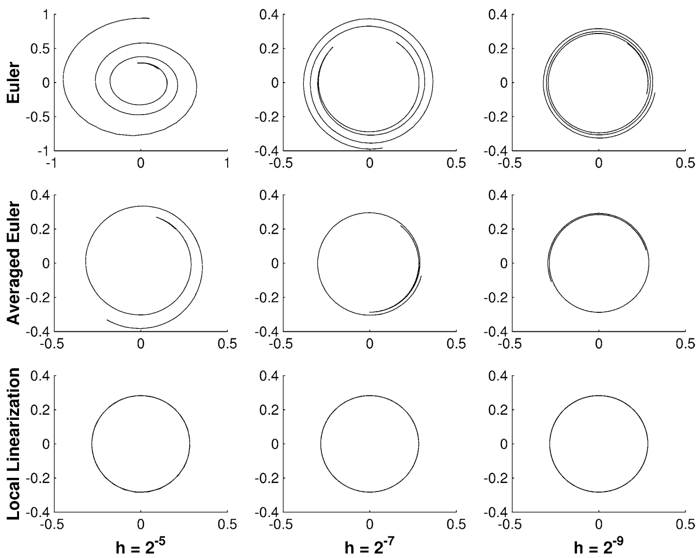

Figure 4.1: Phase portrait of trajectories obtained by the Euler method, the averaged Euler method and the LL method in the integration of the Example 1 with different step sizes.

for $0 \leq t \leq 8$, where $B(t)$ denotes a standard scalar Wiener process. Figure 4.1 shows, for three different values of the step size h, the phase-space of the numerical solution obtained by the Euler scheme, the averaged Euler scheme and the LL scheme. Notice that even for a moderate step size like $h = 2^{-5}$ the LL scheme replicates better the actual dynamics of the systems than the other two schemes.

Example 2. Let $0 \leq t \leq 64$ and consider the RDE defined by

$$\begin{aligned} \dot{x}_1(t) &= -x_2 + x_1(1 - x_1^2 - x_2^2)\sin(B^H(t))^2, \\ \dot{x}_2(t) &= x_1 + x_2(1 - x_1^2 - x_2^2)\sin(B^H(t))^2, \\ x_1(0) &= 0.8, \\ x_2(0) &= 0.1, \end{aligned} \tag{4.1}$$

where $B^H(t)$ denotes a fractional Brownian process with Hurst exponent $H = 0.45$.

In this example, the quantity

$$e(h) = E\Big(\sup_{t_0 \leq t_n \leq T} \|\mathbf{x}(t_n) - \mathbf{y}(t_n)\|\Big)$$

is used to estimate the order β of strong convergence of the LL scheme, where the simulated trajectory $\mathbf{y} = (y_1, y_2)$ of $\mathbf{x} = (x_1, x_2)$ is computed by the LL scheme with step size h. The estimated order $\widehat{\beta}$ is obtained from the slope of the straight line fitted to the set of points $\{\log_2(h_i), \log_2(\widehat{e}(h_i))\}_{i=1,\ldots,p}$, where $\widehat{e}(h_i)$ denote the estimate of $e(h)$ computed as in [12]. For it, the simulations are

Table 4.1: Uniform discretization errors for the LL method applied to the Example 2.

h	$\widehat{e}(h)$	$\pm\Delta(h)$
2^{-4}	0.00850166	$\pm$0.00013116
2^{-5}	0.00464681	$\pm$0.00000740
2^{-6}	0.00253053	$\pm$0.00004795
2^{-7}	0.00135831	$\pm$0.00000253
2^{-8}	0.00070590	$\pm$0.00000901
2^{-9}	0.00035127	$\pm$0.00000464

arranged into M batches with K trajectories $\mathbf{y}(t)$ in each. Thus, computing the error for the jth trajectory of the ith batch by

$$\widehat{e}_{i,j}(h) = \sup_{t_0 \leq t_n \leq T} \|\mathbf{x}(t_n) - \mathbf{y}^{i,j}(t_n)\|,$$

and the sample mean error of the ith batch and of all batches by

$$\widehat{e}_i(h) = \frac{1}{K}\sum_{j=1}^{K} \widehat{e}_{i,j}(h), \quad \text{and} \quad \widehat{e}(h) = \frac{1}{M}\sum_{i=1}^{M} \widehat{e}_i(h)$$

respectively, the confidence interval for $\widehat{e}(h)$ can be computed by

$$[\widehat{e}(h) - \Delta(h), \widehat{e}(h) + \Delta(h)],$$

where

$$\Delta(h) = t_{1-\alpha/2,M-1}\sqrt{\frac{\widehat{\sigma}_e^2(h)}{M}}, \quad \widehat{\sigma}_e^2(h) = \frac{1}{M-1}\sum_{i=1}^{M} |\widehat{e}_i(h) - \widehat{e}(h)|^2,$$

and $t_{1-\alpha/2,M-1}$ denotes the $1-\alpha/2$ percentile of the Student's t distribution with $M-1$ degrees for the significance level $0 < \alpha < 1$.

Specifically, the simulations were arranged into $M = 20$ batches of $K = 100$ trajectories for each step size $h_i = 2^{-(i+3)}$, with $i = 1, \ldots, 6$. The significance level was taken $\alpha = 0.1$. Table 4.1 shows the estimated values of $e(h_i)$ and their respective 90% confidence interval.

Figure 4.2 shows the straight line fitted to the points $\{\log_2(\widehat{e}(h_i))\}_{i=1,\ldots,6}$. The estimated slope of these lines is $\widehat{\beta} = 0.9154 \pm 0.0272$. Note that this result corroborate the theoretical estimate $\beta = 2H = 0.90$ given by Theorem 3.2.

Figure 4.3 shows the comparison between the Euler scheme and the LL scheme for the step size $h = 2^{-5}$. Notice that the phase-space of the LL approximation is very similar to that of the true solution. By the other hand, as time increases, the approximation obtained by the Euler scheme tends to be very different from the actual solution.

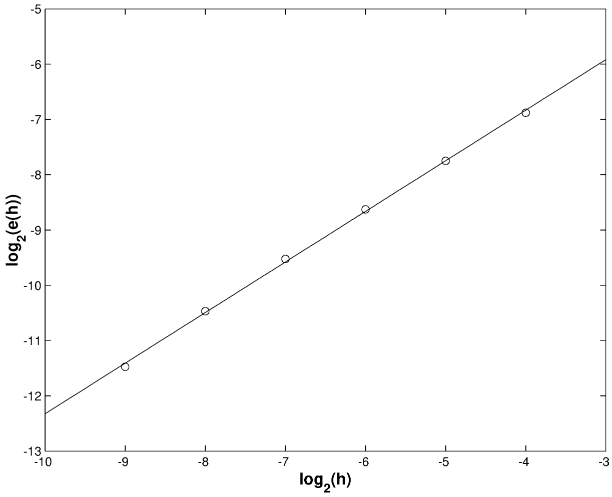

Figure 4.2: Estimated values of the errors $e(h)$ obtained from the application of the LL method in the integration of Example 2 with different step sizes h.

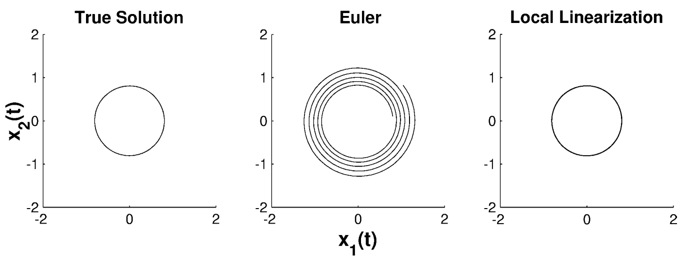

Figure 4.3: Phase portrait of trajectories obtained by the Euler method and the LL method in the integration of the Example 2 with step size $h = 2^{-5}$.

Equation (4.1) as well the equation in the next example are random versions of an ODE consider in [9] to study the dynamics of the LL scheme.

Example 3. Consider the RDE

$$\begin{aligned} \dot{x}_1(t) &= -x_2 + x_1(B(t)^2 - x_1^2 - x_2^2), \\ \dot{x}_2(t) &= x_1 + x_2(B(t)^2 - x_1^2 - x_2^2), \\ x_1(0) &= x_2(0) = 0.1 \end{aligned}$$

in the time interval $0 \leq t \leq 64$, where $B(t)$ is a standard Brownian motion. Figure 4.4 shows a comparison between the Euler scheme and the LL scheme for two different step sizes. Notice that the approximation provided by the Euler scheme does not reconstruct at all the actual dynamics of the systems. In fact the scheme explodes at time $t = 29$. Thus, the left top panel in the figure shows the Euler approximation only for $0 \leq t \leq 29$. In contrast, the LL scheme shows a well performance for both step sizes.

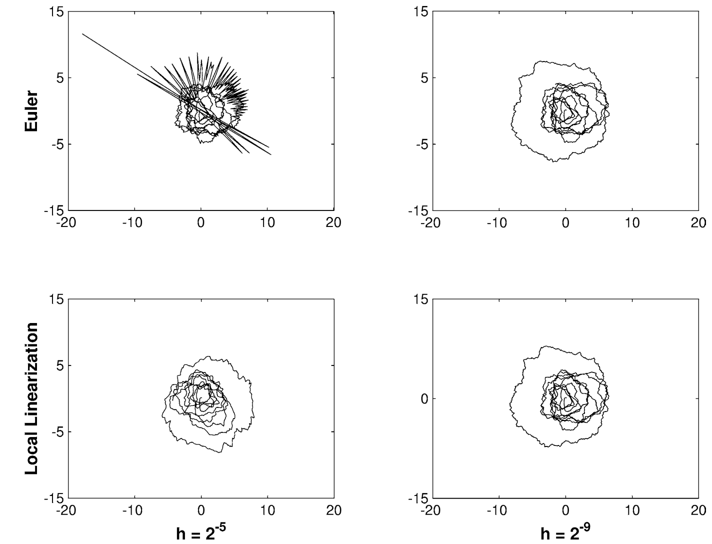

Figure 4.4: Phase portrait of trajectories obtained by the Euler method and the LL method in the integration of the Example 3 with step sizes $h = 2^{-5}$ and $h = 2^{-9}$.

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