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AN ERROR EXPANSION FOR SOME GAUSS–TURÁN QUADRATURES AND L^1-ESTIMATES OF THE REMAINDER TERM*

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Abstract.

Our aim in this paper is to obtain error expansions in the Gauss–Turán quadrature formula $\int_{-1}^{1} f(t)w(t)\,\mathrm{d}t = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R_{n,s}(f)$, in the case when f is an analytic function in some region of the complex plane containing the interval $[-1,1]$ in its interior. Using a representation of the remainder term $R_{n,s}(f)$ in the form of contour integral over confocal ellipses, we obtain $R_{n,1}(f)$ for the four Chebyshev weights and $R_{n,2}(f)$ for the Chebyshev weight of the first kind. Also, we get a few new L^1-estimates of the remainder term, which are stronger than the previous ones. Some numerical results, illustrations and comparisons are also given.

AMS subject classification (2000): 41A55, 65D30, 65D32.

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1 Introduction.

Let w be an integrable weight function on the interval $(-1,1)$. For approximating integrals of the form $I(f) = \int_{-1}^{1} f(t)w(t)\,\mathrm{d}t$ we use the Gauss–Turán quadrature formulas with multiple nodes

$$Q_{n,s}(f) = \sum_{\nu=1}^{n}\sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) \quad (n \in \mathbb{N};\ s \in \mathbb{N}_0), \tag{1.1}$$

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which are exact for all algebraic polynomials of degree at most $2(s+1)n-1$. The object of this paper is to obtain an expansion for the error

$$R_{n,s}(f) = I(f) - Q_{n,s}(f) \tag{1.2}$$

in the case when f is an analytic function in some region of the complex plane containing the interval $[-1,1]$ in its interior.

The nodes τ_ν in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the following integral $\Phi(a_0, a_1, \ldots, a_{n-1}) = \int_{-1}^{1} \pi_n(t)^{2s+2} w(t)\,\mathrm{d}t$, where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. In order to minimize Φ we must have

$$\int_{-1}^{1} \pi_n(t)^{2s+1} t^k w(t)\,\mathrm{d}t = 0, \quad k = 0, 1, \ldots, n-1.$$

These polynomials $\pi_n(t) = \pi_{n,s}(t)$ are known as s-orthogonal (or s-self associated) polynomials with respect to the weight w. For $s = 0$ we have the case of the standard orthogonal polynomials. For details and references about several classes of s-orthogonal polynomials, as well as their generalizations known as σ-orthogonal polynomials, and corresponding quadrature formulae with multiple nodes, see the survey paper [13], and some very recent papers [14, 17, 23].

2 The remainder term for analytic functions.

With $\pi_{n,s}(z)$ we denote a polynomial of degree n with positive leading coefficient which is s-orthogonal with respect to the weight function $w(t)$ over $(-1,1)$.

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and D be its interior. If the integrand f is an analytic function in D and continuous on $\overline{D}$, then we take as our starting point the well-known expression (cf. [19, 18, 15]) of the remainder term (1.2) in the form of the contour integral

$$R_{n,s}(f) = \frac{1}{2\pi i} \oint_\Gamma K_{n,s}(z) f(z)\,\mathrm{d}z. \tag{2.1}$$

The kernel is given by

$$K_{n,s}(z) = \frac{\varrho_{n,s}(z)}{[\pi_{n,s}(z)]^{2s+1}}, \quad z \notin [-1,1], \tag{2.2}$$

where

$$\varrho_{n,s}(z) = \int_{-1}^{1} \frac{[\pi_{n,s}(t)]^{2s+1}}{z-t}\, w(t)\,\mathrm{d}t, \quad n \in \mathbb{N}, \tag{2.3}$$

and $\pi_{n,s}(t)$ is the corresponding (not obligatory monic) s-orthogonal polynomial with respect to the measure $\mathrm{d}\lambda(t) = w(t)\,\mathrm{d}t$ on $(-1,1)$. For $s = 0$, (2.1) and (2.2) reduce to the corresponding formulas for the Gaussian quadratures.

The integral representation (2.1) leads to the error estimate

$$|R_{n,s}(f)| \le \frac{\ell(\Gamma)}{2\pi}\Big(\max_{z\in\Gamma}|K_{n,s}(z)|\Big)\Big(\max_{z\in\Gamma}|f(z)|\Big), \tag{2.4}$$

where $\ell(\Gamma)$ is the length of the contour Γ.

More general, if we apply the Hölder inequality to (2.1), we get

$$\begin{aligned} |R_{n,s}(f)| &= \frac{1}{2\pi}\left|\oint_\Gamma K_{n,s}(z)f(z)\,\mathrm{d}z\right| \\ &\le \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}(z)|^r|\mathrm{d}z|\right)^{1/r}\left(\oint_\Gamma |f(z)|^{r'}|\mathrm{d}z|\right)^{1/r'} \\ &= \frac{1}{2\pi}\|K_{n,s}\|_r\|f\|_{r'}, \end{aligned} \tag{2.5}$$

where $1 \le r \le +\infty$, $1/r + 1/r' = 1$, and

$$\|f\|_r := \begin{cases} \left(\oint_\Gamma |f(z)|^r|\mathrm{d}z|\right)^{1/r}, & 1 \le r < +\infty, \\ \max_{z\in\Gamma}|f(z)|, & r = +\infty. \end{cases}$$

In the case $r = +\infty$ ($r' = 1$), this estimate becomes

$$|R_{n,s}(f)| \le \frac{1}{2\pi}\Big(\max_{z\in\Gamma}|K_{n,s}(z)|\Big)\left(\oint_\Gamma |f(z)||\mathrm{d}z|\right). \tag{2.6}$$

Evidently, from (2.6) it follows the estimate (2.4), which was investigated in details in [15]. The case $s = 0$ was studied by Gautschi and Varga ([7, 8]).

On the other side for $r = 1$ ($r' = +\infty$), the estimate (2.5) reduces to

$$|R_{n,s}(f)| \le \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}(z)||\mathrm{d}z|\right)\Big(\max_{z\in\Gamma}|f(z)|\Big), \tag{2.7}$$

which is evidently stronger than (2.4), because of inequality

$$\oint_\Gamma |K_{n,s}(z)||\mathrm{d}z| \le \ell(\Gamma)\Big(\max_{z\in\Gamma}|K_{n,s}(z)|\Big).$$

The first approach in this sense for Gaussian quadrature rules ($s = 0$) and Chebyshev measures was given by Hunter [11].

Two choices of the contour Γ have been widely used: 1° a circle with center at origin and radius ϱ (> 1), and 2° an ellipse with foci at ± 1.

In this paper we take the contour Γ as an ellipse with foci at the points ± 1 and sum of semiaxes $\varrho > 1$,

$$E_\varrho = \left\{z \in \mathbb{C}\colon\ z = \frac{1}{2}\left(\varrho \mathrm{e}^{i\theta} + \varrho^{-1}\mathrm{e}^{-i\theta}\right),\ 0 \le \theta < 2\pi\right\}, \tag{2.8}$$

and consider the following four weight functions $w(t) = w_i(t)$:

(a) $w_1(t) = (1-t^2)^{-1/2}$, (b) $w_2(t) = (1-t^2)^{1/2+s}$,
(c) $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, (d) $w_4(t) = (1-t)^{1/2+s}(1+t)^{-1/2}$.

S. Bernstein [1] showed that the monic Chebyshev polynomial (orthogonal with respect to w_1) $\hat{T}_n(t) = T_n(t)/2^{n-1}$ minimizes all integrals of the form

$$\int_{-1}^{1} \frac{|\pi_n(t)|^{k+1}}{\sqrt{1-t^2}}\,\mathrm{d}t, \quad k \geq 0.$$

Thus, the Chebyshev polynomials T_n are s-orthogonal on $[-1,1]$ for each $s \geq 0$. Ossicini and Rosatti [19] found three other weights $w_i(t)$, $i = 2,3,4$, for which the s-orthogonal polynomials can be identified as Chebyshev polynomials of the second, third, and fourth kind: U_n, V_n, and W_n, which are defined by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad V_n(t) = \frac{\cos(n+1/2)\theta}{\cos\theta/2}, \qquad W_n(t) = \frac{\sin(n+1/2)\theta}{\sin\theta/2},$$

respectively (cf. [5]), where $t = \cos\theta$. However, such weights depend on s (see (b), (c), (d)). Notice that the weight function in (d) can be omitted from investigation because of $W_n(-t) = (-1)^n V_n(t)$.

Following [7] and [8] we studied in [15] the magnitude of $|K_{n,s}(z)|$ on the contour E_ϱ. Precisely, for the weight functions $w_k(t)$ $(k = 1,2,3)$ we investigated the locations on the confocal ellipses (2.8) where the modulus of the corresponding kernels attain their maximum values.

The cases of Gaussian rules with Bernstein–Szegő weight functions and with some symmetric weights including especially the Gegenbauer weight were studied by Peherstorfer [20] and Schira [22], respectively. Some of the results have been extended to Gauss–Radau and Gauss–Lobatto formulas (cf. Gautschi [5], Gautschi and Li [6], Schira [21], Hunter and Nikolov [12], Milovanović and Spalević [16]).

In this paper we consider error expansions and error estimates for Gauss–Turán quadrature formulae (1.2) for weight functions (a)–(d), based on elliptical contours. The paper is organized as follows. In Section 3 we adapt Hunter's approach [11] for Gaussian quadratures in order to obtain error expansions for Gauss–Turán quadrature formulae, and then in Section 4 we obtain a few new estimates of the remainder term (1.2). In particular, we concentrate our attention on the weight function $w_1(t)$ and obtain some very exact estimates of the remainder term. Some of them are the smallest, including those from [15]. In Sections 5 and 6 we study estimates of the remainder term (1.2) for the generalized Chebyshev weights $w_2(t)$ and $w_3(t)$ (therefore, and for $w_4(t)$), respectively.

3 An error expansion for Gauss–Turán quadrature formulae.

If f is an analytic function in the interior of E_ϱ, then it has the expansion

$$f(z) = \sum_{k=0}^{+\infty}{}' \alpha_k T_k(z), \tag{3.1}$$

where $\alpha_k = (1/\pi)\int_{-1}^{1}(1-t^2)^{-1/2} f(t) T_k(t)\,\mathrm{d}t$, which converges for all z in the interior of E_ϱ. The prim in (3.1) denotes that the first term of the sum is taken with factor 1/2. In terms of $\xi = \varrho \mathrm{e}^{i\theta}$ ($\varrho > 1$), $z = (\xi + \xi^{-1})/2$, $T_k(z)$ is given by

$$T_k(z) = \frac{1}{2}(\xi^k + \xi^{-k}). \tag{3.2}$$

In the sequel we need two auxiliary results (see [11]).

Lemma 3.1. *If $z \notin [-1,1]$, then $1/\pi_{n,s}(z) = \sum_{k=0}^{+\infty} \beta_{n,k}^{(s)} \xi^{-n-k}$. Furthermore, if w is an even function then $\beta_{n,2j+1} = 0$ $(j = 0,1,2,\ldots)$.*

Proof. The zeros of $\pi_{n,s}(z)$ are real, distinct, and all contained in the open interval $(-1,1)$ (cf. [9]). Then, the proof of this statement is the same as the one of Lemma 3 in [11]. □

Now, it is not difficult to see that (cf. [10, Eq. 0.314])

$$\frac{1}{[\pi_{n,s}(z)]^{2s+1}} = \sum_{k=0}^{+\infty} \overline{\beta}_{n,k}^{(s)}\, \xi^{-n(2s+1)-k}, \quad \xi = \varrho \mathrm{e}^{i\theta},\ \varrho > 1, \tag{3.3}$$

where

$$\overline{\beta}_{n,0}^{(s)} = \bigl(\beta_{n,0}^{(s)}\bigr)^{2s+1}, \qquad \overline{\beta}_{n,m}^{(s)} = \frac{1}{m\beta_{n,0}^{(s)}} \sum_{k=1}^{m} (k(2s+1)-m+k)\beta_{n,k}^{(s)}\, \overline{\beta}_{n,m-k}^{(s)}, \quad m \geq 1.$$

In particular, if $w(-t) = w(t)$ then

$$\frac{1}{[\pi_{n,s}(z)]^{2s+1}} = \sum_{k=0}^{+\infty} \overline{\beta}_{n,2k}^{(s)}\, \xi^{-n(2s+1)-2k}, \quad \xi = \varrho \mathrm{e}^{i\theta},\ \varrho > 1.$$

Lemma 3.2. *If $z \notin [-1,1]$, $\varrho_{n,s}(z)$ can be expanded as*

$$\varrho_{n,s}(z) = \sum_{k=0}^{+\infty} \overline{\gamma}_{n,k}^{(s)}\, \xi^{-n-k-1}. \tag{3.4}$$

Furthermore, if w is an even function, then $\overline{\gamma}_{n,2j+1}^{(s)} = 0$ $(j = 0,1,\ldots)$.

PROOF. It is well-known that if $w(t)$ is a weight function then $D_{n,s}(t)$, defined by $D_{n,s}(t) = [\pi_{n,s}(t)]^{2s}\, w(t)$ is also a weight function (see [4, pp. 214–226]). The proof of (3.4) can be given in a similar way as the one of Lemma 4 in [11]. Namely, from (2.3) we have

$$\varrho_{n,s}(z) = \int_{-1}^{1} D_{n,s}(t) \frac{\pi_{n,s}(t)}{z-t}\,\mathrm{d}t = \sum_{k=0}^{+\infty} \overline{\gamma}_{n,k}^{(s)}\, \xi^{-n-k-1},$$

where

$$\overline{\gamma}_{n,k}^{(s)} = 2\int_{-1}^{1} w(t)[\pi_{n,s}(t)]^{2s+1}\, U_{n+k}(t)\,\mathrm{d}t \quad (k=0,1,\ldots). \tag{3.5}$$

If $w(-t) = w(t)$, then for each odd k the integrand in (3.5) is odd, and therefore $\overline{\gamma}_{n,k}^{(s)} = 0$. □

Now, by the substitution (3.3) and (3.4) in (2.2) we obtain

$$K_{n,s}(z) = \sum_{k=0}^{+\infty} \omega_{n,k}^{(s)}\, \xi^{-2n(s+1)-k-1}, \tag{3.6}$$

where

$$\omega_{n,k}^{(s)} = \sum_{j=0}^{k} \overline{\beta}_{n,j}^{(s)}\, \overline{\gamma}_{n,k-j}^{(s)}. \tag{3.7}$$

THEOREM 3.3. *The remainder term $R_{n,s}(f)$ can be represented in the form*

$$R_{n,s}(f) = \sum_{k=0}^{+\infty} \alpha_{2n(s+1)+k}\, \varepsilon_{n,k}^{(s)}, \tag{3.8}$$

where the coefficients $\varepsilon_{n,k}^{(s)}$ are independent of f. Furthermore, if f is an even function then $\varepsilon_{n,2j+1}^{(s)} = 0$ $(j=0,1,\ldots)$.

PROOF. By substitution (3.1) and (3.6) in (2.1) we obtain

$$\begin{aligned} R_{n,s}(f) &= \frac{1}{2\pi i}\int_{E_\varrho}\left(\sum_{j=0}^{+\infty}{}' \alpha_j T_j(z) \sum_{k=0}^{+\infty} \omega_{n,k}^{(s)}\, \xi^{-2n(s+1)-k-1}\right)\mathrm{d}z \\ &= \sum_{k=0}^{+\infty}\left(\frac{1}{2\pi i}\sum_{j=0}^{+\infty}{}' \alpha_j \int_{E_\varrho} T_j(z)\,\xi^{-2n(s+1)-k-1}\,\mathrm{d}z\right)\omega_{n,k}^{(s)}. \end{aligned}$$

Applying Lemma 5 from [11], this reduces to (3.8), with

$$\varepsilon_{n,0}^{(s)} = \frac{1}{4}\omega_{n,0}^{(s)}, \qquad \varepsilon_{n,1}^{(s)} = \frac{1}{4}\omega_{n,1}^{(s)}, \qquad \varepsilon_{n,k}^{(s)} = \frac{1}{4}\big(\omega_{n,k}^{(s)} - \omega_{n,k-2}^{(s)}\big), \quad k=2,3\ldots. \tag{3.9}$$

If $w(-t) = w(t)$ and k is odd it follows from (3.7) and Lemmas 3.1 and 3.2 that $\omega_{n,k}^{(s)} = 0$ and hence $\varepsilon_{n,k}^{(s)} = 0$. □

REMARK 3.1. Setting $f(z) = T_{2n(s+1)+k}(z)$, it immediately follows from (3.8) that

$$\varepsilon_{n,k}^{(s)} = \sigma_{2n(s+1)+k} - \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} T_{2n(s+1)+k}^{(i)}(\tau_\nu) \quad (k = 0, 1, 2, \ldots),$$

where $\sigma_k = \int_{-1}^{1} w(t) T_k(t)\,\mathrm{d}t$ $(k = 0, 1, 2, \ldots)$. Therefore, we conclude that

$$|\varepsilon_{n,k}^{(s)}| \le \int_{-1}^{1} w(t)\,\mathrm{d}t + \sum_{\nu=1}^{n} \sum_{i=0}^{2s} |A_{i,\nu}|\,|T_{2n(s+1)+k}^{(i)}(\tau_\nu)| \quad (k = 0, 1, 2, \ldots).$$

If $s = 0$ then $|\varepsilon_{n,k}^{(0)}| \le 2\int_{-1}^{1} w(t)\,\mathrm{d}t$, and this fact can be used to obtain some global upper bounds of the remainder term (see Hunter [11]). Unfortunately, such a conclusion cannot be made in the general case for $s > 0$, because of the difficulties in finding sharp upper bounds on $|T_{2n(s+1)+k}^{(i)}(\tau_\nu)|$.

4 Error estimates for Gauss–Turán quadrature with Chebyshev weight function of the first kind.

If $u \in \mathbb{C}$, $|u| < 1$, then

$$\frac{1}{(1-u)^{\nu+1}} = \sum_{k=\nu}^{+\infty} \binom{k}{\nu} u^{k-\nu} \quad (\nu = 0, 1, 2, \ldots). \tag{4.1}$$

In this section we consider the weight $w(t) = w_1(t)$, for which $\pi_{n,s}(t) = T_n(t)$. Using (3.2), with $\xi = \varrho e^{i\theta}$, $\varrho > 1$, $z = (\xi + \xi^{-1})/2$, and (4.1), we obtain

$$\begin{aligned} \frac{1}{[T_n(z)]^{2s+1}} &= \left[\frac{1}{2}(\xi^n + \xi^{-n})\right]^{-(2s+1)} = 2^{2s+1}\,\xi^{-n(2s+1)} \left(\frac{1}{1+\xi^{-2n}}\right)^{2s+1} \\ &= 2^{2s+1} \sum_{j=0}^{+\infty} (-1)^j \binom{j+2s}{2s} \xi^{-n(2s+1)-2nj}. \end{aligned}$$

On the other hand, according to (3.3), with $\pi_{n,s}(t) = T_n(t)$, we conclude that

$$\overline{\beta}_{n,k}^{(s)} = \begin{cases} 2^{2s+1}(-1)^j \dbinom{j+2s}{2s}, & k = 2jn \ (j = 0, 1, 2, \ldots), \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

According to (3.5), the coefficients in (3.4) are given by

$$\overline{\gamma}_{n,k}^{(s)} = 2\int_{-1}^{1} \frac{[T_n(t)]^{2s+1}}{\sqrt{1-t^2}} U_{n+k}(t)\,\mathrm{d}t = 2\int_{0}^{\pi} [\cos n\theta]^{2s+1} \frac{\sin{(n+k+1)\theta}}{\sin\theta}\,\mathrm{d}\theta.$$

In order to calculate this integral we use formulas 1.320.5 and 1.320.7 in [10] and combine them with

$$\frac{\sin(m+1)x}{\sin x} = 2 \sum_{k=0}^{[m/2]}{}'' \cos(m-2k)x,$$

where the "second sign" denotes that the last summand has to be halved if m is even. In that way we obtain the coefficients $\overline{\gamma}_{n,k}^{(s)}$ in an explicit form

$$(4.3)\quad \overline{\gamma}_{n,k}^{(s)} = \begin{cases} \dfrac{\pi}{2^{2s-1}} \displaystyle\sum_{\nu=0}^{j} \binom{2s+1}{s-\nu}, & k = 2nj, 2nj+2, \dots, 2n(j+1)-2 \\ & \quad (j = 0, 1, \dots, s-1), \\ 2\pi, & k = 2sn, 2sn+2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 4.1. From (4.3) we conclude that $\overline{\gamma}_{n,k}^{(s)} > 0$ for each even k, as well as

$$\frac{\pi}{2^{2s-1}} \binom{2s+1}{s} \le \overline{\gamma}_{n,k}^{(s)} \le 2\pi,$$

because of $\sum_{\ell=0}^{s} \binom{2s+1}{s-\ell} = 2^{2s}$ (cf. [15]).

4.1 First type of error estimates.

In general, the Chebyshev coefficients α_k in (3.1) are unknown. However, Elliot [3] described a number of ways of estimating or bounding them. In particular, under our assumptions,

$$(4.4)\qquad |\alpha_k| \le \frac{2}{\varrho^k} \Big(\max_{z \in E_\varrho} |f(z)| \Big).$$

In order to present this type of the error estimate, we take $s = 2$. By using (4.2), (4.3), (3.7), and (3.9), we find $\varepsilon_{n,0}^{(2)} = 10\pi$, $\varepsilon_{n,2n}^{(2)} = -45\pi$,

$$\varepsilon_{n,k}^{(2)} = \frac{1}{4}(-1)^j \pi (j^4 + 12j^3 + 49j^2 + 78j + 40) \quad (k = 2jn),$$

and $\varepsilon_{n,k}^{(2)} = 0$ otherwise. Now, by using these results, (3.8) and (4.4), we get

$$\begin{aligned} |R_{n,2}(f)| &= \left| \sum_{k=0}^{+\infty} \alpha_{6n+k}\, \varepsilon_{n,k}^{(2)} \right| = \left| \sum_{j=0}^{+\infty} \alpha_{6n+2jn}\, \varepsilon_{n,2jn}^{(2)} \right| \\ &\le \frac{\pi}{2\varrho^{6n}} \Big(\max_{z \in E_\varrho} |f(z)| \Big) \sum_{j=0}^{+\infty} \frac{j^4 + 12j^3 + 49j^2 + 78j + 40}{\varrho^{2jn}}. \end{aligned}$$

In order to sum the series on the right-hand side in this estimate we need some explicit formulas for the functions $h_k(t) := \sum_{n=1}^{+\infty} n^k t^{n-1}$ ($|t| < 1$). It is easy to prove that for $k \geq 1$, the following recurrence relations are valid

$$\begin{aligned} h_k(t) &= h_{k-1}(t) + t h_{k-1}'(t), \qquad h_k(t) = \frac{1}{1-t}\left[1 + t\sum_{i=0}^{k-1}\binom{k}{i} h_i(t)\right], \\ h_0(t) &= \frac{1}{1-t}. \end{aligned}$$

The previous sums $\sum_{j=0}^{+\infty} j^\nu \varrho^{-2jn}$ ($\nu = 0, 1, 2, 3, 4$) can be calculated by using the expressions for $h_\nu(t)$, putting $t = 1/\varrho^{2n}$, so that we obtain

$$|R_{n,2}(f)| \leq 2\pi\Big(\max_{z\in E_\varrho} |f(z)|\Big) \frac{10\varrho^{4n} - 5\varrho^{2n} + 1}{(\varrho^{2n} - 1)^5}. \tag{4.5}$$

The corresponding result for $s = 1$ is

$$|R_{n,1}(f)| \leq 2\pi\Big(\max_{z\in E_\varrho} |f(z)|\Big) \frac{3\varrho^{2n} - 1}{(\varrho^{2n} - 1)^3}. \tag{4.6}$$

The error estimate for $s = 0$ has been obtained by Hunter [11] (see also Chawla and Jain [2]):

$$|R_{n,0}(f)| \leq 2\pi\Big(\max_{z\in E_\varrho} |f(z)|\Big) \frac{1}{\varrho^{2n} - 1}. \tag{4.7}$$

Remark 4.2. The error estimates (4.5), (4.6), and Hunter's result (4.7) suggest that for a general s ($s \in \mathbb{N}_0$) the estimate could be expressed in the form

$$|R_{n,s}(f)| \leq 2\pi\Big(\max_{z\in E_\varrho} |f(z)|\Big) \frac{\sum_{k=0}^{s}(-1)^k\binom{2s+1}{s-k}\varrho^{2n(s-k)}}{(\varrho^{2n} - 1)^{2s+1}}.$$

4.2 Second type of error estimates.

According to (2.7) we study now the quantity $L_{n,s}(E_\varrho) = \frac{1}{2\pi}\oint_{E_\varrho} |K_{n,s}(z)|\,|\mathrm{d}z|$, where $K_{n,s}(z)$ is given by (2.2). Since $z = \frac{1}{2}(\xi + \xi^{-1})$, $\xi = \varrho \mathrm{e}^{i\theta}$, and $|\mathrm{d}z| = 2^{-1/2}\sqrt{a_2 - \cos 2\theta}\,\mathrm{d}\theta$, where we put

$$a_j = a_j(\varrho) = \frac{1}{2}(\varrho^j + \varrho^{-j}), \quad j \in \mathbb{N},\ \varrho > 1, \tag{4.8}$$

the quantity $L_{n,s}(E_\varrho)$ reduces to

$$L_{n,s}(E_\varrho) = \frac{1}{2\pi\sqrt{2}} \int_0^{2\pi} \frac{|\varrho_{n,s}(z)|(a_2 - \cos 2\theta)^{1/2}}{|\pi_{n,s}(z)|^{2s+1}}\,\mathrm{d}\theta. \tag{4.9}$$

This integral can be evaluated numerically by using a quadrature formula. However, if $w(t) = w_1(t)$ we can obtain explicit expressions for $L_{n,s}(E_\varrho)$ or for their bounds. By using (3.4) and (4.3), after some computation, we find

$$\varrho_{n,s}(z) = \frac{\pi}{2^{2s-1}}\,\frac{1}{\xi^n(\xi-\xi^{-1})}\sum_{\nu=0}^{s}\binom{2s+1}{s-\nu}\frac{1}{\xi^{2\nu n}}, \tag{4.10}$$

and, according to (3.2) and (4.8), we have $|T_n(z)| = 2^{-1/2}\sqrt{a_{2n}+\cos 2n\theta}$, where $z = \frac{1}{2}(\xi+\xi^{-1})$ and $\xi = \varrho e^{i\theta}$. For another approach to getting $K_{n,s}(z) = \varrho_{n,s}(z)/\pi_{n,s}(z)^{2s+1}$ in this Chebyshev case, see [15, §3.1].

Thus, in this Chebyshev case, (4.9) reduces to

$$L_{n,s}(E_\varrho) = \frac{2^{s+1}}{\pi}\int_0^{\pi/2}\frac{|\varrho_{n,s}(z)|(a_2-\cos 2\theta)^{1/2}}{(a_{2n}+\cos 2n\theta)^{s+1/2}}\,d\theta, \tag{4.11}$$

where $\varrho_{n,s}(z)$ is given by (4.10). Now, we define

$$W_s(\varrho,\theta) := \sum_{\nu=0}^{s}\binom{2s+1}{\nu}\varrho^{2\nu-s}e^{i(\nu-s/2)\theta} \tag{4.12}$$

and prove the following auxiliary result:

Lemma 4.1. *Let $r>0$, $\varrho>1$ and $x=\varrho^{4r}$. Then*

$$|W_s(\varrho^r,\theta)|^2 = \sum_{k=0}^{s}A_k\cos k\theta, \tag{4.13}$$

where

$$A_0 = \frac{1}{x^{s/2}}\sum_{\nu=0}^{s}\binom{2s+1}{\nu}^2 x^\nu, \tag{4.14}$$

$$A_k = \frac{2}{x^{(s-k)/2}}\sum_{\nu=0}^{s-k}\binom{2s+1}{\nu}\binom{2s+1}{\nu+k}x^\nu,\quad k=1,\dots,s. \tag{4.15}$$

Proof. Since $|W_s(\varrho^r,\theta)|^2 = W_s(\varrho^r,\theta)W_s(\varrho^r,-\theta)$, i.e.,

$$|W_s(\varrho^r,\theta)|^2 = \sum_{\nu=0}^{s}\sum_{\mu=0}^{s}\binom{2s+1}{\nu}\binom{2s+1}{\mu}\varrho^{2(\nu+\mu-s)r}e^{i(\nu-\mu)\theta},$$

we get (4.13), where

$$A_k = \sum_{\substack{|\nu-\mu|=k\\ \nu,\mu=0,1,\dots,s}}\binom{2s+1}{\nu}\binom{2s+1}{\mu}\varrho^{2(\nu+\mu-s)r},\quad k=0,1,\dots,s.$$

For $k=0$ and $k\ge 1$ these coefficients reduce to (4.14) and (4.15), respectively. □

We also need the following integral

$$J_k(a) = \int_0^{\pi} \frac{\cos k\theta}{(a+\cos\theta)^{2s+1}}\,\mathrm{d}\theta, \quad a>1,\ s\in\mathbb{N}_0.$$

LEMMA 4.2. *Let $x>1$, $a=(x+1)/(2\sqrt{x})$, and $s\in\mathbb{N}_0$. Then*

$$J_k(a) = \frac{2^{2s+1}\pi(-1)^k x^{s-(k-1)/2}}{(x-1)^{4s+1}} \sum_{\nu=0}^{2s} \binom{2s+\nu}{\nu}\binom{2s+k}{k+\nu}(x-1)^{2s-\nu}. \tag{4.16}$$

This result can be found in the book [10, Eq. 3.616.7].

Now, we are ready to prove the following result:

THEOREM 4.3. *Let $x=\varrho^{4n}$ and a_j, A_0 and A_k be defined by* (4.8), (4.14) *and* (4.15)*, respectively. Then, for the Chebyshev weight of the first kind, we have*

$$L_{n,s}(E_\varrho) = \frac{1}{2^{s-1/2}\varrho^{(s+1)n}} \int_0^{\pi} \sqrt{\frac{\sum_{k=0}^{s} A_k \cos k\theta}{(a_{2n}+\cos\theta)^{2s+1}}}\,\mathrm{d}\theta. \tag{4.17}$$

Moreover, an estimate of the form

$$L_{n,s}(E_\varrho) \le 2\pi\Phi_s(\varrho^{4n}), \quad \Phi_s(x) = \sqrt{\frac{Q_s(x)}{(x-1)^{4s+1}}}, \tag{4.18}$$

holds, where $Q_s(x)$ is an algebraic polynomial of degree $3s$, defined by

$$\begin{aligned} Q_s(x) \ :=\ & 2\sum_{k=0}^{s}{}'(-1)^k \left(\sum_{\nu=0}^{s-k}\binom{2s+1}{\nu}\binom{2s+1}{\nu+k}x^\nu\right)\times \\ & \times\left(\sum_{\nu=0}^{2s}\binom{2s+\nu}{\nu}\binom{2s+k}{k+\nu}(x-1)^{2s-\nu}\right). \end{aligned} \tag{4.19}$$

PROOF. Let $x=\varrho^{4n}$. According to (4.10) and (4.8) we have

$$\begin{aligned} |\varrho_{n,s}(z)| \ &=\ \frac{2^{1-2s}\pi}{\varrho^{(s+1)n}\sqrt{2}(a_2-\cos 2\theta)^{1/2}} \left|\sum_{\nu=0}^{s}\binom{2s+1}{\nu}\varrho^{(2\nu-s)n}\mathrm{e}^{i(\nu-s/2)2n\theta}\right| \\ &=\ \frac{2^{1/2-2s}\pi}{\varrho^{(s+1)n}(a_2-\cos 2\theta)^{1/2}}\,|W_s(\varrho^n,2n\theta)|, \end{aligned}$$

where W_s is defined in (4.12). Then (4.11) becomes

$$L_{n,s}(E_\varrho) = \frac{1}{2^{s-3/2}\varrho^{(s+1)n}} \int_0^{\pi/2} \frac{|W_s(\varrho^n,2n\theta)|}{(a_{2n}+\cos 2n\theta)^{s+1/2}}\,\mathrm{d}\theta.$$

Because of the periodicity of the integrand, it reduces to

$$L_{n,s}(E_\varrho) = \frac{1}{2^{s-1/2}\varrho^{(s+1)n}} \int_0^{\pi} \frac{|W_s(\varrho^n,\theta)|}{(a_{2n}+\cos\theta)^{s+1/2}}\,\mathrm{d}\theta,$$

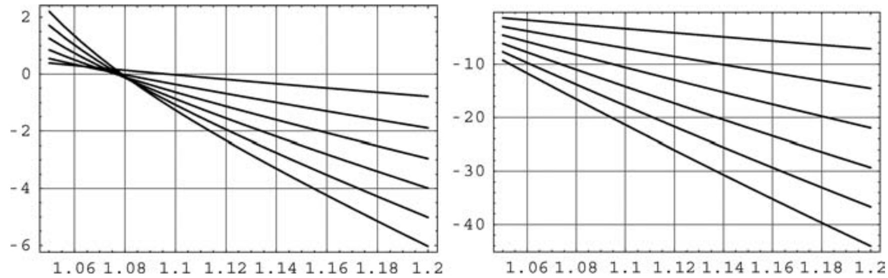

Figure 4.1: Log_{10} of the values $L_{n,s}(E_\varrho)$, $s = 0, 1, \ldots, 5$, as functions of ϱ, for $n = 10$ (left) and $n = 50$ (right).

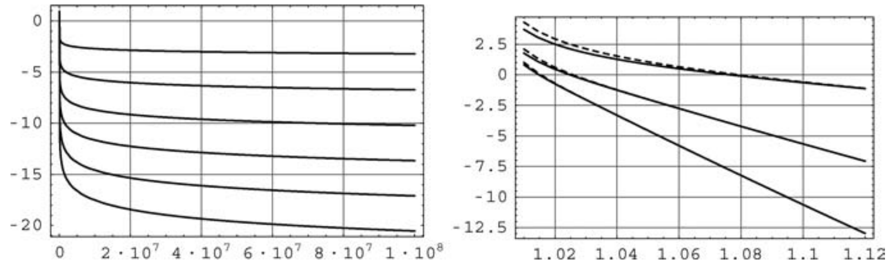

Figure 4.2: The function $x \mapsto \log_{10}(2\pi\Phi_s(x))$ for $s = 0, 1, \ldots, 5$ (left) and Log_{10} of the values $L_{n,2}(E_\varrho)$ (solid lines) and their bounds given by (4.18) (dashed lines) for $n = 10, 30$, and 50 (right).

and then to (4.17), because of (4.13). Applying Cauchy's inequality to (4.17) we obtain

$$L_{n,s}(E_\varrho) \le \frac{\sqrt{\pi}}{2^{s-1/2}\varrho^{(s+1)n}} \left(\int_0^\pi \frac{\sum_{k=0}^s A_k \cos k\theta}{(a_{2n} + \cos\theta)^{2s+1}} \, \mathrm{d}\theta \right)^{1/2}. \tag{4.20}$$

Since $a_{2n} = (\varrho^{2n} + \varrho^{-2n})/2 = (x+1)/(2\sqrt{x})$, using (4.14), (4.15) and (4.16) we obtain

$$\int_0^\pi \frac{\sum_{k=0}^s A_k \cos k\theta}{(a_{2n} + \cos\theta)^{2s+1}} \, \mathrm{d}\theta = \sum_{k=0}^s A_k J_k(a_{2n}) = \frac{2^{2s+1}\pi x^{(s+1)/2}}{(x-1)^{4s+1}} \, Q_s(x),$$

where $Q_s(x)$ is given by (4.19). Note that $\deg Q_s(x) = 3s$. Finally, (4.20) reduces to $L_{n,s}(E_\varrho) \le 2\pi\sqrt{Q_s(x)/(x-1)^{4s+1}}$, where $x = \varrho^{4n}$. □

Remark 4.3. The polynomials $Q_s(x)$ in (4.19) are

$$\begin{aligned} Q_0(x) &= 1, \quad Q_1(x) = 1 - 5x + 19x^2 + 9x^3, \\ Q_2(x) &= 1 - 9x + 36x^2 + 16x^3 + 1251x^4 + 1125x^5 + 100x^6, \\ Q_3(x) &= 1 - 13x + 78x^2 - 286x^3 + 1940x^4 + 32964x^5 + 150578x^6 + \\ &\quad + 148862x^7 + 34251x^8 + 1225x^9, \quad \text{etc.} \end{aligned}$$

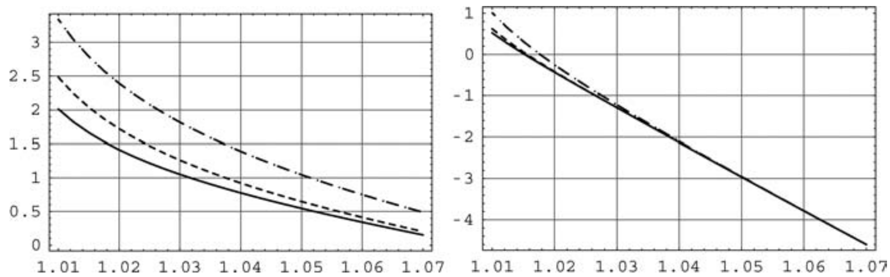

Figure 4.3: Log_{10} of the values $L_{n,1}(E_\varrho)$ (solid line) and its bounds given by (4.6) (dot-dashed line) and (4.18) (dashed line) for $n = 10$ (left) and $n = 50$ (right).

In Figure 4.1 we presented the values of $\log_{10}\big(L_{n,s}(E_\varrho)\big)$, $s = 0, 1, \ldots, 5$, as a function of ϱ, when $n = 10$ and $n = 50$. The upper graphs (for sufficiently large ϱ) correspond to the smaller values of s. The values of $L_{n,s}(E_\varrho)$ were calculated by using (4.17). In Figure 4.2 (left) we presented graphs $x \mapsto \log_{10}(2\pi\Phi_s(x))$ for $s = 0, 1, \ldots, 5$, where $\Phi_s(x)$ is given in (4.18)–(4.19). The graphs $\varrho \mapsto \log_{10}(L_{n,s}(E_\varrho))$ and $\varrho \mapsto \log_{10}(2\pi\Phi_s(\varrho^{4n}))$ for $n = 10, 30, 50$ and $s = 2$ are also displayed in Figure 4.2 (right). The upper graphs correspond to the smaller values of n.

The function $\varrho \mapsto \log_{10}(L_{n,1}(E_\varrho))$, as well as its bounds which appear on the right sides in (4.6) and (4.18), are given in Figure 4.3. As we can see, the second bound (4.18) is very precise especially for larger values of n and ϱ.

Remark 4.4. For $s = 0$ the estimate (4.17) reduces to the corresponding error estimate for Gaussian quadrature obtained by Hunter (see [11, Eq. (5.7)]), which can be expressed in the form $L_{n,0}(E_\varrho) = 4(\varrho^{2n} + 1)^{-1} K(2/(\varrho^n + \varrho^{-n}))$, where $K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-1/2}\, \mathrm{d}\theta$ ($|k| < 1$) is the complete elliptic integral of the first kind.

5 Error estimates for Gauss–Turán quadratures with a generalized Chebyshev weight of the second kind.

In this section we consider a case with the weight function $w_2(t) = (1-t^2)^{1/2+s}$ (the generalized Chebyshev weight of the second kind), for which $\pi_{n,s}(z) = U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi - \xi^{-1})$. Here, we could not find a general pattern for the coefficients $\overline{\beta}_{n,k}^{(s)}$ as in the case of the weight function $w_1(t)$. However, we will show how to find $\overline{\beta}_{n,k}^{(s)}$ for $s = 1$. The cases for $s > 1$ are more complicated.

Taking $s = 1$, we have

$$\frac{1}{U_n(z)^3} = \left(\frac{\xi - \xi^{-1}}{\xi^{n+1} - \xi^{-n-1}}\right)^3 = \xi^{-3n}\left(1 - \frac{3}{\xi^2} + \frac{3}{\xi^4} - \frac{1}{\xi^6}\right)\left(1 - \frac{1}{\xi^{2(n+1)}}\right)^{-3}$$
$$= \xi^{-3n}\left(1 - \frac{3}{\xi^2} + \frac{3}{\xi^4} - \frac{1}{\xi^6}\right)\sum_{k=0}^{+\infty}\binom{k+2}{k}\frac{1}{\xi^{2(n+1)k}}.$$

For $n \geq 3$, it is not difficult to conclude that $\overline{\beta}^{(1)}_{n,2j(n+1)} = -\overline{\beta}^{(1)}_{n,2j(n+1)+6} = \binom{j+2}{j}$, $\overline{\beta}^{(1)}_{n,2j(n+1)+2} = -\overline{\beta}^{(1)}_{n,2j(n+1)+4} = -3\binom{j+2}{j}$, for $j = 0, 1, \ldots$, and $\overline{\beta}^{(1)}_{n,k} = 0$ otherwise. For $n = 2$ we obtain $\overline{\beta}^{(1)}_{2,6j} = \binom{j+2}{j} - \binom{j+1}{j-1} = j + 1$ and $\overline{\beta}^{(1)}_{2,6j+2} = -\overline{\beta}^{(1)}_{2,6j+4} = -3\binom{j+2}{j}$, for $j = 0, 1, \ldots$, and $\overline{\beta}^{(1)}_{2,k} = 0$ otherwise. Finally, for $n = 1$ we have that $\overline{\beta}^{(1)}_{1,2j} = (-1)^j\binom{j+2}{j}$ for $j = 0, 1, \ldots$, and $\overline{\beta}^{(1)}_{1,k} = 0$ otherwise. In the case $n = 3$, we have $\varepsilon^{(1)}_{3,8j} = \frac{\pi}{16}(2j^2 + 6j + 3)$, $\varepsilon^{(1)}_{3,8j+4} = \frac{\pi}{8}(3j^2 + 13j + 9)$, $\varepsilon^{(1)}_{3,8j+2} = \varepsilon^{(1)}_{3,8j+6} = -\frac{\pi}{4}(j^2 + 4j + 3)$, for $j = 0, 1, \ldots$, and $\varepsilon^{(1)}_{n,k} = 0$ otherwise.

For $n > 3$ and $j = 0, 1, \ldots$, we get

$$\begin{aligned}
\omega^{(1)}_{n,2j(n+1)} &= -\omega^{(1)}_{n,2j(n+1)+6} = \frac{3\pi}{4}\binom{j+2}{j} - \frac{\pi}{4}\binom{j+1}{j-1} = \frac{\pi}{4}(j^2 + 4j + 3),\\
\omega^{(1)}_{n,2j(n+1)+2} &= -\omega^{(1)}_{n,2j(n+1)+4} = -3\omega^{(1)}_{n,2j(n+1)} = -\frac{3\pi}{4}(j^2 + 4j + 3),
\end{aligned}$$

and $\omega^{(1)}_{n,k} = 0$ otherwise. Then, the formulas (3.9) become

$$\begin{cases}
\varepsilon^{(1)}_{n,2j(n+1)} = \varepsilon^{(1)}_{n,2j(n+1)+8} = \dfrac{\pi}{16}(j^2 + 4j + 3),\\
\varepsilon^{(1)}_{n,2j(n+1)+2} = \varepsilon^{(1)}_{n,2j(n+1)+6} = -\dfrac{\pi}{4}(j^2 + 4j + 3),\\
\varepsilon^{(1)}_{n,2j(n+1)+4} = \dfrac{3\pi}{8}(j^2 + 4j + 3),
\end{cases}$$

for $j = 0, 1, \ldots$, and $\varepsilon^{(1)}_{n,k} = 0$ otherwise.

In the sequel we derive the first type of error estimates for $n > 3$. Thus,

$$\begin{aligned}
|R_{n,1}(f)| &= \left|\sum_{k=0}^{+\infty} \alpha_{4n+k}\, \varepsilon^{(1)}_{n,k}\right| = \left|\sum_{\nu=0}^{4}\sum_{j=0}^{+\infty} \alpha_{4n+2j(n+1)+2\nu}\, \varepsilon^{(1)}_{n,2j(n+1)+2\nu}\right|\\
&\leq \frac{\pi(\max_{z\in E_\varrho}|f(z)|)}{2\varrho^{4n}}\left(\frac{1}{4} + \frac{1}{\varrho^2} + \frac{3}{2\varrho^4} + \frac{1}{\varrho^6} + \frac{1}{4\varrho^8}\right)\sum_{j=0}^{+\infty}\frac{j^2 + 4j + 3}{\varrho^{2j(n+1)}},
\end{aligned}$$

i.e., by calculating $h_\nu(1/\varrho^{2(n+1)})$ $(\nu = 0, 1, 2)$,

$$|R_{n,1}(f)| \leq \frac{\pi(\varrho + \varrho^{-1})^4(3\varrho^{2n+2} - 1)}{8(\varrho^{2n+2} - 1)^3}\Big(\max_{z\in E_\varrho}|f(z)|\Big). \tag{5.1}$$

The corresponding error estimate for $s = 0$ has been obtained by Hunter [11],

$$|R_{n,0}(f)| \leq \frac{\pi(\varrho + \varrho^{-1})^2}{2(\varrho^{2n+2} - 1)}\Big(\max_{z\in E_\varrho}|f(z)|\Big). \tag{5.2}$$

In order to give the second type of error estimates for the generalized Chebyshev weight of the second kind $w_2(t) = (1 - t^2)^{1/2+s}$, we need the expression

$$\varrho_{n,s}(z) = \frac{\pi}{2^{2s}}\frac{1}{\xi^{n+1}}\sum_{\nu=0}^{s}(-1)^\nu\binom{2s+1}{s-\nu}\frac{1}{\xi^{2(n+1)\nu}}, \tag{5.3}$$

which can be derived from [15, §3.2], as well as the following result for the sum

$$\widetilde{W}_s(\varrho,\theta) := \sum_{\nu=0}^{s} (-1)^\nu \binom{2s+1}{\nu} \varrho^{2\nu-s} \mathrm{e}^{i(\nu-s/2)\theta}. \tag{5.4}$$

LEMMA 5.1. *Let $r > 0$, $\varrho > 1$ and $x = \varrho^{4r}$. Then*

$$\left|\widetilde{W}_s(\varrho^r,\theta)\right|^2 = \sum_{k=0}^{s} (-1)^k A_k \cos k\theta, \tag{5.5}$$

where the coefficients A_k are the same as in Lemma 4.1.

We omit the proof of this lemma.

THEOREM 5.2. *Let $x = \varrho^{4(n+1)}$ and a_j, A_0 and A_k be defined by* (4.8), (4.14) *and* (4.15)*, respectively. Then, for the generalized Chebyshev weight of the second kind $w_2(t) = (1-t^2)^{1/2+s}$, we have*

$$\begin{aligned} L_{n,s}(E_\varrho) &= \frac{\varrho^{-(n+1)(s+1)}}{2^{2s+1/2}} \times \\ &\times \int_0^\pi (a_2 - \cos\theta)^{s+1} \sqrt{\frac{\sum_{k=0}^{s}(-1)^k A_k \cos(n+1)k\theta}{(a_{2n+2} - \cos(n+1)\theta)^{2s+1}}}\, \mathrm{d}\theta. \end{aligned} \tag{5.6}$$

Moreover, the estimate

$$L_{n,s}(E_\varrho) \le \frac{\pi}{2^s} \sqrt{M_{2s+2}(\varrho^2)}\, \Phi_s(\varrho^{4n+4}) \tag{5.7}$$

holds, where

$$M_k(\varrho) := \left(\frac{\varrho - \varrho^{-1}}{2}\right)^k P_k\left(\frac{\varrho + \varrho^{-1}}{\varrho - \varrho^{-1}}\right), \tag{5.8}$$

P_k is the Legendre polynomial of degree k, and Φ_s is defined in (4.18) *and* (4.19).

PROOF. According to (5.3), (4.8), and (5.4) we have

$$\begin{aligned} |\varrho_{n,s}(z)| &= \frac{\pi}{2^{2s}\varrho^{n+1}} \cdot \frac{1}{\varrho^{(n+1)s}} \left| \sum_{\nu=0}^{s} (-1)^\nu \binom{2s+1}{\nu} \varrho^{(2\nu-s)(n+1)} \mathrm{e}^{i(\nu-s/2)2(n+1)\theta} \right| \\ &= \frac{\pi}{2^{2s}\varrho^{(n+1)(s+1)}} \left|\widetilde{W}_s(\varrho^{n+1}, 2(n+1)\theta)\right|, \end{aligned}$$

and also $|U_n(z)| = \sqrt{(a_{2n+2} - \cos(2n+2)\theta)/(a_2 - \cos 2\theta)}$. In this case, because of periodicity of the integrand, (4.11) becomes

$$L_{n,s}(E_\varrho) = \frac{\varrho^{-(n+1)(s+1)}}{2^{2s+1/2}} \int_0^\pi \frac{(a_2 - \cos\theta)^{s+1}}{(a_{2n+2} - \cos(n+1)\theta)^{s+1/2}} \left|\widetilde{W}_s(\varrho^{n+1}, (n+1)\theta)\right| \mathrm{d}\theta.$$

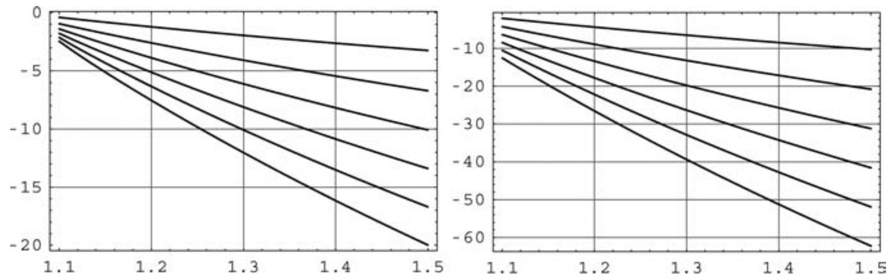

Figure 5.1: Log_{10} of the values $L_{n,s}(E_\varrho)$, $s = 0, 1, \ldots, 5$, as functions of ϱ, for $n = 10$ (left) and $n = 30$ (right).

By using (5.5), it reduces to (5.6). Applying Cauchy's inequality to (5.6) we get

$$(5.9)\qquad L_{n,s}(E_\varrho) \le \frac{\varrho^{-(n+1)(s+1)}}{2^{2s+1/2}} \left(\int_0^\pi (a_2 - \cos\theta)^{2s+2}\, \mathrm{d}\theta \right)^{1/2} \times$$
$$\times \left(\int_0^\pi \frac{\sum_{k=0}^s (-1)^k A_k \cos(n+1)k\theta}{(a_{2n+2} - \cos(n+1)\theta)^{2s+1}}\, \mathrm{d}\theta \right)^{1/2}.$$

From [10, Eq. 3.661.3]), we obtain

$$\frac{1}{\pi} \int_0^\pi (a_2 - \cos\theta)^{2s+2}\, \mathrm{d}\theta = \left(\frac{\varrho^2 - \varrho^{-2}}{2} \right)^{2s+2} P_{2s+2}\left(\frac{\varrho^2 + \varrho^{-2}}{\varrho^2 - \varrho^{-2}} \right) = M_{2s+2}(\varrho^2).$$

On the other side we note that

$$\int_0^\pi \frac{\cos(n+1)k\theta}{(a - \cos(n+1)\theta)^{2s+1}}\, \mathrm{d}\theta = \int_0^\pi \frac{\cos k\theta}{(a - \cos\theta)^{2s+1}}\, \mathrm{d}\theta = (-1)^k J_k(a) \quad (a > 1),$$

because of periodicity of the integrand and Lemma 4.2. According to $a_{2n+2} = (\varrho^{2n+2} + \varrho^{-2n-2})/2$, we put $x = \varrho^{4n+4}$, so that $a_{2n+2} = (x+1)/(2\sqrt{x})$, and then, the last integral in (5.9) becomes

$$\begin{aligned} \int_0^\pi \frac{\sum_{k=0}^s (-1)^k A_k \cos(n+1)k\theta}{(a_{2n+2} - \cos(n+1)\theta)^{2s+1}}\, \mathrm{d}\theta &= \sum_{k=0}^s (-1)^k A_k (-1)^k J_k(a_{2n+2}) \\ &= \frac{2^{2s+1}\pi x^{(s+1)/2}}{(x-1)^{4s+1}}\, Q_s(x), \end{aligned}$$

where A_0 and A_k are given by (4.14) and (4.15), respectively. The polynomial $Q_s(x)$ is defined by (4.19). In this way, inequality (5.9) reduces to (5.7). □

REMARK 5.1. An alternative expression for (5.8) is given by ([10, Eq. 3.616.1]) $M_k(\varrho) = (2\varrho)^{-k} \sum_{\nu=0}^k \binom{k}{\nu}^2 \varrho^{2\nu}$.

In Figure 5.1 we presented the values of $\log_{10}(L_{n,s}(E_\varrho))$, $s = 0, 1, \ldots, 5$, as a function of ϱ, when $n = 10$ and $n = 30$. The upper graphs correspond to the smaller values of s. The values of $L_{n,s}(E_\varrho)$ were calculated by using (5.6).

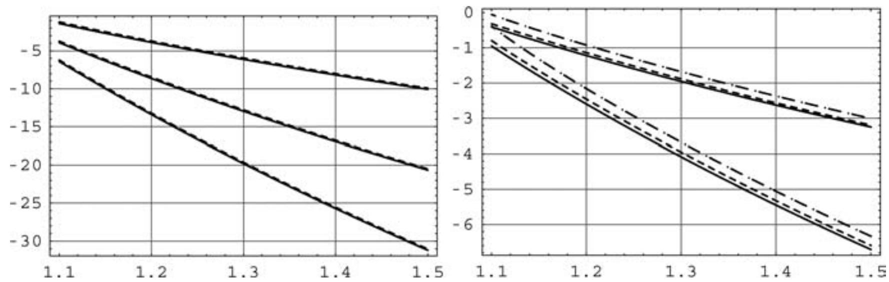

Figure 5.2: Log_{10} of the values $L_{n,2}(E_\varrho)$ (solid lines) and their bounds given by (5.7) (dashed lines) for $n = 10, 20$, and 30 (left) and Log_{10} of the values $L_{n,s}(E_\varrho)$ for $s = 0, 1$ (solid lines) and their bounds given by (5.2) and (5.1) (dot-dashed lines) and (5.7) (dashed lines) for $n = 10$ (right).

The graphs $\varrho \mapsto \log_{10}(L_{n,s}(E_\varrho))$ and $\varrho \mapsto \log_{10}(2^{-s}\pi\sqrt{M_{2s+2}(\varrho^2)}\,\Phi_s(\varrho^{4n+4}))$ for $n = 10, 20, 30$ and $s = 2$ are displayed in Figure 5.2 (left). The upper graphs correspond to the smaller values of n. Beside the bound (5.7) (dashed line), in Figure 5.2 (right) we presented also the Hunter bound (for $s = 0$) and our bound (for $s = 1$), given by (5.2) and (5.1), respectively. These bounds are given as dot-dashed lines. The upper set of graphs correspond to $s = 0$ and lower one to $s = 1$. As we can see, the second type of bounds (5.7) are very precise especially for larger values of n and ϱ.

Remark 5.2. The corresponding error estimate for Gaussian quadrature with the Chebyshev weight function of the second kind has been obtained by Hunter (see [11, Eq. (5.8)]). The quantity $L_{n,s}(E_\varrho)$ given in (5.6), for $s = 0$ reduces to Hunter's result, which can be expressed in terms of the complete elliptic integral of the first kind, $L_{n,0}(E_\varrho) = (\varrho^2 + \varrho^{-2})(\varrho^{2n+2} + 1)^{-1}K(2/(\varrho^{n+1} + \varrho^{-n-1}))$.

6 Error estimates for Gauss–Turán quadratures with a generalized Chebyshev weight of the third kind.

For the special Jacobi weight function $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$ (the generalized Chebyshev weight of the third kind) we can obtain the first type of error estimates in a similar way as for $w_2(t)$.

In this section we consider a problem how to obtain the second type of error estimates for $w_3(t)$. According to [15, §3.3] we can find

$$\varrho_{n,s}(z) = \frac{\pi}{2^{s-1}\xi^{n+1/2}(\xi^{1/2} - \xi^{-1/2})} \sum_{\nu=0}^{s} \binom{2s+1}{s-\nu} \frac{1}{\xi^{(2n+1)\nu}}. \tag{6.1}$$

Since $V_n(t) = \cos((n+1/2)\theta)/\cos(\theta/2)$, $t = \cos\theta$, is a Jacobi polynomial with parameters $\alpha = -1/2$, $\beta = 1/2$ (see [19, 9]), we have the following representation ([19])

$$V_n(z) = \frac{T_{2n+1}(u)}{u}, \quad u = \sqrt{\frac{1+z}{2}}. \tag{6.2}$$

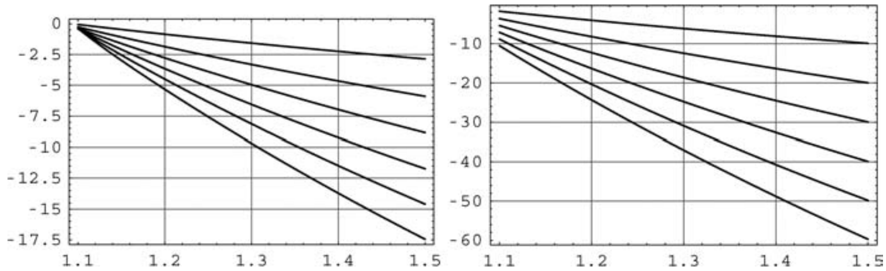

Figure 6.1: Log_{10} of the values $L_{n,s}(E_\varrho)$, $s = 0, 1, \ldots, 5$, as functions of ϱ, for $n = 10$ (left) and $n = 30$ (right).

Theorem 6.1. *Let $x = \varrho^{4n+2}$ and a_j, A_0 and A_k be defined by* (4.8), (4.14) *and* (4.15), *respectively. Then, for the generalized Chebyshev weight of the third kind $w_3(t) = (1-t)^{-1/2}(1+t)^{1/2+s}$, we have*

$$\text{(6.3)}\quad L_{n,s}(E_\varrho) = \frac{2^{1/2-s}}{\varrho^{(n+\frac{1}{2})(s+1)}} \int_0^\pi (a_1 + \cos\theta)^{s+1} \sqrt{\frac{\sum_{k=0}^{s} A_k \cos(2n+1)k\theta}{(a_{2n+1} + \cos(2n+1)\theta)^{2s+1}}}\, \mathrm{d}\theta.$$

Moreover, an estimate of the form

$$\text{(6.4)}\qquad L_{n,s}(E_\varrho) \le 2\pi \sqrt{M_{2s+2}(\varrho)}\, \Phi_s(\varrho^{4n+2})$$

holds, where M_k and Φ_s are defined by (5.8) *and* (4.18) *and* (4.19), *respectively.*

Proof. According to (6.1), (4.8), (4.12) and (6.2) we have

$$|\varrho_{n,s}(z)| = \frac{\pi}{2^{s-1}\varrho^{n+1/2}\sqrt{2}\,(a_1 - \cos\theta)^{1/2}} \cdot \frac{1}{\varrho^{(n+1/2)s}} |W_s(\varrho^{n+1/2}, (2n+1)\theta)|$$

and $|V_n(z)| = \sqrt{(a_{2n+1} + \cos(2n+1)\theta)/(a_1 + \cos\theta)}$. Then (4.11) becomes

$$L_{n,s}(E_\varrho) = \frac{2^{1/2-s}}{\varrho^{(n+1/2)(s+1)}} \int_0^\pi \frac{(a_1 + \cos\theta)^{s+1} |W_s(\varrho^{n+1/2}, (2n+1)\theta)|}{(a_{2n+1} + \cos(2n+1)\theta)^{s+1/2}}\, \mathrm{d}\theta,$$

i.e., (6.3), because of (4.13). Using [10, Eq. 3.661.3]), with $a = a_1 = (\varrho + \varrho^{-1})/2$, $b = 1$, $k = 2s + 2$, we find

$$\frac{1}{\pi} \int_0^\pi (a_1 + \cos\theta)^{2s+2}\, \mathrm{d}\theta = \left(\frac{\varrho - \varrho^{-1}}{2}\right)^{2s+2} P_{2s+2}\left(\frac{\varrho + \varrho^{-1}}{\varrho - \varrho^{-1}}\right) = M_{2s+2}(\varrho),$$

where M_k is defined in (5.8). Now, according to $a_{2n+1} = (\varrho^{2n+1} + \varrho^{-2n-1})/2$, we put $x = \varrho^{4n+2}$, so that $a_{2n+1} = (x+1)/(2\sqrt{x})$.

Finally, applying Cauchy's inequality to (6.3), as in the proof of Theorem 5.2, we obtain (6.4). □

In Figure 6.1 we presented the values of $\log_{10}(L_{n,s}(E_\varrho))$, $s = 0, 1, \ldots, 5$, as a function of ϱ, when $n = 10$ and $n = 30$. The upper graphs correspond to the smaller values of s. The values of $L_{n,s}(E_\varrho)$ were calculated by using (6.3).

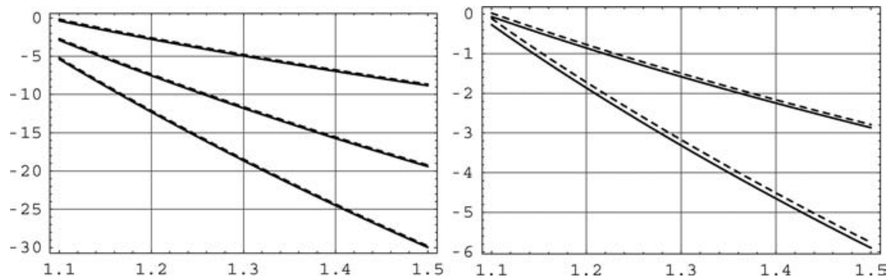

Figure 6.2: Log_{10} of the values $L_{n,2}(E_\varrho)$ (solid lines) and their bounds given by (6.4) (dashed lines) for $n = 10, 20$, and 30 (left) and Log_{10} of the values $L_{n,s}(E_\varrho)$ for $s = 0, 1$ (solid lines) and their bounds given by (6.4) (dashed lines) for $n = 10$ (right).

The graphs $\varrho \mapsto \log_{10}(L_{n,s}(E_\varrho))$ and $\varrho \mapsto \log_{10}(2\pi\sqrt{M_{2s+2}(\varrho)}\,\Phi_s(\varrho^{4n+2}))$ for $n = 10, 20, 30$ and $s = 2$ are displayed in Figure 6.2 (left). The upper graphs correspond to the smaller values of n.

The functions $\varrho \mapsto \log_{10}(L_{10,s}(E_\varrho))$ $(s = 0, 1)$, as well as their bounds (6.4), are also given in Figure 6.2 (right). As before, we can see that the bounds of the second type (6.4) are very precise especially for larger values of n and ϱ.

Remark 6.1. The corresponding error estimate for Gaussian quadrature has been obtained by Hunter. For $s = 0$, (6.3) reduces to Hunter's result (cf. [11, Eq. (5.9)])

$$L_{n,0}(E_\varrho) = \frac{2(\varrho + \varrho^{-1})}{\varrho^{2n+1} + 1}\, K\left(\frac{2}{\varrho^{n+1/2} + \varrho^{-n-1/2}}\right).$$

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