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Response spectrum analysis for non-classically damped linear system with multiple-support excitations

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Abstract A new response spectrum method, which is named complex multiple-support response spectrum (CMSRS) method in this article, is developed for seismic analysis of non-classically damped linear system subjected to spatially varying multiple-supported ground motion. The CMSRS method is based on fundamental principles of random vibration theory and properly accounts for the effect of correlation between the support motions as well as between the modal displacement and velocity responses of structure, and provides an reasonable and acceptable estimate of the peak response in term of peak seismic ground motions and response spectra at the support points and the coherency function. Meanwhile, three new cross-correlation coefficients or cross covariance especially for the non-classically damped linear structures with multiple-supports excitations are derived under the same assumptions of the MSRS method of classically damped system. The CMSRS method is examined and compared to the results of time history analyses in two numerical examples of non-classically damped structures in consideration of the coherences of spatially variable ground motion. The results show that for non-classically damped structure, the cross terms representing the cross covariance between the pseudo-static and dynamic component are also quite small just as same as classically damped system. In addition, it is found that the usual way of neglecting all the off-diagonal elements in transformed damping matrix in modal coordinates in order to make the concerned non-classically damped structure to become remaining

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proportional damping property will bring some errors in the case of subjected to spatially excited inhomogeneous ground motion.

Keywords Non-classical damping · Multiple-support excitations · Response spectrum · Seismic ground motion

Introduction

The seismic energy releases from earthquake source and propagates in earth's crust and then reaches to ground surface in form of seismic waves. Because the seismic waves on different surface points come from different routes and via different topographical formation and geological structures, the ground motion at different point of an area are generally different. For buildings with small plane size, such as normal public and residential buildings, the spatial variation of ground motions could be ignored. But the spatial variation of ground motions could affect significantly for the long span structure (Housner et al. 1990), such as large bridge, nuclear power station, tunnel, dam and aqueduct.

At present, in the structural engineering practice, representation of the input excitation in term of power spectral density function (PSDF) is not as popular as the mean response spectrum from the practice design point of view. Also, in earthquake engineering, frequency characteristics of ground motion are commonly incorporated in the analysis through response spectrum, and many existing structural codes and specification are based on the response spectrum method. However, use of the response spectrum method for long span structures involving multiple-support excitation is not straightforward. Several investigators extended the response spectrum method for the case of multiple-support excitation. A simple and approximate response spectrum technique was proposed for the multiple-support excitation problem by Rutenberg and Heidebrecht (1987). Dong and Wieland (1988) studied the response spectrum method for multiple-support excitations by comparison with the time history using several combination rules. Yamamura and Tanaka (1990) studied the response of flexible MDOF systems to multiple-support excitations by dividing the ground motion of the supports into independent subgroups, whereas inside each subgroup it is considered perfectly correlated. Berrah and Kausel (1992) included the coherency effect in the response spectrum analysis of structures. Der Kiureghian (1991, 1992, 1997) developed a responses spectrum method for multiple-support excitations using the principles of random vibration for classically damped linear system. In addition, much advanced researches on dynamic response analysis method of structure subjected to multiple-support excitations have been done by many researchers (Lou and Ku 1995; Allam and Datta 2000; Kato et al. 2002, 2003; Heredia-Zavoni and Leyva 2003; Liu et al. 2005).

Recently, the dynamic analysis of non-classically damped linear systems has been paid more attention because it is noticed that there are many structures whose damping are non-uniform, for instance, soil-structure interacting system, composite or hybrid structures composed of different materials with different damping and structures equipped with supplemental linear viscous dampers, such as oil dampers. Following similar procedures as deduction of multiple-support response spectrum analysis (MSRS) algorithm (Der Kiureghian, 1991, 1992, 1997) for classically damped linear system, a new response spectrum method is developed for seismic response analysis of non-classically damped linear systems subjected to spatially varying multiple-support ground motion, which is named CMSRS method in this article. This method is based on fundamental principles of random vibration theory and properly accounts for the effect of correlation between the support motions as well as between the modal displacement and velocity responses of structure, and provides an reasonable estimate of the peak response in term of peak ground motions and response spectra at the support points and the coherency function with certain accuracy. Meanwhile, three new cross-correlation coefficients or cross covariance especially for the non-classically damped structures with multiple-support excitations are derived under the same assumptions of the MSRS method for classically damped system, i.e., cross-correlation coefficient between ground displacement and oscillator velocity response, and the cross-correlation coefficients of velocity-velocity and displacement-velocity of oscillators. Furthermore, two numerical examples have been used to examine the desirability of CMSRS method, and the comparison results with time history analysis show this method is acceptable from viewpoint of engineering. Also, it is found that, for non-classically damped linear structure, the cross term representing the covariance between the pseudo-static component and dynamic component is usually smaller compared to the corresponding individual covariance responses of pseudo-static and dynamic components. In addition, if adopting corresponding proportional damping system, which is deduced through neglecting all the off-diagonal elements in transformed damping matrix in modal coordinates of the non-classically damped linear structural system, some errors will be brought in.

Equations of motion

The equations of motion for a discrete, N-degree-of-freedom linear structural system subjected to m support motions can be written in the partitioned matrix form (Clough and Penzien 1975)

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{M}_c \\ \boldsymbol{M}_c^T & \boldsymbol{M}_g \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{y}} \\ \ddot{\boldsymbol{u}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C}_c \\ \boldsymbol{C}_c^T & \boldsymbol{C}_g \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{u}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}_c \\ \boldsymbol{K}_c^T & \boldsymbol{K}_g \end{bmatrix} \begin{Bmatrix} \boldsymbol{y} \\ \boldsymbol{u} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \boldsymbol{P} \end{Bmatrix} \tag{1}$$

where, $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the $N \times N$ mass, damping and stiffness matrices associated with the unconstrained degrees of freedom, respectively; $\boldsymbol{M}_g$, $\boldsymbol{C}_g$ and $\boldsymbol{K}_g$ are the $m \times m$ matrices associated with the support degrees of freedom, and m is the numbers of constrained degrees of freedom; $\boldsymbol{M}_c$, $\boldsymbol{C}_c$ and $\boldsymbol{K}_c$ are $N \times m$ coupling matrices associated with both sets of degrees of freedom; $\boldsymbol{P}$ is the m-vector of reacting forces at the support degrees of freedom; $\boldsymbol{y}$ is the total displacement vector at the unconstrained degrees of freedom, $\boldsymbol{u}$ is the m-vector of prescribed support displacements.

From the Eq. 1, we can get:

$$\boldsymbol{M}\ddot{\boldsymbol{y}} + \boldsymbol{C}\dot{\boldsymbol{y}} + \boldsymbol{K}\boldsymbol{y} = -(\boldsymbol{M}_c\ddot{\boldsymbol{u}} + \boldsymbol{C}_c\dot{\boldsymbol{u}} + \boldsymbol{K}_c\boldsymbol{u}) \tag{2}$$

In the analysis of such system, it is common to decompose the response $\boldsymbol{y}$ into pseudo-static component $\boldsymbol{y}^s$ and dynamic component $\boldsymbol{y}^d$. Following the conventional procedure, we get

$$\boldsymbol{y} = \boldsymbol{y}^s + \boldsymbol{y}^d \tag{3}$$

here pseudo-static component $\boldsymbol{y}^s$ is structural displacement caused by forced supports displacements statistically. Substituting Eq. 3 into Eq. 2, we can obtain the following equation

$$\begin{aligned} &\boldsymbol{M}\ddot{\boldsymbol{y}}^d + \boldsymbol{C}\dot{\boldsymbol{y}}^d + \boldsymbol{K}\boldsymbol{y}^d \\ &\quad = -\left(\begin{bmatrix} \boldsymbol{M} & \boldsymbol{M}_c \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{y}}^s \\ \ddot{\boldsymbol{u}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{C} & \boldsymbol{C}_c \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{y}}^s \\ \dot{\boldsymbol{u}} \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K} & \boldsymbol{K}_c \end{bmatrix} \begin{Bmatrix} \boldsymbol{y}^s \\ \boldsymbol{u} \end{Bmatrix} \right) \end{aligned} \tag{4}$$

In fact for the each instantaneous, the third term in the right-hand of Eq. 4 always remains zero, and then we can gain the following relationship

$$\boldsymbol{y}^s = -\boldsymbol{K}^{-1}\boldsymbol{K}_c\boldsymbol{u} = \boldsymbol{R}\boldsymbol{u} \tag{5}$$

in which $\boldsymbol{R} = -\boldsymbol{K}^{-1}\boldsymbol{K}_c$ is denoted the influence matrix.

Substituting Eq. 5 into Eq. 4, the dynamic component of the response is obtained in the differential form

$$\boldsymbol{M}\ddot{\boldsymbol{y}}^d + \boldsymbol{C}\dot{\boldsymbol{y}}^d + \boldsymbol{K}\boldsymbol{y}^d = -\left(\boldsymbol{M}\boldsymbol{R} - \boldsymbol{M}_c\right)\ddot{\boldsymbol{u}} - \left(\boldsymbol{C}\boldsymbol{R} - \boldsymbol{C}_c\right)\dot{\boldsymbol{u}} \tag{6}$$

Whether classical damping structures or not, the damping forces on the right-side of Eq. 6 might be neglected, which are normally believed to be much smaller than the corresponding inertia forces, the other part standing on the same side. Therefore Eq. 6 can be simplified as

$$\boldsymbol{M}\ddot{\boldsymbol{y}}^d + \boldsymbol{C}\dot{\boldsymbol{y}}^d + \boldsymbol{K}\boldsymbol{y}^d = -\left(\boldsymbol{M}\boldsymbol{R} - \boldsymbol{M}_c\right)\ddot{\boldsymbol{u}} \tag{7}$$

It is noted that $\boldsymbol{M}_c = 0$ if a lumped mass model is used. In this paper, since we assume $\boldsymbol{M}_c = 0$ as usual, and hence Eq. 7 can be written

$$\boldsymbol{M}\ddot{\boldsymbol{y}}^d + \boldsymbol{C}\dot{\boldsymbol{y}}^d + \boldsymbol{K}\boldsymbol{y}^d = -\boldsymbol{M}\boldsymbol{R}\ddot{\boldsymbol{u}} \tag{8}$$

Complex mode superposition method of seismic response for non-classically damped system

For the classically damped linear system, as we know, the free vibration equation corresponding to Eq. 8 is able to be decoupled to classical modes if the damping matrix satisfies necessary and satisfactory condition proved by Caughey (1960) and Caughey and O'Kelly (1965). However, for the non-classically damped linear system, neglecting the effect of damping matrix will produce some errors. Therefore, Eq. 8 can be solved by using decoupled method suggested by Foss (1958), i.e.,

$$\boldsymbol{H}\dot{\boldsymbol{x}} + \boldsymbol{D}\boldsymbol{x} = -\ddot{u}_g\boldsymbol{H}\boldsymbol{E} \tag{9}$$

in which

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{C} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} -\boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} \dot{\boldsymbol{y}}^d \\ \boldsymbol{y}^d \end{bmatrix}, \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{R} \\ \boldsymbol{0} \end{bmatrix} \tag{10}$$

The solution of eigenproblem corresponding Eq. 9 can be transformed into the solution of following equation

$$\boldsymbol{D}\boldsymbol{\Phi} = -\mu\boldsymbol{H}\boldsymbol{\Phi} \tag{11}$$

in which μ and $\boldsymbol{\Phi}$ are generic eigenvalue and eigenvector respectively, and according to formula (10), $\boldsymbol{\Phi}$ can be given as

$$\boldsymbol{\Phi} = \begin{bmatrix} \mu\phi & \phi \end{bmatrix}^T \tag{12}$$

where ϕ is the complex mode shape vector which is regarded $\phi = \varphi + i\psi$.

Because matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are symmetric in general and so the eigenvalues and the eigenvectors of Eq. 11 normally occurs in complex conjugate pairs, but for highly damped systems, an even number of them can be real (Inman and Andry Jr 1980). However, in some other particular cases, some of the eigenvalues could be multi-fold, which means the

characteristics equation of the non-classically damped system comprises re-roots. These cases will not be handled in this article.
Suppose:

$$\mu_j = -\alpha_j + i\beta_j \tag{13}$$

where $\alpha_j = \zeta_j\omega_j, \beta_j = \omega_{Dj} = \omega_j\sqrt{1-\zeta_j^2}$ are damping coefficient and damping frequency of the jth mode respectively, the free vibration frequency ω_j and the corresponding critical damping ratio ζ_j can be deduced from the general orthogonality relations (Foss 1958).
Substitute transformation

$$\boldsymbol{x} = \sum_{j=1}^{2N} \boldsymbol{\Phi}_j s_j(t) \tag{14}$$

into Eq. 9 and employ the generated orthogonal relation of eigenvectors, the decoupled equations of motion are obtained as

$$\dot{s}_j(t) + \mu_j s_j(t) = -\sum_{k=1}^{m} \eta_{kj} \ddot{u}_{gk}(t) \tag{15}$$

in which, $s_j(t)$ is the displacement response of the jth SDOF oscillator with frequency ω_j and damping ratio ζ_j at the given input force. The index k denotes the degree of freedom associated with the prescribed support motion, the subscript j denotes the mode number, and μ_j represents the structural complex eigenvalue, i.e.,

$$\mu_j = \frac{\left(\boldsymbol{\Phi}_j\right)^T \boldsymbol{D}\boldsymbol{\Phi}_j}{\left(\boldsymbol{\Phi}_j\right)^T \boldsymbol{H}\boldsymbol{\Phi}_j} \tag{16}$$

and η_{kj} is the modal participation factor given by

$$\eta_{kj} = \frac{(\boldsymbol{\Phi}_j)^T \boldsymbol{H}\boldsymbol{E}_k}{L_j} \tag{17}$$

where $\boldsymbol{E}_k$ is the kth column of the matrix $\boldsymbol{E}$, a composite matrix of influence matrix $\boldsymbol{R}$ and zero matrix, and denominator L_j is given by

$$L_j = \left(\boldsymbol{\Phi}_j\right)^T \boldsymbol{H}\boldsymbol{\Phi}_j \tag{18}$$

Separating the right part of Eq. 18 into real and imaginary parts, the following formula is available after some simplification.

$$L_j = e_j + if_j \tag{19}$$

in which

$$\begin{aligned} e_j = &-2\alpha_j((\varphi_j)^T \boldsymbol{M}\varphi_j - (\psi_j)^T \boldsymbol{M}\psi_j) - 4\beta_j(\varphi_j)^T \boldsymbol{M}\psi_j + (\varphi_j)^T \boldsymbol{C}\varphi_j \\ &-(\psi_j)^T \boldsymbol{C}\psi_j \end{aligned} \tag{20}$$

$$f_j = 2\beta_j\left((\varphi_j)^T \boldsymbol{M}\varphi_j - (\psi_j)^T \boldsymbol{M}\psi_j\right) - 4\alpha_j(\varphi_j)^T \boldsymbol{M}\psi_j + 2(\varphi_j)^T \boldsymbol{C}\psi_j \tag{21}$$

Substituting Eq. 19 into Eq. 17 and separating the numerator of right part of Eq. 17, modal participation factor can be expressed as

$$\eta_{kj} = \frac{1}{e_j^2 + f_j^2}\left[e_j(\varphi_j)^T \boldsymbol{M}\boldsymbol{R}_k + f_j(\psi_j)^T \boldsymbol{M}\boldsymbol{R}_k + i\left(e_j(\psi_j)^T \boldsymbol{M}\boldsymbol{R}_k - f_j(\varphi_j)^T \boldsymbol{M}\boldsymbol{R}_k\right)\right] \tag{22}$$

It is convenient to define a normalized response $q_{kj}(t)$, representing the response of a single-degree-of-freedom oscillator with unit mass, frequency ω_j and damping ratio ζ_j, which is subjected to the base motion $\ddot{u}_k(t)$. Substituting Eq. 22 into Eq. 14 and combining the terms consisted of a pair of conjugated complex modes, we can get

$$\boldsymbol{y}^d(t) = \sum_{k=1}^{m}\sum_{j=1}^{N}\left[\boldsymbol{A}_{kj} q_{kj}(t) + \boldsymbol{B}_{kj}\dot{q}_{kj}(t)\right] \tag{23}$$

in which

$$\boldsymbol{A}_{kj} = -\frac{2}{e_j^2 + f_j^2}\left[(\zeta_j p_{kj} + \sqrt{1-\zeta_j^2}w_{kj})\varphi_j + (\zeta_j w_{kj} - \sqrt{1-\zeta_j^2}p_{kj})\psi_j\right]\omega_j \tag{24}$$

$$\boldsymbol{B}_{kj} = -\frac{2}{e_j^2 + f_j^2}\left(p_{kj}\varphi_j + w_{kj}\psi_j\right) \tag{25}$$

$$\begin{aligned} p_{kj} &= e_j c_{kj} + f_j d_{kj}, \quad w_{kj} = f_j c_{kj} - e_j d_{kj}, \\ c_{kj} &= (\varphi_j)^T \boldsymbol{M} R_k, \quad d_{kj} = (\psi_j)^T \boldsymbol{M} R_k \end{aligned} \tag{26}$$

and $q_{kj}(t)$ can be expressed as solution of the following equation.

$$\ddot{q}_{kj}(t) + 2\zeta_j\omega_j\dot{q}_{kj}(t) + \omega_j^2 q_{kj}(t) = -\ddot{u}_k(t) \tag{27}$$

It is worth pointing out that Eq. 23 in this article is apparently the same as Eq. 25 in Igusa et al.'s paper (1984), except an extra summation is introduced here to consider the influence of multiple-support excitation, and Wang (1994) also gave similar results. However we use real value form formula (Zhou et al. 2004) to calculate the coefficient vectors involved in Eq. 23 efficiently. A generic response quantity or effect of interest, $\boldsymbol{z}(t)$ (e.g., a nodal displacement, an internal force, stress or strain component), in general can be expresses as a linear function of the nodal displacements $\boldsymbol{y}(t)$, i.e.,

$$\boldsymbol{z}(t) = \boldsymbol{v}^T\boldsymbol{y}(t) = \boldsymbol{v}^T\left[\boldsymbol{y}^s(t) + \boldsymbol{y}^d(t)\right] \tag{28}$$

where $\boldsymbol{v}$ is a response transfer vector which usually depends on the geometry and stiffness properties of the structure. Substituting for the pseudo-static component and for the dynamic component in terms of the normalized modal responses, the generic response $\boldsymbol{z}(t)$ is written as

$$\boldsymbol{z}(t) = \sum_{k=1}^{m} g_k u_k(t) + \sum_{k=1}^{m}\sum_{j=1}^{N}\left[a_{kj}q_{kj}(t) + b_{kj}\dot{q}_{kj}(t)\right] \tag{29}$$

where, $g_k = \boldsymbol{v}^T\boldsymbol{R}_k, a_{kj} = \boldsymbol{v}^T\boldsymbol{A}_{kj}, b_{kj} = \boldsymbol{v}^T\boldsymbol{B}_{kj}$ are denoted effective influence coefficients and effective modal participation factors, respectively. It is important to note that g_k, a_{kj} and b_{kj} are functions only of the structural properties, and that $q_{kj}(t)$ is dependent only on the jth modal frequency and damping ratio and the kth input motion. Clearly, the first sum on the right-hand side of Eq. 29 represents the pseudo-static component of the response and the double-sum term represents the dynamic component.

Mean-square stationary response of the system under random disturbance

The response spectrum formulation will be developed based on the random vibration theory. We firstly assume that the support motions $\ddot{\boldsymbol{u}}_k$ are jointly stationary processes with zero means, and that the response in each mode of the structure is also stationary. These assumptions are reasonable for the intended purpose as long as the fundamental period of vibration of structure is short in relation to the duration of excitation. The stationary assumption will be relaxed later when the response spectrum method is developed. Using Eq. 29, the power spectral density of the generic steady state response $z\,(t)$ can be written as

$$\begin{aligned} G_{zz}\,(\omega) = &\sum_{k=1}^{m}\sum_{l=1}^{m} g_k g_l G_{u_k u_l}\,(i\omega) + 2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{N}\left(g_k a_{lj} + i\omega g_k b_{lj}\right) H_j\,(-i\omega)\, G_{u_k \ddot{u}_l} \\ &+ \sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(a_{ki}a_{lj} + 2i\omega b_{ki}a_{lj} + \omega^2 b_{ki}b_{lj}\right) \\ &\times H_i\,(i\omega)\, H_j\,(-i\omega)\, G_{\ddot{u}_k\ddot{u}_l}\,(i\omega) \end{aligned} \tag{30}$$

in which $G_{xy}\,(i\omega)$ denotes the cross-power spectral density of processes x and y, and $H_i\,(i\omega) = \left(\omega_i^2 - \omega^2 + 2i\zeta_i\omega_i\omega\right)^{-1}$ represents the frequency response function of mode i. Integrating over the frequency domain $-\infty < \omega < \infty$, the mean-square response is obtained as

$$\begin{aligned} \sigma_z^2 = &\sum_{k=1}^{m}\sum_{l=1}^{m} g_k g_l \rho_{u_k u_l}\sigma_{u_k}\sigma_{u_l} + 2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{N}\left[g_k a_{lj}\rho_{u_k q_{lj}}\sigma_{u_k}\sigma_{q_{lj}} + g_k b_{lj}\rho_{u_k \dot{q}_{lj}}\sigma_{u_k}\sigma_{\dot{q}_{lj}}\right] \\ &+ \sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[a_{ki}a_{lj}\rho_{q_{ki}q_{lj}}\sigma_{q_{ki}}\sigma_{q_{lj}} + 2b_{ki}a_{lj}\rho_{\dot{q}_{ki}q_{lj}}\sigma_{\dot{q}_{ki}}\sigma_{q_{lj}}\right. \\ &\left.+ b_{ki}b_{lj}\rho_{\dot{q}_{ki}\dot{q}_{lj}}\sigma_{\dot{q}_{ki}}\sigma_{\dot{q}_{lj}}\right] \end{aligned} \tag{31}$$

in which σ_{u_k} and $\sigma_{q_{ki}}$ are the mean-square-root of ground displacement $u_k\,(t)$ and normalized modal displacement response $q_{ki}\,(t)$, and the terms $\rho_{u_k u_l}$, $\rho_{u_k q_{lj}}$ and $\rho_{q_{ki}q_{lj}}$ are the corresponding cross-correlation coefficient, which were discussed by Der Kiuerghian (1991, 1992). In addition, $\sigma_{\dot{q}_{ki}}$ represents the mean-square-root of velocity response $\dot{q}_{ki}\,(t)$ and can be given by the integral

$$\sigma_{\dot{q}_{ki}}^2 = \int_{-\infty}^{\infty} \omega^2\,|H_i\,(i\omega)|^2\, G_{\ddot{u}_k\ddot{u}_k}\,(\omega) d\omega \tag{32}$$

in which $G_{\ddot{u}_k\ddot{u}_k}\,(\omega)$ is the real-valued power spectral densities of acceleration processes, which is input motion here. And the cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$, $\rho_{\dot{q}_{ki}q_{lj}}$ and $\rho_{\dot{q}_{ki}\dot{q}_{lj}}$ related to the modal velocity response in Eq. 31 can be defined by

$$\rho_{u_k \dot{q}_{lj}} = \frac{1}{\sigma_{u_k} \sigma_{\dot{q}_{lj}}} \int_{-\infty}^{\infty} i\omega H_j\left(-i\omega\right) G_{u_k \ddot{u}_l}\left(i\omega\right) d\omega \tag{33}$$

$$\rho_{\dot{q}_{ki} \dot{q}_{lj}} = \frac{1}{\sigma_{\dot{q}_{ki}} \sigma_{\dot{q}_{lj}}} \int_{-\infty}^{\infty} \omega^2 H_i\left(i\omega\right) H_j\left(-i\omega\right) G_{\ddot{u}_k \ddot{u}_l}\left(i\omega\right) d\omega \tag{34}$$

$$\rho_{\dot{q}_{ki} q_{lj}} = \frac{1}{\sigma_{\dot{q}_{ki}} \sigma_{q_{lj}}} \int_{-\infty}^{\infty} i\omega H_i\left(i\omega\right) H_j\left(-i\omega\right) G_{\ddot{u}_k \ddot{u}_l}\left(i\omega\right) d\omega \tag{35}$$

Each of the above integrands has an anti-symmetric imaginary part and, hence, their integrals have real values. The three cross-correlation coefficients related to velocity response, $\rho_{u_k \dot{q}_{lj}}$, $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$ and $\rho_{\dot{q}_{ki} q_{lj}}$ are newly introduced in this article.

The cross-correlation coefficients in Eqs. 33–35 can be interpreted in terms of a pair of oscillators representing modes i and j of the structure, which are respectively subjected to the support motions $u_k\left(t\right)$ and $u_l\left(t\right)$. Specifically, $\rho_{u_k \dot{q}_{lj}}$ denotes the cross-correlation coefficient between the forced displacement at support k and the modal velocity response of the oscillator corresponding to mode j at support l, $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$ denotes the cross-correlation coefficient between the velocity responses of the two oscillators associated to modes i and j, $\rho_{\dot{q}_{ki} q_{lj}}$ denotes the cross-correlation coefficient between the displacement-velocity responses of the two oscillators associated to modes i and j. These coefficients incorporate all the effects of cross-modal and cross-support correlations that arise in the response of the structure to the spatially varying ground motion.

Discussions of cross-correlation coefficients

For the non-classically damped linear structure system, there are six cross-correlation coefficients included in the Eq. 31. But the variations of cross-correlation coefficients, $\rho_{u_k u_l}$, $\rho_{u_k q_{lj}}$ and $\rho_{q_{ki} q_{lj}}$, are exactly the same as those discussed by Der Kiureghian (1991, 1992), so that we will only give the other cross-correlation coefficients related to velocity response, such as $\rho_{u_k \dot{q}_{lj}}$, $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$ and $\rho_{\dot{q}_{ki} q_{lj}}$, in this article.

In order to conveniently compare our results with the results calculated by Der Kiuerghian (1991, 1992), Clough and Penzien's double filter model (1975) is adopted in this paper and the filter parameters for firm soil, soft soil and the intermediate soil are the same as the reference (Der Kiuerghian 1991). Also, the following ground motion coherency function (Luco and Wong 1986) is used

$$\gamma_{kl} = \exp\left[-\left(\frac{a\omega d_{kl}}{v_s}\right)^2\right] \exp\left(i\frac{\omega d_{kl}^L}{v_{app}}\right) \tag{36}$$

in which a is an incoherence factor, d_{kl} denotes the horizontal distance between stations or supports k and l, d_{kl}^L denotes the projected horizontal distance in the longitudinal direction of wave propagation. v_s is the shear wave velocity of the medium, and v_{app} is the surface apparent wave velocity. This model is used throughout the study because of its simplicity and frequent usage by other investigators.

We still assume that v_s and v_{app} are some kind of 'average' values for each pair of supports k and l or for the entire region. Based on reported values of a and a/v_s (Luco and Wong 1986; Zerva 1990), and considering a reasonable range of distances between stations that might be of engineering interest, the parameter ad_{kl}/v_s is varied between 0 and 2, and d_{kl}^L/v_{app} is varied 0, 0.5 and 1.

Cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$ between ground displacement at station k and oscillator velocity response of different modes at station l.

There is a cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$ existing in non-classically damped linear structure system, which is different from classically damped linear structure system. Figures 1 and 2 respectively show the plots of the cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$ when the two supports k and l have roughly firm or soft soil conditions, and the wave traveling direction is along the route from support k to support l. For intermediate soil condition, $\rho_{u_k\dot{q}_{lj}}$ takes values between that of firm and soft sites, and the corresponding curves are omitted here for the sake of brevity.

Because $\rho_{u_k\dot{q}_{lj}}$ is a function of the oscillator frequency ω_j and the damping ratios ζ_j for arbitrary mode j, we have to firstly determinate the values of damping ratio. The curves of this function versus possible ω_j values for a preset damping ratio $\zeta_j = 0.05$ respectively for roughly firm soil and roughly soft soil condition are shown in Figs. 1 and 2. These three pairs of curves from left to right are for increasing values of the wave passage effect as defined by parameter d_{kl}^L/v_{app}, whereas each pair of curves are respectively for values $ad_{kl}/v_s = 0$ and 0.5, representing cases without and with the effect of incoherence. As mentioned previously, the considered ranges of parameter values have included the values of interest in most application. It can be seen from the left plots in Figs 1 and 2 that the cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}} = 0$ at the starting value of $\omega_j = 0$, and then takes negative

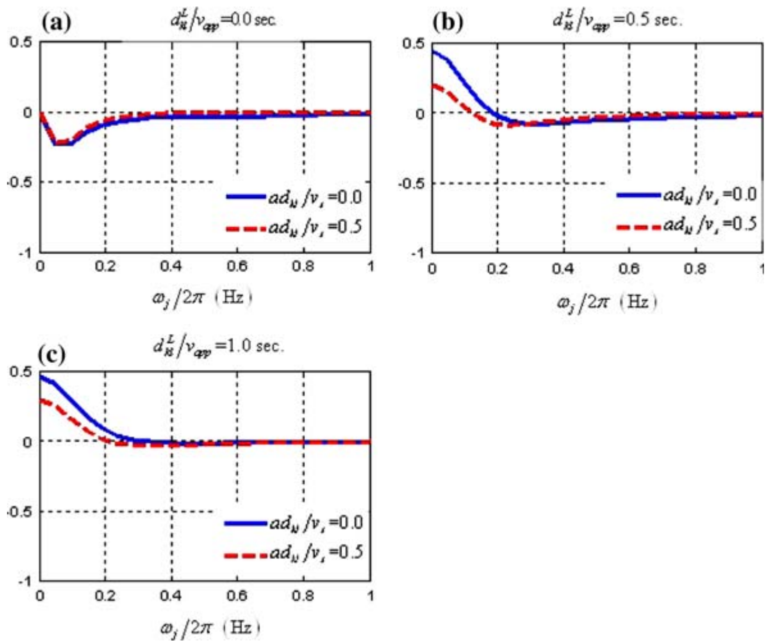

Fig. 1 Cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$ between ground displacement at support k and modal oscillator velocity response at Support l (in the case of roughly firm soil condition)

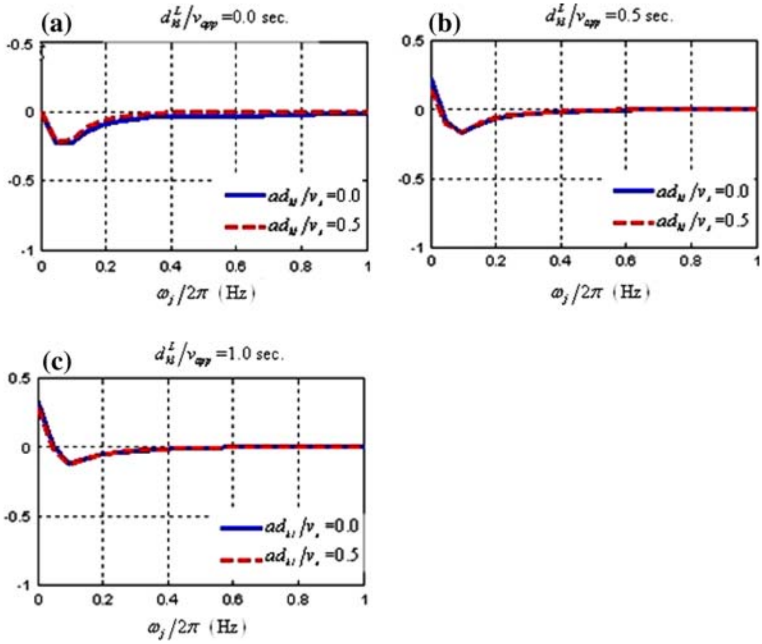

Fig. 2 Cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$ between ground displacement at support k and modal oscillator velocity response at support l (in the case of roughly soft soil condition)

values, reaches minimum and tends zero as increasing ω_j, but for intermediate and right plots the cross-correlation coefficient $\rho_{u_k\dot{q}_{lj}}$ takes maximum positive values at the starting value of $\omega_j = 0$ and gradually tends zero as increasing ω_j. Furthermore some differences of the $\rho_{u_k\dot{q}_{lj}}$ curves can be observed in the case of firm soil as shown in the plots of Fig. 1, whereas the two curves corresponding to $ad_{kl}/v_s = 0$ and $ad_{kl}/v_s = 0.5$ shown in the plots of Fig. 2 are almost same. However, for the values of ω_j for normal structures, say $\omega_j/2\pi > 0.5\,\text{Hz}$, $\rho_{u_k\dot{q}_{lj}}$ is very small in all the cases shown in Figs. 1 and 2 that means only for long period structures the influences coming from cross correlation $\rho_{u_k\dot{q}_{lj}}$ between ground displacement at support k and modal oscillator velocity response at support l are of significance.

Cross-correlation coefficient $\rho_{\dot{q}_{ki}\dot{q}_{lj}}$ between velocity responses of oscillators of different modes at supports k and l.

The cross-correlation coefficient $\rho_{\dot{q}_{ki}\dot{q}_{lj}}$ in Eq. 31 for non-classically damped linear structure system is a function of the frequencies and damping ratios of the two modal oscillators, ω_i, ω_j and ζ_i, ζ_j corresponding to two different modes, respectively, and the parameters defining the coherency function and the site soil conditions at supports k and l, when the two supports have similar firm soil conditions, and waves travel first at support k and then at support$l$. Figures 3, 4 show the plots of this function in terms of ω_j at given damping ratios $\zeta_i = \zeta_j = 0.05$. Since many parameters are involved, only two values for the incoherence and wave passage parameters and three values of frequency of modal oscillator are considered, i.e., $ad_{kl}/v_s = 0$ and 0.1, $d_{kl}^L/v_{app} = 0$ and 0.5, and $\omega_i/2\pi = 1, 2$ and 4 Hz.

Figure 3 is for the case of $d^L_{kl}/v_{app} = 0$ without the wave passage effect. It can be seen that in the case of $ad_{kl}/v_s = 0$, the incoherence uniformly reduces the value of the cross-correlation coefficient especially when $\omega_i/2\pi = 2$ and 4 Hz. For the firm soil condition with broad-band excitation, it is easy to find that the cross-correlation coefficient for the case without the incoherence effect is identical to the coefficient used in the CCQC combination rule (Zhou et al. 2004), which is based on the white noise approximation and steady state response assumption. It is also worth noting that the coefficient $\rho_{\dot{q}_{ki}\dot{q}_{lj}}$ is basically positive if absence of the wave passage effect due to $d^L_{kl}/v_{app} = 0$. Figure 4 is for roughly firm soil conditions and including the effect of wave passage in terms of $d^L_{kl}/v_{app} = 0.5$. The curves now can be oscillatory and taken on positive as well as negative values depending upon the value of d^L_{kl}/v_{app}.

Cross-correlation coefficient $\rho_{\dot{q}_{ki}q_{lj}}$ between displacement-velocity responses of oscillators of different modes at supports k and l.

The variation law of displacement-velocity response cross-correlation coefficient $\rho_{\dot{q}_{ki}q_{lj}}$ between modes i and j, which has positive and negative peaks, is obviously different and more complex compared to that of velocity-velocity response as we have seen from the curves shown in Figs. 3 and 4. By using the same method, we have deduced the variation curve of cross-correlation coefficient $\rho_{\dot{q}_{ki}q_{lj}}$ against ω_j as shown in Figs. 5 and 6. It can be seen from the figures that both wave passage effect and incoherence effect can reduce the value of coefficient $\rho_{\dot{q}_{ki}q_{lj}}$.

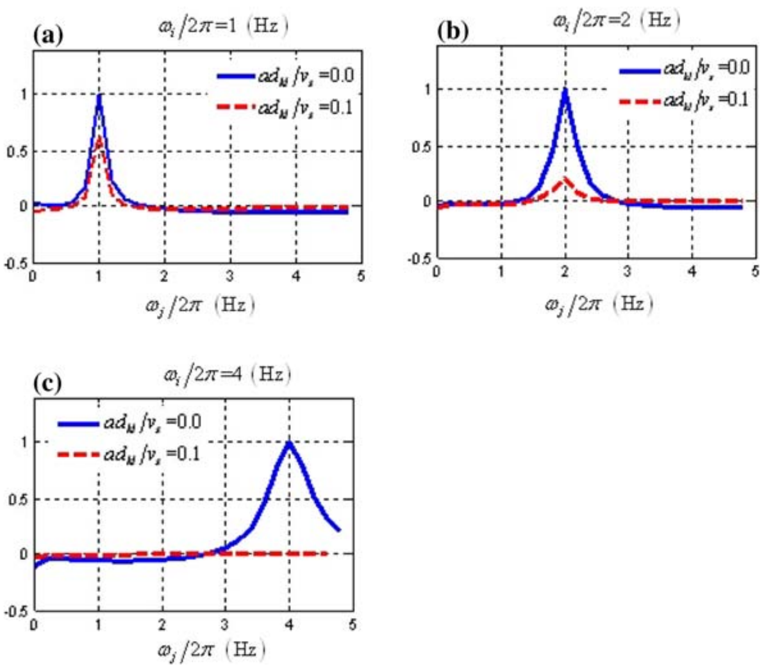

Fig. 3 Cross-correlation coefficient $\rho_{\dot{q}_{ki}\dot{q}_{lj}}$ between velocity response of modal oscillators at supports k and l (in the case of roughly firm soil condition and $d^L_{kl}/v_{app} = 0$)

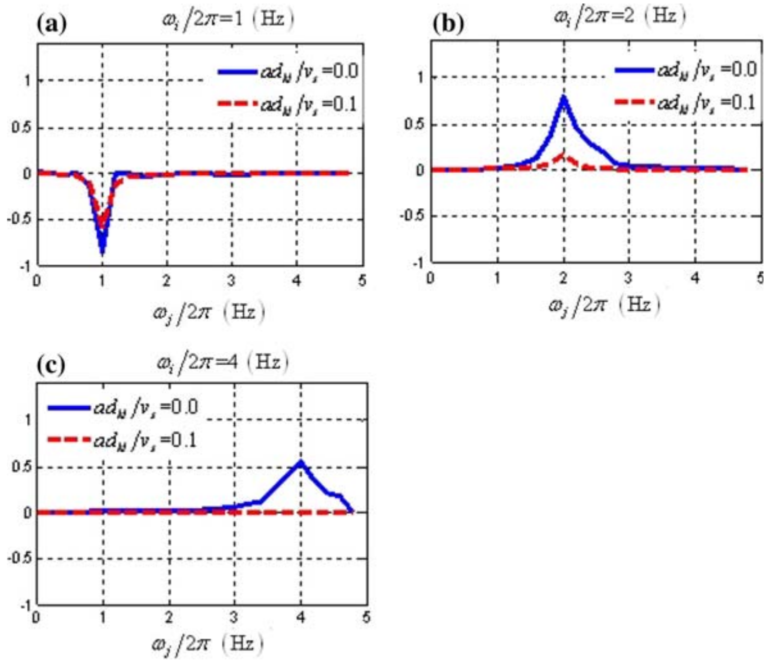

Fig. 4 Cross-correlation coefficient $\rho_{\dot{q}_{ki}\dot{q}_{lj}}$ between velocity responses of modal oscillators at supports k and l (in the case of roughly firm soil condition and $d_{kl}^{L}/v_{app} = 0.5$)

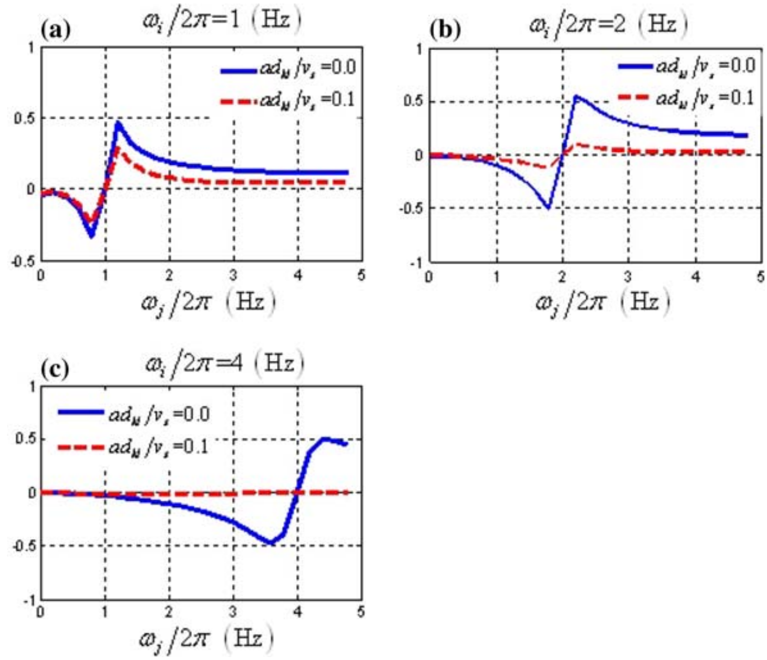

Fig. 5 Cross-correlation coefficient $\rho_{\dot{q}_{ki}q_{lj}}$ between Displacement-velocity responses of oscillators of different modes at supports k and l (in the case of roughly firm soil condition and $d_{kl}^{L}/v_{app} = 0)$

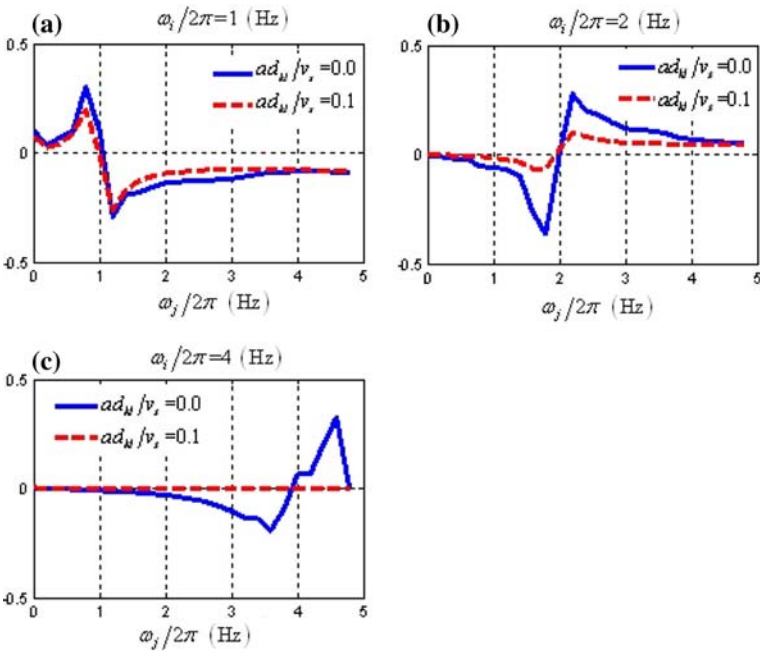

Fig. 6 Cross-correlation coefficient $\rho_{\dot{q}_{ki}q_{lj}}$ between displacement-velocity responses of oscillators of different modes at supports k and l (in the case of roughly firm soil condition and $d_{kl}^{L}/v_{app} = 0.5$)

Multiple support response spectrum method for non-classically damped linear system

Based on above discussions in Sect. 5, the multiple-support response spectrum method for non-classically damped linear system is deduced following previous works (Der Kiuerghian 1991, 1992; Zerva 1990), which is named CMSRS method in this paper. Now let $u_{k,\max} = E[\max|u_k(t)|]$ denote the mean value of the peak displacement at support k, and $D_k(\omega_i, \zeta_i) = E[\max|q_{ki}(t)|]$ denote the mean response spectrum ordinate for the oscillator of mode i, representing the mean peak relative displacement response of the oscillator of frequency ω_i and damping ratio ζ_i to the base motion $u_k(t)$. If assume the root-mean-squares of the ground displacement, oscillator displacement response and velocity response for different modes, i.e., $\sigma_{u_{gk}}$, $\sigma_{q_{ki}}$ and $\sigma_{\dot{q}_{ki}}$ are proportional to the peak values of the seismic response (Zerva 1990), then we can get

$$
\begin{aligned}
|z(t)|_{\max} = \Bigg\{ & \sum_{k=1}^{m}\sum_{l=1}^{m} g_k g_l \rho_{u_k u_l} u_{k,\max} u_{l,\max} \\
& + 2\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{j=1}^{N}\left[\left(g_k a_{lj}\rho_{u_k q_{lj}} + g_k b_{lj}\omega_j \rho_{u_k \dot{q}_{lj}}\right) u_{k,\max} D_l\left(\omega_j, \zeta_j\right)\right] \\
& + \sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\left(a_{ki}a_{lj}\rho_{q_{ki}q_{lj}} + 2b_{ki}a_{lj}\omega_i\rho_{\dot{q}_{ki}q_{lj}} + b_{ki}b_{lj}\omega_i\omega_j\rho_{\dot{q}_{ki}\dot{q}_{lj}}\right) \right. \\
& \left. \times D_k\left(\omega_i, \zeta_i\right) D_l\left(\omega_j, \zeta_j\right)\right] \Bigg\}^{1/2}
\end{aligned} \tag{37}
$$

The above formula represents the response spectrum combination rule for the peak response of the multiple-support structure for the non-classically damped linear system, that is, CMSRS method as precedent mentioned. When coefficients $\rho_{u_{gk}\dot{q}_{lj}} = \rho_{\dot{q}_{ki}q_{lj}} = \rho_{\dot{q}_{ki}\dot{q}_{lj}} = 0$, Eq. 37 will be reduced to the results deduced by Der Kiureghian (1991, 1992, 1997), which is based on classically damped linear systems.

Furthermore, since the cross-correlation coefficients between ground displacement and the modal displacement and velocity response of oscillator, $\rho_{u_k q_{lj}}$ and $\rho_{u_k \dot{q}_{lj}}$, are relatively small, which can be seen from preceding discussions in Sect. 5.1, the Eq. 37 can be simplified as

$$
\begin{aligned}
|z(t)|_{\max} = \Bigg\{ & \sum_{k=1}^{m}\sum_{l=1}^{m} g_k g_l \rho_{u_k u_l} u_{k,\max} u_{l,\max} \\
& + \sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{i=1}^{N}\sum_{j=1}^{N} \left[\left(a_{ki} a_{lj} \rho_{q_{ki} q_{lj}} + 2 b_{ki} a_{lj} \omega_i \rho_{\dot{q}_{ki} q_{lj}} + b_{ki} b_{lj} \omega_i \omega_j \rho_{\dot{q}_{ki}\dot{q}_{lj}} \right) \right. \\
& \left. \times D_k(\omega_i, \zeta_i)\, D_l(\omega_j, \zeta_j) \right] \Bigg\}^{1/2}
\end{aligned}
\tag{38}
$$

Numerical example

In this paper, it is assumed that the soil conditions at the supports are identical, and the NS component of the El Centro earthquake acceleration recorded on May 18, 1940 earthquake in California, which contains energy over a broad range of frequencies and has been broadly used in earthquake response analyses, is selected as ground motion input. The earthquake velocity and displacement used in this paper can be attained through integration for the earthquake acceleration recorded. Furthermore, the displacement response spectra for the different natural periods and corresponding damping ratios will be calculated in terms of the selected El Centro record. The consistent power spectral density is assumed to be Clough and Penzien's double filter model (1975), which is only used in determining the cross-correlation coefficients.

Furthermore, five different cases of input support motion are discussed, i.e.,

Case 1: Uniform support input

This is the case where all support inputs along the same direction are identical. In this case, the spatial coherence function of ground motion is defined as: $\gamma_{kl}(i\omega) = 1.0$ for any supports k and l.

Case 2: Only wave passage effect included

The support motion may essentially be correlated because only the phase effect is considered. In this case the incoherence factor $a = 0$ with surface apparent wave velocity $v_{app} = 400\,m/s$;

Case 3: Only incoherence effect included

In this case, the phase effect in the spatial coherence function caused by wave propagation affect is neglected, that is, only incoherence effect is included and let $v_{app} = \infty$ with $v_s/\alpha = 600\,m/s$;

Case 4: General input

Both wave passage and incoherence effects are included and suppose $v_{app} = 400\,m/s$ and $v_s/\alpha = 600\,m/s$;

Case 5: Independent support input

This is the case where the support inputs are considered to be statistically independent and uncorrelated with each other. And in this case the coherence function is set equal to zero for $k \neq l$, i.e., $\gamma_{kl}(i\omega) = 0.0$ for $k \neq l$.

Example 1

Here is an exemplary structure originally taken from reference (Clough and Penzien 1993). Thus we consider a rigid bar in Fig. 7, which has additional lumped mass $m/2$ at each end and length $L = 10$ ft and total uniformly distributed mass is m. This bar is rigidly attached to the top of a weightless column of length L and in addition there is a lateral spring support at mid-height of the bar, as shown in Fig. 7. The mass matrix and the stiffness matrix responding to the Eq. 1 are shown as,

$$M = \frac{m}{6}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix},$$
$$K = \frac{EI}{L^3}\begin{bmatrix} 30.5 & -7.5 \\ -7.5 & 6.5 \end{bmatrix},$$
$$K_c = \frac{EI}{L^3}\begin{bmatrix} -5 & -18 \\ -5 & 6 \end{bmatrix},$$
$$K_g = \frac{EI}{L^3}\begin{bmatrix} 10 & 0 \\ 0 & 12 \end{bmatrix}.$$

where $m = 0.4 kip.s^2/ft$, $EI/L^3 = 3.0 kips/ft$. And the displacement vector of the structure corresponding Eq. 1 can be obtained as $\left[y_1 \; y_2 \; u_{ga} \; u_{gb} \right]^T$, here y_1, y_2 and u_{ga}, u_{gb} are the displacements at both the ends of the bar with uniformly distributed mass and the support displacements respectively as shown in Fig. 7. In this example there are two supports to be disturbed. Firstly suppose the damping matrix C complies following Rayleigh rule, i.e., $C = \alpha M + \beta K$, where $\alpha = 0.2$(1/s.), $\beta = 0.00173$(s).

Now transform the exemplary structure into a non-proportionally damped system by equipping a supplemental damper on the first inter-storey, which results in abrupt changes both in

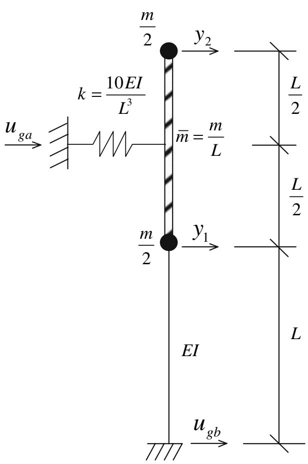

Fig. 7 Exemplary structure

the stiffness and damping. The corresponding changes to K and C matrices are such that

$$K(1,1) \leftarrow 1.05K(1,1), C(1,1) \leftarrow 41.0C(1,1)$$

and remain all the other elements in matrix $\boldsymbol{K}$ and $\boldsymbol{C}$ unchanged. Via complex mode analysis procedure, the modal properties of the damper-added structure are obtained, as given in columns 2 and 4 in Table 1. Columns 3 and 5 of Table 1 show modal properties of the damper-added structure if proportional damping is assumed, that means the off-diagonal elements in the modal-transformed damping matrix, which is calculated from the mode matrix of the corresponding non-damping system, are ignored. These results show that the damping ratio of the 2-th mode is greatly underestimated although their natural period is approximate to each other.

In order to examine the correctness of complex mode superposition method and the involved parameters, Eq. 2 is solved by Newmark-β numerical integration (integration step 0.02s and parameters $r = 0.05$, $\beta = 0.25$) under the uniform ground displacement of the selected El Centro record, in which we assume $\boldsymbol{M}_c = 0$ and $\boldsymbol{C}_c = 0$ as mentioned in Sect. 2. The column 2 of Table 2 lists the results calculated using Newmark-β method. The results obtained from complex mode superposition method are listed in columns 3 and 4 of Table 2. Column 3 gives the results considering the damping matrix $\boldsymbol{C}$ in calculation of dynamic response component $\boldsymbol{y}^d$, as shown in Eq. 6, which are identical with the results in column 2 calculated by Newmark-β numerical integration and illustrate the analytical method for calculating seismic response of non-classically damped linear system with multiple-support excitations is validate. The results in column 4 do not consider the effect of damping matrix associated the support ground motion velocity, as shown in right side of Eq. 6 that means those results are obtained from Eq. 7. Compare the results in columns 3 and 4, it can be seen that neglecting the damping forces on the right-hand of Eq. 6 will not produce large errors, as shown in column 5 of Table 2, though the structure is non-classical damping system in this example.

Table 1 Modal properties of the non-proportional damping structure

Mode number	Modal periods (s)		Modal damping ratios (%)	
	Exact	Based on proportional damping assumption	Exact	Based on proportional damping assumption
1	0.9619	1.0580	15.48	15.57
2	0.3625	0.3409	78.31	54.98

Table 2 Maximum displacement of y_1 and y_2 (cm)

Response	Eq. 2			
	Numerical integration using Newmark-β	Complex mode superposition method		
		In consideration of damping	Without consideration of damping	Errors(%)
y_1	59.8023	59.8023	61.0880	2.15
y_2	59.0580	59.0580	61.1718	3.58

Now, let us examine the six cross-correlation coefficients $\rho_{u_k u_l}$, $\rho_{u_k q_{lj}}$, $\rho_{u_k \dot{q}_{lj}}$, $\rho_{q_{ki} q_{lj}}$, $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$ and $\rho_{\dot{q}_{ki} q_{lj}}$. Table 3 lists the calculation results for $\rho_{u_k u_l}$, the cross-correlation coefficient between the ground displacements at the various supports. These coefficients are independent of the direction of wave propagation and hence identical results are obtained for support pairs such as 1, 2 and 2, 1. For Case 1, because the motions are completely correlated, the correlation coefficient must be unity, and for Case 5 they are zero because the motions are statistically independent. The remaining cases corresponding to conditions of partial correlation, and $\rho_{u_k u_l}$ is found to be between zero and 1.0 and in this example near 1.0.

Table 4 gives the cross-correlation coefficients $\rho_{u_k q_{lj}}$ between the displacement at support k and the displacement response of an oscillator represent a given mode at support$l$. Table 5 shows the $\rho_{u_k \dot{q}_{lj}}$, the cross-correlation coefficient between the displacement at support k and the velocity response of a modal oscillator at support l. These coefficients are not symmetric with respect to k and l when the wave passage effect is present and, hence, for Cases 2 and 4 different values are obtained for each ordered combination of support pairs, e.g., for (1, 2) and (2, 1). The correlation coefficient $\rho_{u_k q_{lj}}$ is found to be relatively small (<0.2) for all cases, and $\rho_{u_k \dot{q}_{lj}}$ is much smaller than cross-correlation $\rho_{u_k q_{lj}}$ as shown in Tables 4 and 5. The influences of the wave passage and incoherence effects can be observed separately by comparing the results for Cases 1 and 2 and Cases 1 and 3, respectively, whereas their combined influence can be observed via comparing Cases 1 and 4. Here i and j represent numbers of modes and k and l are numbers of support as mentioned.

Table 6 lists the cross-correlation coefficients $\rho_{q_{ki} q_{lj}}$ between the displacement responses of the modal oscillators at supports k and l. Table 7 shows the cross-correlation coefficients $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$ between the velocity responses of the modal oscillators at supports k and l. For each case and pair of supports, a 2×2 correlation matrix is given in the tables. In Cases 2 and 4 where the wave passage effect is included, the matrix is asymmetric and is given in form of full matrix both for $\rho_{q_{ki} q_{lj}}$ and $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$. In Cases 1, 3 the matrix is symmetric and Case 5 is zero matrix as shown in Tables 6 and 7. In Cases 2 and 4, the correlation matrices for support combinations (2, 1) can be obtained by using of the symmetry rule $\rho_{q_{ki} q_{lj}} = \rho_{q_{lj} q_{ki}}$

Table 3 Cross-correlation coefficients $\rho_{u_k u_l}$

Case		1	2	3	4	5
Supports (k, l)	(1,1); (2,2)	1	1	1	1	1
	(1,2); (2,1)	1	0.9983	0.9985	0.9969	0

Table 4 Cross-correlation coefficients $\rho_{u_k q_{lj}}$

Case	Concerned supports pair (k, l)	Mode of non-classical damping Structure		Mode of proportional damping assumption	
		1	2	1	2
1 All = 1–5	Any k, l $k = l$	0.2164	0.3142	0.0905	0.0622
2	(1,2)	0.1960	0.2377	0.0824	0.0566
	(2,1)	0.2843	0.3384	0.0990	0.0652
3	(1, 2) and (2, 1)	0.2167	0.3117	0.0906	0.0610
4	(1, 2)	0.1967	0.2839	0.0827	0.0562
	(2, 1)	0.2372	0.3336	0.0989	0.0632
5	Any $k \neq l$	0	0	0	0

Table 5 Cross-correlation coefficients $\rho_{u_k \dot{q}_{lj}}$

Case	Concerned supports pair (k, l)	Mode of non-classical damping Structure		Mode of proportional damping assumption	
		1	2	1	2
1 All = 1–5	Any $k, l \quad k = l$	−0.0838	−0.0598	−0.0374	−0.0068
2	(1,2)	−0.0783	−0.0659	−0.0348	−0.0092
	(2,1)	−0.0840	−0.0412	−0.0382	−0.0014
3	(1, 2) and (2, 1)	−0.0815	−0.0546	−0.0366	−0.0056
4	(1, 2)	−0.0774	−0.0622	−0.0345	−0.0083
	(2, 1)	−0.0806	−0.0371	−0.0369	−0.0004
5	Any $k \neq l$	0	0	0	0

Table 6 Cross-correlation coefficients $\rho_{q_{ki} q_{lj}}$

Case	Concerned supports pairs (k, l)	Mode number	Mode of non-classical damping Structure		Mode of proportional damping assumption	
			1	2	1	2
1 All = 1–5	Any $k, l \quad k = l$	1	1	0.3348	1	−0.0125
		2	0.3348	1	−0.0125	1
2	(1,2)	1	0.9687	0.4824	0.9750	0.0012
		2	0.1759	0.8959	−0.0226	0.7785
3	(1,2)	1	0.9726	0.3300	0.9780	−0.0109
		2	0.3300	0.9138	−0.0109	0.8154
4	(1,2)	1	0.9427	0.4670	0.9537	0.0016
		2	0.1836	0.8305	−0.0198	0.6364
5	Any $k \neq l$	1	0	0	0	0
		2	0	0	0	0

Table 7 Cross-correlation coefficients $\rho_{\dot{q}_{ki} \dot{q}_{lj}}$

Case	Concerned supports pairs (k, l)	Mode number	Mode of non-classical damping structure		Mode of proportional damping assumption	
			1	2	1	2
1 All = 1–5	Any $k, lk = l$	1	1	0.1018	1	−0.0232
		2	0.0108	1	−0.0232	1
2	(1,2)	1	0.9453	0.3143	0.9716	−0.0046
		2	−0.1663	0.7169	−0.0432	0.7684
3	(1,2)	1	0.9537	0.0824	0.9752	−0.0234
		2	0.0824	0.7856	−0.0234	0.8077
4	(1,2)	1	0.9065	0.2741	0.9483	−0.0067
		2	−0.1329	0.6183	−0.0387	0.6226
5	Any $k \neq l$	1	0	0	0	0
		2	0	0	0	0

and $\rho_{\dot{q}_{ki} \dot{q}_{lj}} = \rho_{\dot{q}_{lj} \dot{q}_{ki}}$. That is, the correlation matrix for support combinations (2, 1) is the transpose of the matrix for the combinations (1, 2).

Table 8 is the cross-correlation coefficients $\rho_{\dot{q}_{ki} q_{lj}}$ between the velocity and displacement responses of the oscillators at supports k and l. Similarly, for each case and pair of supports,

Table 8 Cross-correlation coefficients $\rho_{\dot{q}_{ki} q_{lj}}$

Case	Supports	Mode number	Mode of non-classical damping structure		Mode of proportional damping assumption	
			1	2	1	2
1 All = 1–5	Any $k, l \quad k = l$	1	0	0.6297	0	0.0554
		2	−0.3373	0	−0.0183	0
2	(1,2)	1	−0.2466	0.5282	−0.2217	0.0646
		2	−0.3287	−0.4209	−0.0108	−0.6258
3	(1,2)	1	0	0.5804	0	0.0497
		2	−0.3109	0	−0.0164	0
4	(1,2)	1	−0.2357	0.4945	−0.2163	0.0598
		2	−0.3042	−0.3416	−0.0093	−0.5060
5	Any $k \neq l$	1	0	0	0	0
		2	0	0	0	0

a 2×2 correlation matrix is given in the table. For this coefficient $\rho_{\dot{q}_{ki} q_{lj}}$, the matrix is asymmetric and the full elements are given in Cases 1, 2, 3 and 4. In Cases 2 and 4, the correlation matrices for support combinations 2, 1 can be obtained by used of the symmetry rule $\rho_{\dot{q}_{ki} q_{lj}} = \rho_{q_{lj} \dot{q}_{ki}}$.

The peak responses y_1 and y_2 are listed in Table 9. The third column lists the results of non-classically damped linear system for each case calculated by Eq. 37, and the fourth column is the result coming from the simplified Eq. 38. It can be seen that neglecting the cross-correlation coefficients between ground displacement and the displacement and velocity response of oscillator, or the cross term representing the cross covariance between the pseudo-static component and dynamic component will not produce large errors. Furthermore, the results obtained from proportional damping assumption are listed in columns 5 and 6. From the comparison of these results, we can see that for this simple non-classical damping system, the results calculated by proposed CMSRS method is closer to exact solution although proportional damping assumption does not bring large errors. It is worth pointing out that the results for the above example show that the influence of spatial variability of the ground motion on the response of a multiple-support structure is fairly significant. It can be seen that the peak response of the non-classical damping structure y_1 and y_2 computed from CMSRS

Table 9 The peak response y_1 and y_2 of the non-classical damping structure (cm)

Response	Case	CMSRS method		Based on proportional damping assumption	
		Eq. 37	Eq. 38	Eq. 37	Eq. 38
y_1	1	60.1234	61.0456	65.0875	65.3322
	2	60.0170	61.0166	65.0500	65.3051
	3	60.1030	61.0236	65.0634	65.3080
	4	60.0392	60.9956	65.0266	65.2816
	5	42.5929	43.2041	46.1057	47.0391
y_2	1	59.8639	61.8484	65.5097	66.1767
	2	60.1891	61.9098	65.5765	66.2267
	3	59.9241	61.9044	65.5547	66.2213
	4	60.0428	61.9641	65.6201	66.2700
	5	43.2357	44.3397	46.2762	47.3475

method are nearer to the results calculated from Newmark-β numerical integration and the proportional damping assumption is somewhat overestimate the displacement responses in this example.

This simple example can be regarded as an introduction to considerate the response of non-classically damped linear system under multiple-support seismic excitation. However, the influences of the inhomogeneous support ground motion on the peak responses in this example are not very significant. In order to clarify the influences of the inhomogeneous support ground motion on the peak responses of the non-classically damped linear structural systems, in consideration of large span structure is needed.

Example 2

Now the three-span continuous beam in Fig. 8 is considered, which has uniform mass and stiffness properties and simple support. As shown in Fig. 8, the beam is discretized into eight elements along each $L = 40\,m$ span and the mass of each element is lumped half at each end of the element. Assume that $EI/m = 4.28 \times 10^5\, m^4/s^2$, where EI denotes the flexural rigidity and m denotes the mass per unit length of the beam. The modal damping ratio is assumed to be 5modes at first. The mid-span placements u_1, u_2 and the bending moment M at the support 2 indicated in Fig. 8 are optioned for comparing the calculated results from various methods and different cases. For the convenience of notation, these response quantities are scaled and collected in a dimensionless response vector z defined as

$$z = [z_1 z_2 z_3] = 10^3 \times \left[\frac{u_1}{L} \frac{u_2}{L} \frac{LM}{EI} \right]$$

Now additional isolators are installed in the middle-bearings 2 and 3, which are simplified as columns with low stiffness, $EI = 1.48 \times 10^9\ \mathrm{N{\cdot}m^2}$ and short height, $H = 0.3\mathrm{m}$ equipped with viscous dampers in parallel with each of the two columns respectively, thus the original structure will be transformed into a non-proportionally damped system as that have been done in example 1. In this example we also consider five cases as mentioned in example 1, but in Case 2 to Case 4 we let $v_{app} = 200\,m/s$ and $v_s/\alpha = 300\,m/s$. Table 10 lists the modal periods and damping ratios of the first four modes for the non-classically damped linear

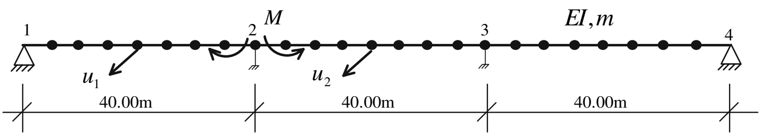

Fig. 8 Example structure

Table 10 Modal properties of the non-proportional damping structure

Modal number	Modal periods (s)		Modal damping ratios (%)	
	Exact	Based on proportional damping assumption	Exact	Based on proportional damping assumption
1	0.7131	0.7851	3.41	5.00
2	0.6955	0.7203	4.50	2.11
3	0.6867	0.4048	5.00	1.41
4	0.2582	0.2312	4.09	5.00

system. Similarly, the corresponding results obtained from proportional damping assumption are also listed in columns 3 and 5 of Table 10. It can be seen from Table 10 that both the period and damping ratio of the corresponding proportional damping system comprise certain errors compared to exact values calculated from non-classically damped system.

For this example, we still use Newmark-β numerical integration to solve Eq. 2, in which we assume $\boldsymbol{M}_c = 0, \boldsymbol{C}_c = 0$ and the input motion is the same as example 1, i.e., the uniform ground displacement of the selected El Centro record, the computational results $z_{Newmark}$ are listed in the column 2 of Table 11. The corresponding results obtained from complex mode superposition method according to Eq. 2 are listed in columns 3 and 4 of Table 11. These results once more illustrate the analytical method for calculating seismic response of non-classically damped linear system with multiple support excitations is correct. The different results are given respectively in columns 3 and 4 according to whether considering the damping matrix $\boldsymbol{C}$ in calculation of dynamic response component $\boldsymbol{y}^d$ or not, it can be seen that neglecting the damping forces on the right-hand of Eq. 6 will not produce large errors, as shown in column 5 of Table 11, though the structure is non-classical damping system.

The peak responses of z_1, z_2 and z_3 are listed in Table 12. The 3rd column lists the results z_{CMSRS} of non-classically damped linear system for each case calculated by Eq. 37, and the 4th column is the ratios obtained by calculating $z_{CMSRS}/z_{Newmark}$, in which the value of $z_{Newmark}$ is listed in column 2 of Table 11. It can be seen that the results obtained by CMSRS

Table 11 Maximum responses of z_1, z_2 and z_3

Response	Eq. 2			
	$z_{Newmark}$	Complex mode superposition method		
		In consideration of damping	Without consideration of damping	Errors(%)
z_1	12.71	12.71	12.72	0.09
z_2	17.89	17.89	17.91	0.11
z_3	8.02	8.02	8.04	0.26

Table 12 The peak response of the non-classical damping structure

Response	Case	CMSRS method			Based on proportional damping assumption			
		z_{CMSRS}	$\frac{z_{CMSRS}}{z_{Newmark}}$	$\frac{z_{CMSRS}}{z_{Case1}}$	z_{MSRS}	$\frac{z_{MSRS}}{z_{Newmark}}$	$\frac{z_{MSRS}}{z_{CMSRS}}$	$\frac{z_{MSRS}}{z_{Case1}}$
z_1	1	13.11	1.03	1	15.98	1.26	1.22	1
	2	11.98	0.94	0.914	14.69	1.16	1.23	0.919
	3	12.22	0.96	0.932	14.83	1.17	1.21	0.928
	4	12.01	0.94	0.916	14.72	1.16	1.23	0.921
	5	10.09	0.79	0.770	12.85	1.01	1.27	0.804
z_2	1	17.96	1.00	1	20.68	1.16	1.15	1
	2	16.67	0.93	0.928	19.27	1.08	1.16	0.932
	3	16.70	0.94	0.930	19.29	1.08	1.16	0.932
	4	16.68	0.93	0.928	19.31	1.08	1.17	0.933
	5	14.11	0.79	0.786	17.51	0.98	1.24	0.847
z_3	1	8.53	1.06	1	10.86	1.35	1.27	1
	2	7.34	0.92	0.861	9.44	1.18	**1.29**	0.869
	3	7.44	0.93	0.873	9.54	1.19	1.28	0.878
	4	10.74	1.34	1.259	12.95	1.61	1.21	1.192
	5	24.68	3.00	2.893	31.64	3.95	1.28	2.913

method will not produce great errors except the Case 5. Furthermore, the 5th column gives the ratios of each case and the Case 1. It can be seen that for the displacement response z_1 and z_2, the results for the different cases are close each other except bending moment response z_3 in Case 5 that means the internal force responses are more sensible than displacement responses as observed in references (Der Kiuerghian, 1991, 1992, 1997).

Moreover, the results z_{MSRS} based on the proportional damping assumption are listed in column 6, which is calculated by MSRS method (Der Kiuerghian, 1991, 1992), and the ratio $z_{MSRS}/z_{Newmark}$ is listed in column 7. For the convenient to illustrate the effect of adopting proportional damping assumption on the structure response, the ratios z_{MSRS}/z_{CMSRS} is showed in column 8. It can be seen that the results of adopting proportional damping assumption will be greater than that of CMSRS method, which means the proportional damping assumption overestimate both the displacement and internal force response in this example. The amplitude of overestimation normally reaches 10 percent more or less except the moment responses z_3 in Case 2 and Case 3 where the ratio z_{MSRS}/z_{CMSRS} mostly reaches 1.29 as shown by bold figures in Table 12 in this example. In addition, the ratios of each case and Case 1 based on proportional damping assumption are listed in column 9. Compare these results, we found that the results calculated by proposed CMSRS method is closer to exact solution. However this result is gotten from one particular example. So we can not gain the general rule about the possible errors caused by proportional assumption in dealing with seismic responses of non-classically damped linear structural system subjected to multiple supports ground motion excitation. In order to get more general conclusion further study about practical examples seems to be very necessary. However in the case of non-proportionally damped linear structural system with multiple supports seismic excitation, using the proposed CMSRS method is not only more reasonable but also excludes the unknown errors due to proportional damping assumption. In addition the influences of the inhomogeneous support ground motion on the peak responses of the non-classically damped linear structural systems are not very great in this example but this is not particular for the discussed non-classically damped linear structural systems. The main purpose of us is to discuss and compare the difference between proportional and non-proportional damping systems, the detailed discuss about the influences of inhomogeneous support ground motion on the peak responses beyond the scope of this paper and retaining for further study.

Conclusions

According to theoretical analysis and numerical examination in this paper, some conclusions can be obtained as follows:

1. For the non-classically damped linear system, a new response spectrum method is proposed for seismic response analysis of multiple-support structures subjected to spatially varying ground motion, which is named CMSRS method. This method is based on fundamental principles of random vibration theory and properly accounts for the effect of correlation between the support motions as well as between the modal displacement and velocity responses of structure, and the three new cross-correlation coefficients or cross covariance especially for the non-classically damped structures with multiple-support excitations are deduced in this article. Furthermore, for non-proportionally damped long span linear structures the proposed CMSRS method provides a reasonable estimate of the peak response in term of peak ground motions and response spectra at the support points

and the coherency function and excludes possible errors caused by using proportional damping assumption.
2. For the non-classically damped linear structure, the cross term representing the cross covariance between the pseudo-static and dynamic components is also usually small in relation to the individual covariance responses of pseudo-static and dynamic components.
3. The deduced proportional damping system through neglecting all the off-diagonal elements in transformed damping matrix in modal coordinates of the non-classically damped linear system will bring some unknown errors in the case of subjected to spatially excited inhomogeneous ground motion.
4. The CMSRS method is examined by using time history analyses in two numerical examples of non-classically damped structures in consideration of the coherences spatially variable ground motion. The results of CMSRS method in two exemplary structures illustrate good coincidence with that of time history analyses for displacement responses but for internal force responses still imply certain errors which might be unavailable in response spectrum mode superposition method.
5. In the discussed two simple examples, the influences of the inhomogeneous support ground motion on the peak responses of the non-classically damped linear structural systems are not very great. However this result draws just from these particular example and we can not assert what situation will be seen in other examples. In order to clarify the influences of the inhomogeneous support ground motion on the peak responses of the non-classically damped linear structural systems, more researches have to be done in future.

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