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SEMIDISCRETE GALERKIN APPROXIMATION FOR A LINEAR STOCHASTIC PARABOLIC PARTIAL DIFFERENTIAL EQUATION DRIVEN BY AN ADDITIVE NOISE⋆

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Abstract.

We study the semidiscrete Galerkin approximation of a stochastic parabolic partial differential equation forced by an additive space-time noise. The discretization in space is done by a piecewise linear finite element method. The space-time noise is approximated by using the generalized L_2 projection operator. Optimal strong convergence error estimates in the L_2 and $\dot{H}^{-1}$ norms with respect to the spatial variable are obtained. The proof is based on appropriate nonsmooth data error estimates for the corresponding deterministic parabolic problem. The error estimates are applicable in the multi-dimensional case.

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1 Introduction.

In this paper we will study the finite element approximation of the linear stochastic parabolic partial differential equation

$$\mathrm{d}u + Au\,\mathrm{d}t = \mathrm{d}W, \quad \text{for } 0 < t \le T, \text{ with } u(0) = u_0, \tag{1.1}$$

in a Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $u(t)$ is an H-valued random process, A is a linear, selfadjoint, positive definite, not necessarily bounded operator with a compact inverse, densely defined in $\mathcal{D}(A) \subset H$, where $W(t)$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $u_0 \in H$.

For the sake of simplicity, we shall concentrate on the case $A = -\Delta$ subject to homogeneous Dirichlet boundary conditions, where Δ stands for the Laplacian operator and $H = L_2(\mathcal{D})$, where $\mathcal{D}$ is a bounded convex domain in $\mathbf{R}^d, d = 1, 2, 3,$ with a sufficiently smooth boundary $\partial\mathcal{D}$.

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Such equations are common in applications. Many models in physics, chemistry, biology, population dynamics, neurophysiology, etc., are described by stochastic partial differential equations, see Da Prato and Zabczyk [7], Walsh [28], etc. The existence, uniqueness, and properties of the solutions of such equations have been well studied, see Curtain and Falb [4], Da Prato [5], Da Prato and Lunardi [6], Da Prato and Zabczyk [7], Dawson [9], Gozzi [11], Peszat and Zabczyk [22], Walsh [28], etc.

Let $E(t) = \mathrm{e}^{-tA}$, $t \geq 0$. Then (1.1) admits a unique mild solution, see Da Prato and Zabczyk [7, Theorems 5.2, 5.4],

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad \text{for } 0 < t \leq T, \tag{1.2}$$

where the integral is understood in Itô sense. We will review the Hilbert space valued Itô integral in Section 2.

The numerical approximation for (1.1) started with the work by Greksch and Kloeden [12], Gyöngy and Nualart [15]. Further contributions include Allen, Novosel and Zhang [1], Benth and Gjerde [2], Davie and Gaines [8], Du and Zhang [10], Gyöngy [13, 14], Hausenblas [16, 17], Kloeden and Shott [18], Lord and Rougemont [19], Printems [23], Shardlow [24], Theting [25, 26], Yan [29], etc.

The difficulty of the numerical approximation of (1.1) is to approximate the noise in a suitable way. Let us review some ways to approximate the noise used in literature. Consider the one-dimensional problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) - \frac{\partial^2 u}{\partial x^2}(t,x) = \frac{\partial^2 W}{\partial t \partial x}(t,x), \quad 0 < t \leq T, \\ &u(0,x) = u_0(x), \quad 0 < x < 1, \\ &u(t,0) = u(t,1) = 0, \quad t \geq 0, \end{aligned} \tag{1.3}$$

where $\partial^2 W/\partial t \partial x$ denotes the mixed second-order derivative of the Brownian sheet. The integral formulation of (1.3) has the form

$$u(t,x) = \int_0^1 G_t(x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G_{t-s}(x,y)\,\mathrm{d}W(s,y),$$

where $G_t(x,y) = 2\sum_{n=1}^{\infty} \sin n\pi x \sin n\pi y \mathrm{e}^{-(n\pi)^2 t}$ is the fundamental solution of

$$v_t(t,x) - v_{xx}(t,x) = 0, \qquad v(0,x) = \phi(x), \qquad v(t,0) = v(t,1) = 0,$$

so that

$$v(t,x) = \int_0^1 G_t(x,y)\phi(y)\,\mathrm{d}y.$$

Let $0 = t_0 < t_1 \cdots < t_N = T$ be a partition of $[0,T]$, $t_n = nk$, $n = 0,1,2,\ldots,N$, where k is the time step. Let $0 = x_0 < x_1 < \cdots < x_J = 1$ be a partition of $[0,1]$, $x_j = jh$, $j = 0,1,2,\ldots,J$, where h is the space step. Allen, Novosel and Zhang [1] approximate the space-time white noise W by using the following piecewise constant functions on a partition $[t_{n-1}, t_n] \times [x_{j-1}, x_j]$, $1 \le n \le N$, $1 \le j \le J$ of $[0, T] \times [0, 1]$,

$$\mathrm{d}W(t,x) \approx \mathrm{d}\hat{W}(t,x) = \frac{\partial^2 \hat{W}(t,x)}{\partial t \partial x}\,\mathrm{d}t\,\mathrm{d}x = \frac{1}{kh}\sum_{n=1}^{N}\sum_{j=1}^{J}\eta_{nj}\sqrt{kh}\chi_n(t)\chi_j(x)\,\mathrm{d}t\,\mathrm{d}x,$$

where

$$\chi_n(t) = \begin{cases} 1, & t_{n-1} \le t \le t_n, \\ 0, & \text{otherwise,} \end{cases} \qquad \chi_j(x) = \begin{cases} 1, & x_{j-1} \le x \le x_j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta_{nj} = \frac{1}{kh}\int_{t_{n-1}}^{t_n}\int_{x_{j-1}}^{x_j}\mathrm{d}W(t,x) = \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ is the standard real-valued Gaussian random variable and η_{nj} are independent and identically distributed (iid). It is obvious that $\partial^2\hat{W}/\partial t\partial x \in L_2(0,1)$ for fixed $t \in [0,T]$, $\omega \in \Omega$. Applying the standard finite element and finite difference methods to (1.3) with $\partial^2 W/\partial t\partial x$ replaced by $\partial^2\hat{W}/\partial t\partial x$, they obtain the corresponding error estimates. See also Davie and Gaines [8], Gyöngy [13, 14] for a quasi-linear parabolic stochastic partial differential equations with finite difference method, Du and Zhang [10] for some special noises.

Shardlow [24] approximates the noise by spectral method. Let P_J denote the operator taking f to its first J Fourier modes, i.e.,

$$P_J f = \sum_{j=1}^{J}(f, e_j)e_j,$$

where $e_j = \sqrt{2}\sin j\pi x$, $j = 1, 2, \ldots$, are the eigenvectors of $A = -\partial^2/\partial x^2$ subject to Dirichlet boundary condition. Then he approximates the Wiener process over the time step (t_{n-1}, t_n) by

$$\mathrm{d}W_k(n) := \int_{t_{n-1}}^{t_n} P_J\,\mathrm{d}W(s),$$

which is a $L_2(0,1)$ function. The numerical method evaluates this function at the grid points $x_j = jh$, $j = 1, 2, \ldots, J$. See also Hausenblas [16, 17] for a quasi-linear parabolic stochastic partial differential equations in a very general approach.

Moreover, Benth and Gjerde [2], Theting [25, 26] use the chaos expansion theory and finite element methods to consider the approximation of (1.1).

In the present paper, we approximate the space-time noise by using the generalized L_2-projection operator (1.4) and then introduce the finite element formulation for (1.1) in the semidiscrete case. By using the error estimates for deterministic parabolic problem, we can prove optimal strong error estimates in both L_2 and $\dot{H}^{-1}$ norms. Our proof is quite simple and applicable in the multi-dimensional case.

Let S_h be a family of linear finite element spaces, i.e., S_h consists of continuous piecewise polynomials of degree ≤ 1 with respect to triangulation $\mathcal{T}$ of $\mathcal{D}$. For simplicity, we always assume that $\{S_h\} \subset H_0^1 = H_0^1(\mathcal{D}) = \{v \in L_2(\mathcal{D}), \nabla v \in L_2(\mathcal{D}), v|_{\partial\mathcal{D}} = 0\}$. Let $\dot{H}^s = \dot{H}^s(\mathcal{D}) = \mathcal{D}(A^{s/2})$ for any $s \in \mathbf{R}$ and denote its norm by $|\cdot|_s = \|A^{s/2}\cdot\|$.

We find that $W(t) \in \dot{H}^{-1}$, see Lemma 3.5 in Section 3. To introduce the finite element formulation for (1.1), we will use the generalized L_2-projection operator $P_h : \dot{H}^{-1} \to S_h$ defined by, see Chrysafinos and Hou [3],

$$(P_h v, \chi) = \langle v, \chi\rangle, \quad \forall \chi \in S_h \subset \dot{H}^1, \ \forall v \in \dot{H}^{-1}, \tag{1.4}$$

where $\langle \cdot,\cdot\rangle$ denotes the pairing between $\dot{H}^{-1}$ and $\dot{H}^1$. One can easily show that P_h is well defined by introducing a basis $\{\varphi_i\}_{i=1}^{N_h}$ and solve for $P_h v = \sum_{j=1}^{N_h} \alpha_j \varphi_j$ from the equations $(\sum_{j=1}^{N_h} \alpha_j \varphi_j, \varphi_i) = \langle v, \varphi_i\rangle$. Also it is evident that when $v \in L_2(\mathcal{D})$, $P_h v$ is the standard L_2 projection operator, see Thomée [27].

The semidiscrete problem corresponding to (1.1) is to find the process $u_h(t) = u_h(\cdot,t) \in S_h$, such that

$$\mathrm{d}u_h + A_h u_h \,\mathrm{d}t = P_h \,\mathrm{d}W, \quad \text{for } 0 < t \leq T, \text{ with } u_h(0) = P_h u_0, \tag{1.5}$$

where A_h is the discrete analogue of $A = -\Delta$ with the Dirichlet boundary condition defined by

$$(A_h\psi, \chi) = (\nabla\psi, \nabla\chi), \quad \forall \psi, \chi \in S_h.$$

With $E_h(t) = \mathrm{e}^{-tA_h}$, $t \geq 0$, (1.5) admits a unique mild solution

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h \,\mathrm{d}W(s).$$

Let $\mathbf{E}$ be the expectation. For any Hilbert space H_1, we define $L_2(\Omega; H_1)$ by

$$L_2(\Omega; H_1) = \left\{ v : \mathbf{E}\|v\|_{H_1}^2 = \int_\Omega \|v(\omega)\|_{H_1}^2 \,\mathrm{d}\mathbf{P}(\omega) < \infty \right\},$$

with the norm $\|v\|_{L_2(\Omega;H_1)} = (\mathbf{E}\|v\|_{H_1}^2)^{1/2}$.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert–Schmidt operators from $Q^{1/2}(H)$ to H, where Q is the covariance operator of $W(t)$, see Section 2. Our main results in this paper are the following:

THEOREM 1.1. *Let u_h and u be the solutions of* (1.5) *and* (1.1), *respectively. Assume that $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$. Then we have, for $t \geq 0$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq Ch^\beta \big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L_2^0}\big). \tag{1.6}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, *then we have, for $t \geq 0$ and $u_0 \in L_2(\Omega; \dot{H}^1)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq Ch\big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \mathrm{Tr}(Q)^{1/2}\big). \tag{1.7}$$

THEOREM 1.2. *Let u_h and u be the solutions of* (1.5) *and* (1.1), *respectively. Assume that $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$. Then we have, for $0 \le t \le T$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_h = \log(T/h^2)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^{\beta+1}\big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \ell_h \|A^{(\beta-1)/2}\|_{L_2^0}\big). \tag{1.8}$$

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, *then we have, for $0 \le t \le T$ and $u_0 \in L_2(\Omega; \dot{H}^1)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \le Ch^2\big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \ell_h \,\mathrm{Tr}(Q)^{1/2}\big). \tag{1.9}$$

We remark that similar error estimates can be obtained in the fully discrete case. The proofs are similar to the semidiscrete case. We will not discuss them here. For example, the backward Euler scheme is to find $U^n \in S_h$, $U^n \approx u(t_n)$, such that,

$$\frac{U^n - U^{n-1}}{k} + A_h U^n = \frac{1}{k}\int_{t_{n-1}}^{t_n} P_h \,\mathrm{d}W(t), \quad n \ge 1, \qquad U^0 = P_h u_0, \tag{1.10}$$

where we approximate the noise over (t_{n-1}, t_n) by

$$\int_{t_{n-1}}^{t_n} P_h \,\mathrm{d}W(t) = P_h\big(W(t_n) - W(t_{n-1})\big).$$

With $r(\lambda) = (1+\lambda)^{-1}$, we can rewrite (1.10) in the form

$$\begin{aligned} U^n &= r(kA_h)U^{n-1} + \int_{t_{n-1}}^{t_n} r(kA_h)P_h \,\mathrm{d}W(s), \quad n \ge 1, \\ U^0 &= P_h u_0. \end{aligned} \tag{1.11}$$

Following the proof of error estimates for $u_h - u$, we can prove similar error estimates for $U^n - u(t_n)$. This was first done in the temporally semidiscrte case by Printems [23] and later in the completely discrete case by Yan [29]. In this paper, we will only focus on the proofs of the error estimates in the spatially semidiscrete case.

This paper is organized as follows. In Section 2, we discuss the Itô integral with respect to the Wiener process in Hilbert space. In Section 3, we consider the regularity of the solution of (1.1). In Section 4, we prove error estimates for a deterministic problem. We then give, in Section 5, the proofs of our main results.

2 Preliminaries.

In this section, we will give a short discussion of the Itô integral with respect to the Wiener process $W(t)$ in Hilbert space.

2.1 The stochastic integral with respect to an H-valued Wiener process.

A family $W(t)$, $t \ge 0$, of H-valued random variables is called a Wiener process on H, if and only if, see [7] and [30],

(i) $W(0) = 0$,
(ii) for almost all $\omega \in \Omega$, $t \mapsto W(t, \omega)$ is a continuous function,
(iii) $W(t)$ has independent increments,
(iv) $\mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t-s))$, $0 \leq s \leq t$.

Here $\mathcal{L}(X)$ denotes the law, or the distribution, of the H-valued random variable X, i.e., the probability measure on H defined by

$$\mathcal{L}(X)(A) = \mathbf{P}\{\omega : X(\omega) \in A\}, \quad \text{for any } A \in \mathcal{B}(H),$$

where $\mathcal{B}(H)$ is the Borel σ-algebra of H, i.e., the smallest σ-algebra containing all closed (or open) sets of H.

It turns out that if $W(t)$ is a Wiener process on H, then, for arbitrary t, $\mathcal{L}(W(t))$ is a Gaussian probability measure on H with the mean 0 and the covariance operator tQ, i.e.,

$$\mathcal{L}(W(t)) = \mathcal{N}(0, tQ),$$

where Q is a linear, self-adjoint, positive definite, bounded operator with finite trace, i.e., $\mathrm{Tr}(Q) < \infty$. We then call the above $W(t)$ an H-valued Wiener process with covariance operator Q, $\mathrm{Tr}(Q) < \infty$.

There is a natural class of operator-valued processes, which can be stochastically integrated with respect to an H-valued Wiener process $W(t)$. Denote by $Q^{1/2}(H)$ the image of the operator $Q^{1/2}$ on H. Denote by $L(H)$ the space of bounded linear operators on H, and by $L_2^0(Q^{1/2}(H), H)$ the space of all Hilbert–Schmidt operators from $Q^{1/2}(H)$ into H, i.e.,

$$L_2^0(Q^{1/2}(H), H) = \left\{ \psi \in L\big(Q^{1/2}(H), H\big) : \sum_{j=1}^{\infty} \|\psi g_j\|^2 < \infty \right\},$$

where $\{g_j\}_{j=1}^{\infty}$ is an arbitrary orthonormal basis of $Q^{1/2}(H)$. Its norm is denoted by

$$\|\psi\|_{L_2^0} = \left(\sum_{j=1}^{\infty} \|\psi g_j\|^2 \right)^{1/2},$$

where $L_2^0 = L_2^0(Q^{1/2}(H), H)$.

Denote by $L_{\mathcal{F}}^2([0,T]; L_2^0)$ the separable Hilbert space of all measurable processes x, with values in L_2^0, such that

$$\|x\|_{L_{\mathcal{F}}^2([0,T]; L_2^0)} = \left(\int_0^T \mathbf{E}\|x(t)\|_{L_2^0}^2 \,\mathrm{d}t \right)^{1/2} < \infty.$$

For any $\psi(\cdot) \in L_{\mathcal{F}}^2([0,T]; L_2^0)$, we can define the stochastic integral

$$\int_0^T \psi(t) \,\mathrm{d}W(t) \tag{2.1}$$

in the standard way as in the stochastic integral with respect to the scalar Wiener process $W(t)$.

2.2 The stochastic integral with respect to a cylindrical Wiener process.

The construction of the stochastic integral for an H-valued Wiener process $W(t)$ above requires that $W(t)$ is H-valued, which implies that Q is a trace class operator. Here we shall extend the definition of the stochastic integral to the case of a cylindrical Wiener process. Let Q be a linear, self-adjoint, positive definite, bounded operator on H, not necessarily in the trace class, but with a bounded sequence of positive eigenvalues $\{\gamma_l\}_{l=1}^{\infty}$ and a corresponding orthonormal basis of eigenvectors $\{e_l\}_{l=1}^{\infty}$ in H. Thus Q is not necessarily compact, for example, $Q = I$. By a cylindrical Wiener process with covariance operator $Q, \mathrm{Tr}(Q) \leq \infty$, we mean the series, see Da Prato and Zabczyk [7], Peszat [21], Peszat and Zabczyk [22],

$$W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t), \quad t \geq 0, \tag{2.2}$$

where $\{\beta_l(t)\}$ is a family of real-valued, independent, Brownian motions. In the special case $Q = I$, $W(t)$ is defined by

$$W(t) = \sum_{l=1}^{\infty} e_l \beta_l(t), \quad t \geq 0. \tag{2.3}$$

We observe that (2.2) is divergent in $L_2(\Omega; H)$ if Q is not in the trace class, in which case $W(t)$ is not an H-valued process. In fact, for arbitrary $t > 0$,

$$\mathbf{E}\left\|\sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t)\right\|^2 = \sum_{l=1}^{\infty} \gamma_l \mathbf{E}\beta_l(t)^2 = t \sum_{l=1}^{\infty} \gamma_l = t\,\mathrm{Tr}(Q) = \infty.$$

However, let H_L be an arbitrary Hilbert space such that the embedding of $Q^{1/2}(H)$ into H_L is Hilbert–Schmidt. Then we have the following lemma, see [7, Proposition 4.11].

LEMMA 2.1. *The cylindrical Wiener process* (2.2) *defines a H_L-valued Wiener process with some covariance operator Q_L.*

For arbitrary $h \in H$, the process

$$\langle h, W(t) \rangle := \sum_{l=1}^{\infty} \gamma_l^{1/2} (h, e_l) \beta_l(t) \tag{2.4}$$

is a real-valued Brownian motion and

$$\mathbf{E}\big(\langle h_1, W(t)\rangle \langle h_2, W(s)\rangle\big) = \min(t,s)(Qh_1, h_2), \quad \textit{for } h_1, h_2 \in H. \tag{2.5}$$

For any $\psi(\cdot) \in L^2_{\mathcal{F}}([0,T]; L^0_2)$, we can define the stochastic integral with respect to the cylindrical Wiener process as follows:

$$\int_0^T \psi(t)\,\mathrm{d}W(t) = \sum_{l=1}^{\infty} \int_0^T \psi(t) g_l \,\mathrm{d}\langle g_l, W(t)\rangle, \tag{2.6}$$

where $\{g_l\}_{l=1}^{\infty}$ is an arbitrarily orthonormal basis in $Q^{1/2}(H)$, and the integral on the right is the standard Itô integral.

Let us consider three special cases.

(i) If $Q = I$, then we can choose $g_l = e_l$, and hence $\langle g_l, W(t)\rangle = \beta_l(t)$ by (2.4), therefore the stochastic integral is

$$\int_0^T \psi(t)\,\mathrm{d}W(t) = \sum_{l=1}^{\infty} \int_0^T \psi(t) e_l \,\mathrm{d}\beta_l(t).$$

(ii) If $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then $Q^{1/2}$ is Hilbert–Schmidt and $H_L = H$. In this case, the stochastic integral defined by (2.6) is consistent with the stochastic integral defined in (2.1).

(iii) More generally, in the present paper we assume that $\|A^{(\beta-1)/2}\|_{L^0_2} < \infty$ for some $\beta \in [0,1]$, i.e.,

$$\|A^{(\beta-1)/2}\|^2_{L^0_2} = \sum_{l=0}^{\infty} \gamma_l \|A^{(\beta-1)/2} e_l\|^2 < \infty,$$

which implies that $H_L = \dot{H}^{\beta-1}$, see Lemma 3.5 in Section 3. Thus $W(t)$ is $\dot{H}^{-1}$-valued, which suggests that we should use the generalized L_2-projection operator in the formulation of finite element method for (1.1). We remark that the following isometry property holds for the cylindrical Wiener process $W(t)$

$$\mathbf{E}\Big\|\int_0^T \psi(t)\,(t)\Big\|^2 = \int_0^T \mathbf{E}\|\psi(t)\|^2_{L^0_2}\,\mathrm{d}t. \tag{2.7}$$

3 Regularity of the mild solution.

In this section we will consider the regularity of the mild solution of (1.1). We have

THEOREM 3.1. *Let $u(t)$ be the mild solution* (1.2) *of* (1.1)*. If $\|A^{(\beta-1)/2}\|_{L^0_2} < \infty$ for some $\beta \in [0,1]$, then we have, for fixed $t \in [0,T]$,*

$$\|u(t)\|_{L_2(\Omega;\dot{H}^\beta)} \le C\big(\|u_0\|_{L_2(\Omega;\dot{H}^\beta)} + \|A^{(\beta-1)/2}\|_{L^0_2}\big), \quad \text{for } u_0 \in L_2(\Omega;\dot{H}^\beta). \tag{3.1}$$

In particular, if $W(t)$ is an H-valued Wiener process with covariance operator $Q, \mathrm{Tr}(Q) < \infty$, then we have

$$\|u(t)\|_{L_2(\Omega;\dot{H}^1)} \le C\big(\|u_0\|_{L_2(\Omega;\dot{H}^1)} + \mathrm{Tr}(Q)^{1/2}\big), \quad \text{for } u_0 \in L_2(\Omega;\dot{H}^1). \tag{3.2}$$

To prove this theorem, we need some regularity results which are related to the fact that $E(t) = \mathrm{e}^{-tA}$ is an analytic semigroup on H. For later use, we collect some results in the next two lemmas, see Thomée [27] or Pazy [20].

LEMMA 3.2. *Let $\alpha, \beta \in \mathbf{R}$ and let $l \geq 0$ be any integer. We have*

$$|D_t^l E(t)v|_\beta \leq Ct^{-(\beta-\alpha)/2-l}|v|_\alpha, \quad \textit{for } t > 0,\ 2l + \beta \geq \alpha, \tag{3.3}$$

and

$$\int_0^t s^\alpha |D_t^l E(s)v|_\beta^2 \,\mathrm{d}s \leq C|v|_{2l+\beta-\alpha-1}^2, \quad \textit{for } t \geq 0,\ \alpha \geq 0. \tag{3.4}$$

LEMMA 3.3. *For arbitrary $\alpha \geq 0$, $0 \leq \beta \leq 1$, we have*

$$\|A^\alpha E(t)\| \leq Ct^{-\alpha}, \quad \textit{for } t > 0, \tag{3.5}$$

and

$$\|A^{-\beta}(I - E(t))\| \leq Ct^\beta, \quad \textit{for } t \geq 0. \tag{3.6}$$

PROOF OF THEOREM 3.1. By (1.2), we have, for arbitrary $\beta \geq 0$, using stability property of $E(t)$ and isometry property,

$$\begin{aligned} \mathbf{E}(|u(t)|_\beta^2) &\leq 2\mathbf{E}(|E(t)u_0|_\beta^2) + 2\mathbf{E}\left\| \int_0^t A^{\beta/2}E(t-s)\,\mathrm{d}W(s) \right\|^2 \\ &\leq 2\mathbf{E}(|u_0|_\beta^2) + 2\mathbf{E}\int_0^t \|A^{\beta/2}E(t-s)\|_{L_2^0}^2 \,\mathrm{d}s. \end{aligned} \tag{3.7}$$

With $\{e_l\}_{l=1}^\infty$ an arbitrary orthonormal basis on H, we have, using Lemma 3.2,

$$\begin{aligned} \int_0^t \|A^{\beta/2}E(t-s)\|_{L_2^0}^2 \,\mathrm{d}s &= \sum_{j=1}^\infty \int_0^t \|A^{\beta/2}E(t-s)Q^{1/2}e_j\|^2 \,\mathrm{d}s \\ &= \sum_{j=1}^\infty \int_0^t |E(s)Q^{1/2}e_j|_\beta^2 \,\mathrm{d}s \\ &\leq C\sum_{j=1}^\infty |Q^{1/2}e_j|_{\beta-1}^2 = C\|A^{(\beta-1)/2}\|_{L_2^0}^2. \end{aligned}$$

Together with (3.7) this shows (3.1).

In particular, if $W(t)$ is an H-valued Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ because

$$\|I\|_{L_2^0}^2 = \sum_{j=1}^\infty \|Q^{1/2}e_j\|^2 = \sum_{j=1}^\infty \gamma_j = \mathrm{Tr}(Q). \qquad \square$$

If $d = 1$, then we may specialize to $Q = I$:

Corollary 3.4. *Let $u(t)$ be the solution of* (1.1) *and $A = -\partial^2/\partial x^2$ with $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. Assume that $W(t)$ is a cylindrical Wiener process with $Q = I$. Then we have, for every $\beta \in [0, 1/2)$,*

$$\|u(t)\|_{L_2(\Omega;\dot{H}^\beta)} \le C(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \textit{for } u_0 \in L_2(\Omega; \dot{H}^\beta).$$

Proof. It is well known that A has eigenvalues $\lambda_j = j^2\pi^2$, $j = 1, 2, \dots$, and corresponding eigenfunctions $\varphi_j = \sqrt{2}\sin j\pi x$, $j = 1, 2, \dots$, which form an orthonormal basis in $H = L_2(0,1)$. Thus, we have

$$\|A^{(\beta-1)/2}\|^2_{L_2^0} = \sum_{j=1}^\infty \|A^{(\beta-1)/2}\varphi_j\|^2 = \sum_{j=1}^\infty \lambda_j^{\beta-1},$$

which is convergent if $\beta \in [0, 1/2)$.

The proof is complete. □

We note that in Theorem 3.1, we require the condition $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for $\beta \in [0,1]$. The following lemma shows that this condition is equivalent to saying that $W(t)$ is $\dot{H}^{\beta-1}$-valued. In particular, $W(t) \in \dot{H}^{-1}$, which is important for the finite element formulation of (1.1).

Lemma 3.5. *Assume that $W(t)$ is a Wiener process with covariance operator Q. Assume that A and Q have the same eigenvectors. Then there exists an operator $\tilde{Q}$,* $\mathrm{Tr}(\tilde{Q}) < \infty$, *such that the following statements hold.*

(i) *If $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, then*

$$W(t) = \sum_{l=1}^\infty Q^{1/2} e_l \beta_l(t), \quad t \ge 0,$$

defines an $\dot{H}^{\beta-1}$-valued Wiener process with covariance operator $\tilde{Q}$. In particular, $\tilde{Q} = Q$ if $\mathrm{Tr}(Q) < \infty$.

(ii) *If $W(t) = \sum_{l=1}^\infty Q^{1/2} e_l \beta_l(t)$, $t \ge 0$, is an $\dot{H}^{\beta-1}$-valued Wiener process with the covariance operator $\tilde{Q}$, then*

$$\|A^{(\beta-1)/2}\|_{L_2^0} < \infty, \quad \textit{for some } \beta \in [0,1].$$

Proof. We first prove (i). With $\{\gamma_l, e_l\}_{l=1}^\infty$ the eigensystem of Q in H, it is easy to show that $g_l = Q^{1/2}e_l = \gamma_l^{1/2}e_l$ is an orthonormal basis of $Q^{1/2}(H)$. In fact,

$$(g_l, g_k)_{Q^{1/2}(H)} = (Q^{-1/2}g_l, Q^{1/2}g_k) = (e_l, e_k) = \delta_{l,k}.$$

Note that

$$\sum_{l=1}^\infty |g_l|^2_{\beta-1} = \sum_{l=1}^\infty \|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 = \|A^{(\beta-1)/2}\|_{L_2^0} < \infty,$$

which means that the embedding of $Q^{1/2}(H)$ into $\dot{H}^{\beta-1}$ is Hilbert–Schmidt. By Lemma 4.11 in Da Prato and Zabczyk [7], $W(t)$ defines an $\dot{H}^{\beta-1}$-valued Wiener process with covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$. It is obvious that $\tilde{Q} = Q$ if $\mathrm{Tr}(Q) < \infty$.

We now turn to (ii). Since $W(t) = \sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t)$, $t \geq 0$, is an $\dot{H}^{\beta-1}$-valued Wiener process with the covariance operator $\tilde{Q}$, $\mathrm{Tr}(\tilde{Q}) < \infty$, we have

$$\mathbf{E}|W(t)|_{\beta-1}^2 < \infty.$$

With $\{\lambda_l, e_l\}_{l=1}^{\infty}$ the eigensystem of A in H, we have

$$\begin{aligned} \mathbf{E}|W(t)|_{\beta-1}^2 &= \mathbf{E}\left|\sum_{l=1}^{\infty} Q^{1/2} e_l \beta_l(t)\right|_{\beta-1}^2 \\ &= \mathbf{E}\sum_{l=1}^{\infty} \lambda_l^{\beta-1} \gamma_l \beta_l(t)^2 = t\|A^{(\beta-1)/2}\|_{L_2^0}, \end{aligned}$$

which implies that $\|A^{(\beta-1)/2}\|_{L_2^0} < \infty$ for $\beta \in [0,1]$. The proof is complete. □

4 Error estimates for a deterministic problem.

In order to prove our error estimates for the stochastic partial differential equations, we need some nonsmooth data error estimates for the corresponding homogeneous deterministic parabolic equation.

Let us first consider the stationary problem

$$-\Delta u = f \quad \text{in } \mathcal{D}, \text{ with } u = 0 \text{ on } \partial\mathcal{D}, \tag{4.1}$$

where $f \in \dot{H}^{-1}$.

The variational form of (4.1) is to find $u \in H_0^1$ such that

$$(\nabla u, \nabla \phi) = \langle f, \phi \rangle, \quad \forall \phi \in H_0^1, \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\dot{H}^{-1}$ and H_0^1.

Let $S_h \subset H_0^1$ be the finite element space. The semidiscrete problem of (4.2) is to find $u_h \in S_h$ such that

$$(\nabla u_h, \nabla \chi) = \langle f, \chi \rangle, \quad \forall \chi \in S_h. \tag{4.3}$$

By the Lax–Milgram lemma, there exist unique solutions $u \in H_0^1$ and $u_h \in S_h$ such that (4.2) and (4.3) hold. Moreover the following stability result holds:

$$|u|_1 \leq C|f|_{-1}, \quad \forall f \in \dot{H}^{-1}. \tag{4.4}$$

The standard error estimates read:

$$\|u_h - u\| \leq Ch^s |u|_s, \quad s = 1, 2. \tag{4.5}$$

Let $G : \dot{H}^{-1} \to H_0^1$ denote the exact solution operator of (4.1), i.e., $u = Gf$. We define the linear operator $G_h : \dot{H}^{-1} \to S_h$ by $G_h f = u_h$, so that $u_h = G_h f \in S_h$ is the approximate solution of (4.2). It is easy to see that G_h is selfadjoint, positive semidefinite on H, and positive definite on S_h. Introducing the elliptic projection $R_h : H_0^1 \to S_h$ by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall v \in H_0^1.$$

We see that $G_h = R_h G$, and $R_h v$ is the finite element approximation of the solution of the corresponding elliptic problem with exact solution v. By (4.5), we get

$$\|R_h v - v\| \leq Ch^s |v|_s, \quad s = 1, 2.$$

Hence, using (4.4) and the elliptic regularity estimate, we have

$$\|(G_h - G)f\| = \|(R_h - I)Gf\| \leq Ch^s |Gf|_s = Ch^s |f|_{s-2}, \quad \text{for } s = 1, 2, \tag{4.6}$$

which we need below.

Let $E_h(t) = \mathrm{e}^{-tA_h}$ with $A_h = G_h^{-1}$, and let $E(t) = \mathrm{e}^{-tA}$ with $A = G^{-1}$. We then have the following error estimates for the deterministic parabolic problem.

Lemma 4.1. *Let $F_h(t) = E_h(t)P_h - E(t)$. Then*

$$\|F_h v\|_{L_\infty([0,T];H)} \leq Ch^\beta |v|_\beta, \quad \textit{for } v \in \dot{H}^\beta, \ 0 \leq \beta \leq 1, \tag{4.7}$$

and

$$\|F_h v\|_{L_2([0,T];H)} \leq Ch^\beta |v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1}, \ 0 \leq \beta \leq 1. \tag{4.8}$$

Further, in the weak norm,

$$\|F_h v\|_{L_\infty([0,T];\dot{H}^{-1})} \leq Ch^\beta |v|_{\beta-1}, \quad \textit{for } v \in \dot{H}^{\beta-1}, \ 1 \leq \beta \leq 2, \tag{4.9}$$

and, with $\ell_h = \log(T/h^2)$,

$$\|F_h v\|_{L_2([0,T];\dot{H}^{-1})} \leq Ch^\beta \ell_h |v|_{\beta-2}, \quad \textit{for } v \in \dot{H}^{\beta-2}, \ 1 \leq \beta \leq 2. \tag{4.10}$$

Proof. We denote $u(t) = E(t)v$, $u_h(t) = E_h(t)P_h v$, and $e(t) = u_h(t) - u(t) = F_h(t)v$. We first show (4.7). By the stability properties of the L_2 projection operator P_h and the solution operators $E_h(t)$ and $E(t)$, we have

$$\|e(t)\| = \|E_h(t)P_h v - E(t)v\| \leq 2\|v\|, \quad \text{for } t \geq 0, \ v \in H. \tag{4.11}$$

We will show that

$$\|e(t)\| \leq Ch|v|_1, \quad \text{for } t \geq 0, \ v \in \dot{H}^1. \tag{4.12}$$

Combining this with interpolation theory, we get (4.7).

To show (4.12), let us consider the error equation

$$G_h e_t + e = \rho, \tag{4.13}$$

where $\rho = -(G_h - G)u_t$. We note that $G_h e(0) = 0$ for

$$(G_h e(0), w) = (P_h v - v, G_h w) = 0, \quad \text{for } w \in H, \tag{4.14}$$

since $G_h w \in S_h$.

By the energy method, we can show, see Thomée [27, Lemma 3.3],

$$\|e(t)\| \le C \sup_{s \le t} \bigl(s\|\rho_t(s)\| + \|\rho(s)\|\bigr), \quad t \ge 0.$$

Obviously, by (4.6) and Lemma 3.2,

$$\|\rho(s)\| = \|(G_h - G)u_t\| \le Ch|u_t|_{-1} \le Ch|v|_1,$$

and

$$s\|\rho_t(s)\| \le Chs|u_t(s)|_1 \le Ch|v|_1.$$

Hence (4.12) follows and therefore we get (4.7).

We next show (4.8). By interpolation theory, it suffices to show that

$$\|e\|_{L_2([0,T];H)} \le C|v|_{-1}, \tag{4.15}$$

and

$$\|e\|_{L_2([0,T];H)} \le Ch\|v\|. \tag{4.16}$$

Taking the inner product of (4.13) with e, we get

$$(G_h e_t, e) + (e, e) = (\rho, e).$$

Integrating with respect to t, we get, noting that $G_h e(0) = 0$ and using the inequality $(\rho, e) \le \frac{1}{2}(\|\rho\|^2 + \|e\|^2)$,

$$(G_h e(T), e(T)) + \int_0^T \|e\|^2 \, \mathrm{d}t \le \int_0^T \|\rho\|^2 \, \mathrm{d}t. \tag{4.17}$$

Obviously, by (4.6) and Lemma 3.2,

$$\int_0^T \|\rho\|^2 \, \mathrm{d}t \le \int_0^T \|(G_h - G)u_t\|^2 \, \mathrm{d}t \le Ch^2 \int_0^T |u|_1^2 \, \mathrm{d}t \le Ch^2\|v\|^2, \tag{4.18}$$

which implies that (4.16) holds.

To show (4.15), we note that, by Lemma 3.2 and its discrete counterpart,

$$\int_0^T \|e\|^2 \, \mathrm{d}t \le 2\int_0^T \bigl(\|u_h\|^2 + \|u\|^2\bigr) \, \mathrm{d}t \le 2|v|_{-1,h}^2 + 2|v|_{-1}^2, \tag{4.19}$$

where $|v|_{-1,h}$ is a discrete seminorm defined by

$$|v|_{-1,h} = (G_h v, v)^{1/2} = \|G_h^{1/2} v\|.$$

Since $|v|_{-1} = \sup\{(v,w)/|w|_1 : w \in \dot{H}^1\}$, see Thomée [27, Chapter 6], we thus have, with $w = G_h v,\ v \in \dot{H}^{-1}$,

$$|v|_{-1} = \sup_{w\in \dot{H}^1} \frac{(v,w)}{|w|_1} \geq \frac{(v,G_hv)}{|G_hv|_1} = \frac{(v,G_hv)}{(v,G_hv)^{1/2}} = |v|_{-1,h},$$

since

$$|G_hv|_1^2 = (AG_hv, G_hv) = A(G_hv,G_hv) = (A_hG_hv,G_hv) = (v,G_hv),$$

where $A_h = G_h^{-1}$. Hence by (4.19), we get $\int_0^T \|e\|^2\,\mathrm{d}t \leq 4|v|_{-1}^2$, which implies that (4.15) holds.

We now turn to (4.9). It suffices to show that

$$|e(t)|_{-1} \leq Ch\|v\|, \tag{4.20}$$

and

$$|e(t)|_{-1} \leq Ch^2|v|_1. \tag{4.21}$$

By (4.17) and (4.18), we have

$$(G_he,e) = |e|_{-1,h}^2 \leq Ch^2\|v\|^2. \tag{4.22}$$

Using

$$|e|_{-1} \leq |e|_{-1,h} + Ch\|e\|, \tag{4.23}$$

which follows from, by (4.6),

$$|e|_{-1}^2 = (G_he,e) + ((G-G_h)e,e) \leq |e|_{-1,h}^2 + Ch^2\|e\|^2,$$

we obtain, by (4.11)

$$|e|_{-1} \leq |e|_{-1,h} + Ch\|e\| \leq Ch\|v\|,$$

which is (4.20).

By (4.17) and (4.6), we obtain

$$|e(t)|_{-1,h}^2 = (G_he(t),e(t)) \leq \frac{1}{2}\int_0^t \|\rho\|^2\,\mathrm{d}s \leq Ch^4\int_0^t |u_t|^2\,\mathrm{d}s \leq Ch^4|v|_1^2.$$

Combining this with (4.12) and (4.23), we get (4.21).

It remains to show (4.10). Integrating (4.13) with respect to t, we have, with $\tilde{e}(t) = \int_0^t e(s)\,\mathrm{d}s,\ \tilde{\rho}(t) = \int_0^t \rho(s)\,\mathrm{d}s$,

$$G_he + \tilde{e} = \tilde{\rho}, \qquad \tilde{e}(0) = 0. \tag{4.24}$$

Taking the inner product of (4.24) with e, we get, since $e = \tilde{e}_t$,

$$(G_he,e) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{e}\|^2 = (\tilde{\rho},e) = \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\rho},\tilde{e}) - (\rho,\tilde{e}).$$

After integration, we have, noting that $\tilde{e}(0)=0$,

$$\begin{aligned}\int_0^T |e|_{-1,h}^2\,\mathrm{d}s + \frac{1}{2}\|\tilde{e}(T)\|^2 &= \int_0^T (\tilde{\rho}, e)\,\mathrm{d}s = \big[(\tilde{\rho},\tilde{e})\big]_0^T - \int_0^T (\rho,\tilde{e})\,\mathrm{d}s \\ &\le \|\tilde{\rho}(T)\|\|\tilde{e}(T)\| + \Big(\int_0^T \|\rho\|\,\mathrm{d}s\Big) \sup_{0\le s\le T}\|\tilde{e}(s)\| \\ &\le 2\Big(\int_0^T \|\rho\|\,\mathrm{d}s\Big)\sup_{0\le s\le T}\|\tilde{e}(s)\|.\end{aligned}$$

By a kick-back argument, we obtain

$$\int_0^T |e|_{-1,h}^2\,\mathrm{d}s \le C\Big(\int_0^T \|\rho\|\,\mathrm{d}s\Big)^2.$$

Noting that

$$\begin{aligned}\int_0^T \|\rho\|\,\mathrm{d}s &= \int_0^{h^2}\|\rho\|\,\mathrm{d}s + \int_{h^2}^T \|\rho\|\,\mathrm{d}s \\ &\le C\int_0^{h^2} s^{-1/2}|v|_{-1}\,\mathrm{d}s + C\int_{h^2}^T h|u|_1\,\mathrm{d}s \le Ch\ell_h|v|_{-1},\end{aligned}$$

and, similarly,

$$\begin{aligned}\int_0^T \|\rho\|\,\mathrm{d}s &= \int_0^{h^2}\|\rho\|\,\mathrm{d}s + \int_{h^2}^T \|\rho\|\,\mathrm{d}s \\ &\le Ch^2\|v\| + Ch^2\int_{h^2}^T |u|_2\,\mathrm{d}s \\ &\le Ch^2\|v\| + Ch^2\log(T/h^2)\|v\| \le Ch^2\ell_h\|v\|,\end{aligned}$$

we therefore get

$$\int_0^T |e|_{-1,h}^2\,\mathrm{d}s \le Ch^2\ell_h^2|v|_{-1}^2, \tag{4.25}$$

and

$$\int_0^T |e|_{-1,h}^2\,\mathrm{d}s \le Ch^4\ell_h^2\|v\|^2. \tag{4.26}$$

By (4.19), (4.23), and (4.25), we obtain

$$\begin{aligned}\int_0^T |e|_{-1}^2\,\mathrm{d}s &\le C\int_0^T |e|_{-1,h}^2\,\mathrm{d}s + Ch^2\int_0^T \|e\|^2\,\mathrm{d}s \\ &\le Ch^2\ell_h^2|v|_{-1}^2 + Ch^2|v|_{-1}^2 \le Ch^2\ell_h^2|v|_{-1}^2,\end{aligned}$$

and, by (4.26)

$$\int_0^T |e|_{-1}^2\,\mathrm{d}s \le Ch^4\ell_h^2\|v\|^2 + Ch^4\|v\|^2 \le Ch^4\ell_h^2\|v\|^2.$$

Now (4.10) follows from the interpolation theory. The proof is complete. □

5 Proofs of Theorems 1.1 and 1.2.

In this section, we prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We have, with $E(t) = \mathrm{e}^{-tA}$,

$$u(t) = E(t)u_0 + \int_0^t E(t-s)\,\mathrm{d}W(s),$$

and, with $E_h(t) = \mathrm{e}^{-tA_h}$,

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t-s)P_h\,\mathrm{d}W(s).$$

Denoting $e(t) = u_h(t) - u(t)$ and $F_h(t) = E_h(t)P_h - E(t)$, we write

$$\begin{aligned} e(t) &= E_h(t)P_h u_0 - E(t)u_0 + \int_0^t \big(E_h(t-s)P_h - E(t-s)\big)\,\mathrm{d}W(s) \\ &= F_h(t)u_0 + \int_0^t F_h(t-s)\,\mathrm{d}W(s) = I + II. \end{aligned}$$

Thus

$$\|e(t)\|_{L_2(\Omega;H)} \leq 2\big(\|I\|_{L_2(\Omega;H)} + \|II\|_{L_2(\Omega;H)}\big).$$

For I, we have, by (4.7) with $v = u_0$,

$$\|I\| = \|F_h(t)u_0\| \leq Ch^{\beta}|u_0|_{\beta}, \quad \text{for } 0 \leq \beta \leq 1,$$

which implies that $\|I\|_{L_2(\Omega;H)} \leq Ch^{\beta}\|u_0\|_{L_2(\Omega;\dot{H}^{\beta})}$, for $0 \leq \beta \leq 1$.

For II, we have, by the isometry property,

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &= \left\|\mathbf{E}\int_0^t F_h(t-s)\,\mathrm{d}W(s)\right\|^2 = \int_0^t \|F_h(t-s)\|^2_{L_2^0}\,\mathrm{d}s \\ &= \sum_{l=1}^{\infty}\int_0^t \|F_h(t-s)Q^{1/2}e_l\|^2\,\mathrm{d}s, \end{aligned}$$

where $\{e_l\}$ is any orthonormal basis in H.

Using (4.8) with $v = Q^{1/2}e_l$, we obtain

$$\begin{aligned} \|II\|^2_{L_2(\Omega;H)} &\leq C\sum_{l=1}^{\infty} h^{2\beta}\|Q^{1/2}e_l\|^2_{\beta-1} = C\sum_{l=1}^{\infty} h^{2\beta}\|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 \\ &= Ch^{2\beta}\|A^{(\beta-1)/2}\|^2_{L_2^0}, \end{aligned}$$

which completes the proof of (1.6).

In particular, if $W(t)$ is a Wiener process with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in (1.6) and obtain (1.7), since $\|I\|^2_{L_2^0} = \mathrm{Tr}(Q)$. □

As in Corollary 3.4, we may specialize to $Q = I$ if $d = 1$:

COROLLARY 5.1. *Let u_h and u be the solutions of* (1.5) *and* (1.1)*, respectively. Assume that $A = -\partial^2/\partial x^2$ with $\mathcal{D}(A) \subset H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $t \geq 0$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;H)} \leq Ch^\beta(1 + \|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \textit{for } 0 \leq \beta < 1/2.$$

Now we turn to consider the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Using the same notation as in Theorem 1.1, we have, by (4.9),

$$\|I\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^{\beta+1}\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}, \quad \text{for } 0 \leq \beta \leq 1.$$

For II, we have, by the isometry property, and (4.10) with $v = Q^{1/2}e_l$,

$$\begin{aligned}
\|II\|^2_{L_2(\Omega;\dot{H}^{-1})} &= \mathbf{E}\Big|\int_0^t F_h(t-s)\,\mathrm{d}W(s)\Big|^2_{-1} = \mathbf{E}\Big\|\int_0^t A^{-1/2}F_h(t-s)\,\mathrm{d}W(s)\Big\|^2 \\
&= \int_0^t \|A^{-1/2}F_h(t-s)\|^2_{L_2^0}\,\mathrm{d}s \leq Ch^{2\beta}\ell_h^2 \sum_{l=1}^\infty \|A^{(\beta-1)/2}Q^{1/2}e_l\|^2 \\
&\leq Ch^{2(\beta+1)}\ell_h^2\|A^{(\beta-1)/2}\|^2_{L_2^0},
\end{aligned}$$

which completes the proof of (1.8).

In particular, if $W(t)$ is a Wiener process on H with $\mathrm{Tr}(Q) < \infty$, then we can choose $\beta = 1$ in (1.8) and obtain (1.9). □

COROLLARY 5.2. *Let u_h and u be the solutions of* (1.5) *and* (1.1)*, respectively. Assume that $A = -\partial^2/\partial x^2$ and $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$. If $W(t)$ is a cylindrical Wiener process with $Q = I$, then we have, for $0 \leq t \leq T$ and $u_0 \in L_2(\Omega; \dot{H}^\beta)$, with $\ell_h = \log(T/h^2)$,*

$$\|u_h(t) - u(t)\|_{L_2(\Omega;\dot{H}^{-1})} \leq Ch^{\beta+1}(1 + \ell_h\|u_0\|_{L_2(\Omega;\dot{H}^\beta)}), \quad \textit{for } 0 \leq \beta < 1/2.$$

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