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ORIGINAL PAPER

The Border Space between Logic and Aesthetics in Mathematics

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Abstract
The main thesis defended in this paper is that, interpreted in the light of reflections of Peirce and Poincaré, one can found in mathematical reasoning a non-logical symptom that may be aesthetic in Goodman's sense. This symptom is called exemplification and serves to distinguish between only logically correct and even explanatory proofs. It broadens the scope of aesthetics to include all activities involving symbolic systems and blurs the boundaries between logic and aesthetics in mathematics. It gives a better understanding of Poincaré's thesis that to affect aesthetic value to certain properties is not simply an added value, a bonus that somehow rewards the mathematician's mechanical labor, but on the contrary, taking the aesthetic value into account can be helpful to mathematical practice. As an example, three proofs of the irrationality of $\sqrt{2}$ are compared for their aesthetic functioning.

Keywords Mathematical reasoning · Aesthetics · Goodman · Exemplification · Poincaré · Peirce

1 Introduction

The question of the relationship between mathematics and aesthetics is often understood in terms of the role mathematics can play in artistic fields: mathematics in art. This is the case, for example, when we consider the mathematical underpinning of a piece of music, or perspective in a painting. At the level of the mathematical community, the question takes another direction, that of the role of aesthetics within mathematics. Indeed, it's well known that mathematicians like to defend the aesthetic aspect of their science and it's rare not to find a mention of the beauty of this science, the elegance of a theory or the harmonious symmetry of a particular

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equation in retrospective writings, public lectures, popular mathematical works, or honorary speeches:

> "The true spirit of joy, of exaltation, the feeling of being more than a man, which are the touchstone of the highest excellence, are to be found in mathematics as in poetry",

notes le Lionnais (1962, 438).[1] In associating mathematics with the arts, he makes an art-historical distinction between "romanticism and classicism", the former indulging in "striking effects and aiming at paroxysm", the latter.

> "fulfills us, either through its simplicity, or through a controlled variety, or even when it combines these two impressions in a harmoniously arranged construction" (Le Lionnais 1962, 438-440).

So, for mathematicians, aesthetics does not confine itself to a subjective feeling but is responsible for a harmonious combination of simplicity and variety, which could contribute to better understanding. However, as as Ulianov Montano Juarez remarks (2014, XIV), such an attitude is based on the premise that mathematics is considered an art from the outset, so that Le Lionnais' distinction between classical and romantic aesthetics concerns rather a style of art. Indeed, I think it is too undifferentiated to simply reduce the 'knowing that' of science to the 'knowing how' of the arts, because acquaintance (*kennen*) also plays a role in the sciences and recognizing (*erkennen*) in the arts.

Now, the use of aesthetics in mathematics "is best addressed in the context of an aesthetic theory" (Montano, ibid.). However, contrary to Montano, my approach is not a naturalistic one which should cohere as much as possible with empirical findings, studying for example "phenomena involved in mathematician's judgments, like 'Cantor's notion of infinity is beautiful' or 'proofs by cases are cumbersome'" (*ibid.*). "The question of how to interpret what mathematicians mean when they use aesthetic expressions" (Montano 2014, XIV) is not the main focus here, but rather a consequence of my considerations. I ask if there are processes in mathematics which are symptomatic of an aesthetic expression, *regardless* of whether mathematicians have recognized this or not, and if so, whether the recognized phenomenon was described as aesthetic or otherwise.

The aesthetic dimension of mathematics has two facets, one of which is evaluative while the second is functional. In the evaluative perspective we recognize the beauty of a 'Monet' or the mathematician recognize the beauty of a proof. Both feelings, although quite different,[2] are characterized in modern aesthetics "by […] the view that the phenomena related to our perception of beauty are independent of any practical or cognitive concerns" (Montano 2014, XII). This aspect is obviously no longer to be treated according to more or less subjective criteria of taste and aesthetic preferences, but in a full theoretical way (Montano 2014, XIII). The

[1] If no translation is given in the bibliography, all translations from French and German are mine. I thank an anonymous reviewer for corrections and suggestions.

[2] The latter, unlike the former, requires mathematical knowledge; cf. de Rosa (2023, 122).

second aspect concerns the attribution of a cognitive role to the consideration of the aesthetic element and engages philosophical and epistemological questions. Within mathematics, it concerns the process of mathematical functioning, such as proof construction, and implies, as we shall see, that mathematical reasoning is in some cases complemented by an approach possessing aesthetic symptoms.

Montano and others raised a widely discussed problem in the literature: do mathematical entities or processes *exist* that have certain properties, such as harmony, architecture, simplicity, symmetry or order, into which one project genuine or metaphorical and reductive aesthetic values in the context of a background-understanding? (Montano (2014, 170); McAllister (2005), Novaes (2019)).

My question is similar but different in an essential point: Adopting the functional point of view in aesthetics, I am interested in the question of whether there are processes in mathematics, for example mathematical proofs, whose aesthetic evaluation follows on from a special *symbol functioning* and is *not* the consequence of *perceiving* properties such as harmony or order. In this perspective, the aesthetic evaluation, is just the tip of the iceberg.

2 The Relevance of Aesthetic Intuition in Mathematical Reasoning According to Poincaré

For my argument,[3] it is a historically fortunate fact that Henri Poincaré is a mathematician, which sometimes suggests ideas in the direction of my last question. Nevertheless, he considers that we need an *intuition* "that makes us divine hidden harmonies and relations".[4] This sounds ambiguous regarding an evaluative and cognitive view of aesthetics. In any case, for Poincaré to affect aesthetic value to certain properties is not simply an added value, a bonus that somehow rewards the mathematician's mechanical labor, but on the contrary, taking the aesthetic value into account can be helpful to mathematical practice. It is not even historically impossible that he might have influenced the mentor of my subsequently related conceptual apparatus, Nelson Goodman. I quote a booklet written in 1911:

> "Since the spirit of finesse is necessary to everyone [...] we will conclude that literary culture is necessary to scientists [...]. But it is generally believed that they need it to become men, not to become scientists; and therein lies the mistake" (Poincaré 1911, 25-26).

Goodman seems to insist:

> "My argument that the arts must be taken no less seriously than the sciences is not that the arts 'enrich' us or contribute something warmer and more human, but that the sciences as distinguished from technology, and the arts distin-

[3] I am returning here to an idea that I developed when the award of *Docteur honoris causa* of the University of Nancy 2 was conferred to Nelson Goodman (Heinzmann 1997). Since then, I have taken up the topic again and again and improved it significantly in this paper.

[4] Poincare(1908, 385) and Poincare (1905, 218).

guished from fun, have as their common function the advancement of understanding" (Goodman 1979, 619).

For the further interpretation, everything depends on whether intuition mentioned by Poincaré only expresses an aesthetic feeling or can be interpreted as the hidden expression of a symbolic use that provokes the aesthetic feeling.

In his dispute with "logicians", Poincaré refuses their conviction, that a logical correct proof is always sufficient for our understanding of this proof. Mathematical reasoning and derivation in formal logic are not in a simple correspondence relationship. To translate an existing mathematical proof in a formal derivation is not unambiguous: it might tell us more about the method of reconstruction than about what gets reconstructed. The logical structure of a reconstructing formal system is, at best, underdetermined by the reconstructed fragment of mathematics (Leitgeb 2009). This may be the reason that today's admitted validity of a logical, i. e. topic-neutral, inference from a proposition p to a proposition q (in Tarski's conception: every model of *p* is a model of *q*) is not enough to sustain epistemic growth of *mathematical* knowledge. According to Poincaré, such a growth requires rather that *p* and *q* are seen as united by "epistemic condensers". Detlefsen (1992, p. 360) formulated this in a pointed way: "Poincaré was not so much opposed to logicism as to the logicization of mathematical proof". The following question arises: What does it mean that the premises and the conclusion of mathematical reasoning are unified by "epistemic condensers"—that are devices which serve to "abridge our reasonings and our calculations"? (Detlefsen 1992, 360 and Poincaré 1908, 440).

Poincaré's answer is complex and multifaceted:

He insists on the non-invariance of mathematical reasoning with respect to its content and advances, so to speak, a *local* conception of reasoning. So, the epistemic condensation "signifies a grasp of how the movement from premises to conclusion contributes to the 'development' of some architectural theme of local subject-matter. In short, it marks the presence of a comprehending 'universal' in the differences through which it persists" (Detlefsen 1992, 361). Thanks to "*fortunate inventions of language*" that introduce an order-structure, the complexity of a domain of objects is made more harmonious by introducing of an invariance (see Poincaré 1908, 375). In mathematical reasoning, understanding is placed in the context of "mathematical architectures".

In other words, Poincaré does not limit himself to "blind (ungrounded) correct judgments". Stated differently, he considers the Bolzanian reduction "of the validity of an inference between judgements to a corresponding logical consequence among suitable propositions" (Sundholm 2012, 945) insufficient for understanding. For Poincaré all genuinely mathematical inferences—i.e., those which are not syllogistic or elementary combinatorial—are synthetic, since they all close a gap by 'putting together' the premises and conclusion, which 'goes behind' the premises' (Detlefsen 1992, 363).[5]

[5] The general universal of the architecture closing such gaps is especially emphasized by Poincaré for the principle of complete induction on $$\mathbb{N}$$ but the axiom of choice has the same synthetic a priori character as the induction: The axiom of choice "is a synthetic a priori judgment […] although I am favorably disposed to accept Zermelo's axiom, I reject his proof" (Poincare 1906), 313, 315).

Unfortunately, Poincaré did neither explain his metaphor of architecture, his criterion of harmony reinforced by a principle of economy of thought nor his use of intuition, central to mathematical reasoning, nor their interrelationship, so that their modern interpretation depends on the general way to extend and partially to transform Poincaré's reflections.

In the following I use the term 'explanatory reasoning' to express that reasoning gives a proof-understanding what does not yet imply that it gives an understanding of the proved result. According to Poincaré's position the explanatory function is only a disguised element of any mathematico-logical proof that should be made explicit: in *Science and Hypothesis*, he uses the term "implicit axioms" for what we now call tacit presuppositions, that are premises "that geometers [i. e. the mathematicians] accept without explicitly stating" (Poincaré 1902*,* 38). In the *Value of Science,* Poincaré explains:

> "The logician cuts up, so to speak, each demonstration into a very great number of elementary operations; when we have examined these operations one after the other and ascertained that each is correct, are we to think we have grasped the real meaning of the demonstration? Shall we have understood it even when, by an effort of memory, we have become able to repeat this proof by reproducing all these elementary operations in just the order in which the inventor has arranged them; that *I know not what* (emphasis G.H.), which makes the unity of the demonstration, will completely elude us. […]
> Pure analysis puts at our disposal a multitude of procedures whose infallibility it guarantees. […] We need a faculty which makes us see the end from afar, and intuition is this faculty" (Poincaré 1905, 217-218).

Although the rigor of the proof is all the greater than the gaps are smaller and a proof without gaps is explicit and can be verified step by step, such a verification is sufficient to understand *that* a proof is right but not yet sufficient to understand *why* it is right. This requires aesthetic intuition:

> "It may be surprising to see emotional sensibility invoked *à propos* of mathematical demonstrations which, it would seem, can interest only the intellect. This would be to forget the feeling of mathematical beauty, of the harmony of numbers and forms, of geometrical elegance. This is a true aesthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility […] This harmony is at once a satisfaction of our aesthetic needs and an aid to the mind, sustaining and guiding" (Poincaré, 1908, 391).

Good proofs characterize the border space between logic and aesthetics. Because aesthetics is "responsible for generating new ideas and insights that could not be described by logical aspects alone", Nathalie Sinclair calls "generative aspect" this distinguishing feature of the mathematical mind (Sinclair 2004, 264).

The *main goal,* what needs to be solved, is to have a look for the general characterization for the generative aspect of aesthetic sensibility in mathematics as a symbolic system. While defending the functional aspect of aesthetics, the traditional trait of Poincaré consists in the fact that he associates aesthetic in mathematics with properties to which he attributes an aesthetic merit, i.e. intellectual beauty. Leaving aside this essential trait, the lines I draw here use Nelson Goodman's research of the way aesthetics works, by adding some features of Peirce's pragmaticism.

3 Goodman's Languages of Art[6]

The theoretical foundation of Goodman's approach to aesthetics is Poincaré's insight that aesthetic experience in mathematics can be linked to cognitive experience. This is nothing new: the link between aesthetics and cognition has been established since Greek antiquity and persisted right up to the birth of aesthetics and underwent a clear break after Kant's Third Critique. However, from the beginning of the twentieth century and the development of conceptual art, aesthetics was once again thought of as a means of cognition and Goodman occupies a major place in the advocacy of this position:

> "Rating works of art or scientific discoveries according to their greatness matters much less then comprehending and projecting them. Delight is a dividend that comes with the achievement of new insights by means of either science or art. […] The philosophy of science and the philosophy of art are embraced within epistemology conceived as the philosophy of the understanding" (Goodman 1984, 148).

Although Goodman displaces the problems of aesthetics into the language of art, emotions and evaluations are not excluded, but he creates a new link between cognition and emotion. Critics often ask:

> "But isn't [Goodman] cutting out all that is really important in aesthetics, all the so-called pleasure and value?" (Goodman 1997, 19)

His answer is: The intention is precisely to describe the general considerations on which such pleasure and value are based (Goodman 1997, 19). Pleasure is subordinated to cognition in the sense of understanding not through the consideration of specific properties but through the specific use of a symbolic system.

To this end, it replaces the essentialist question "what is art?" with the question "when is there art?". In doing so, he establishes his theory outside the framework usually reserved for aesthetics, that delimited by the fine arts. In fact, he notes,

> "we have to avoid confusing the notion of art with the notion of good art. The said truth is that, if you think of it, most works of art are bad. All you have to do is to go looking at most exhibitions" (Goodman 1997, 18).

[6] Cf. on this section the excellent thesis of Jullien (2008) and Heinzmann (2007).

The shift from the analysis of aesthetic merit to that of aesthetic functioning, and the cognitive conception of aesthetic experience, are two closely related points in Goodman's theory of extending the fields of application of aesthetics to all activities involving symbolic systems[7]:

> "Even among works of art and aesthetic experiences of evident excellence, the emotive component varies widely—from, say, a late Rembrandt to a late Mondrian, or from a Brahms to a Webern quartet. The Mondrian and the Webern are not obviously more emotive than Newton's or Einstein's laws; and a line between emotive and cognitive is less likely to mark off the aesthetic neatly from the scientific than to mark off some aesthetic objects and experiences from others" (Goodman 1969, 246–247)."

Goodman's semiotic approach to aesthetics blurs the boundaries between science and art. Of course, he is by no means concerned with unification, but merely shows that the difference is not to be found in the contrast between affective and cognitive but concerns a different use of symbols. The difference should rather be sought in the manner of construction than in the assumption that science discovers, and art creates. In this sense, Poincaré recommended a humanistic education above all for the creators in science. This is why Goodman's theory of mathematics seems for me the obvious choice to interpret Poincaré.

Goodman gives five syntactic and semantic symptoms that often distinguish the aesthetic from the non-aesthetic (Goodman 1978, 67–68), although he notes that "it's true that some of these symptoms are common to certain arts and others are not, […so] we still don't have a distinction" (Goodman 1997, 18): hence a symptom being neither a sufficient nor a necessary criterion, I want to limit myself in mathematics to the symptom of "exemplification".[8]

It is distinguished from *denotation*, which, as an instrument of representation, goes from a predicate to an object to which this predicate can be applied, by its direction: exemplification, as the instrument of *expression*, goes from the object to the predicate. More precisely, the object that exemplifies a predicate that applies to it possesses the properties, literally or metaphorically, that the predicate denotes. But this does not mean that an object exemplifies all the features it possesses. An inscription of 'red' written in blue denotes all red objects but does not exemplify the predicate 'to be red', whereas it can exemplify the predicate 'to be blue', provided that reference is made to it. A metaphorical exemplification is, for instance, a painting which expresses sadness in spite of the fact that paintings cannot literally

[7] On this point see Morizot (1996, 14).

[8] The other three symptoms are "syntactic density", "relative repleteness", and "multiple and complex reference": intuitively, a symbolic system is syntactically dense when there's no way to exhaustively list each character, no way to go through them, no way to isolate them from each other (Goodman 1969, 135–137). By introducing syntactic density, Goodman allows an initial classification between images and non-images. To distinguish images from diagrams, he introduces another syntactic symptom: relative repleteness, which makes of the difference between images and diagrams a matter of degree (Goodman 1969, 230); multiple and complex reference is present "where a symbol performs several integrated and interacting referential functions" (Goodman 1978, 68).

be sad. An object to which the predicate applies without reference from the object to the predicate is an instance of the predicate. We shouldn't think that symbolic systems can be neatly and definitively divided into denotative and exemplificative: a tailor's sample book usually functions as an exemplification system, with each piece of cloth exemplifying its color, texture and so on. But it can also be used to show what a tailor's sample is.

4 Proof-Understanding as an Aesthetic Function

First of all, it is necessary to draw attention to a possible misinterpretation which can be described as follows with Wagner (2001, 369–370):

> "[I] argue that since topic-neutral logical inference might not suffice for insight, one needs a topic-specific route from a premise *A* to a proposition p that the 'universal medium' of logic cannot provide. […] The argument given does not […] suggest that for some knowable *p*, no deduction of *p* from *A* would provide adequate knowledge or insight [for an agent *S*]. It suggests rather that the state of having sufficient knowledge cannot be *identical with* or *causally guaranteed* by the state of having deduced *p*. […] [We have to make it explicit]. If [some proof] enlightens *S* while [another] does not, there is a cognitive epistemological difference that logicism leaves unexplained".

This is my subject.

In the literature, one can find a broad range of attempts to precise what makes a proof more understandable than another with respect to the theorem proved. [9] In contrast, the understanding of the proof itself received much less attention, if only there were no logical gaps.

In Heinzmann (1997), I proposed a "*pragmatic"* way out based on Peirce's distinction between corollarial and theorematic reasoning analyzed in Heinzmann (1994), that I would like to use to distinguish between logically correct and also 'explanatory reasoning'' in the sense I defined this expression. But I don't pass judgment on the appropriateness of all the steps in Peirce's proposal and interpret it with Goodman's eyes.

Peirce uses his distinction between corollarial and theorematic reasoning to explain the extensive character of mathematical cognition without having to resort to the device of assuming a synthetic a priori element. The deductive process leading to a corollary is limited to conceptual level and corresponds to formal inferences realized only in marginal cases, e.g. in syllogistics and elementary arithmetic (Peirce, NE. IV, 237). A theorematic proof, instead, requires retrogression to the level of action: "Thinking in general terms is not enough. It is necessary that something should be done" (CP 4.233) he says.

According to a suggestion by Hintikka (1973, 1980), "a valid deductive step is theorematic, if it increases the number of layers of quantifiers in the propositions in

[9] e.g. Steiner (1978), Kitcher (1989), Tappenden (2005), Frans and Weber (2014).

question." Although this solution is technically brilliant, the translation of the difference in question into a formal language neglects Peirce's pragmatic context.

Following my own interpretation, the characterization of 'theorematic' defines reasoning where in the pragmatic context the semiotically mediated reference of syntax takes effect. A theorematic proof of a proposition is not given by a finite column of propositions but requires a diagrammatic interpretation of the premise[10] yielding to a procedure of preference of possible interpretations, and so to a modal interpretation of reasoning. Given the fact, that an axiomatic system can be conceived as a pragmatic determination of the meaning of a 2nd order concept, this concept can be understood as a sign. So, it is possible to define the claim formulated in Peirce celebrated maxim[11] as semiosis, in which, through an illustrative sequence of interpretants, a more and more differentiated semiotic classification of the object will be achieved.

By this process, the premise of a reasoning is translated into a diagram, by which one considers and "experiments" different forms of the premise. To determine a theorematic consequence means to choose one such form in eliminating the others; so theorematic acceptability is relative to a semantic purpose. The conclusion is a necessary one, if the transformations have a schematic character and are not only particular modifications embodying the premise.

Goodman's inspiration from Peirce's semiotics being well noted by his interpreters (cf. e.g. Cometti 1997, 39), I propose to argue that Peirce's diagram could be understood as an exemplification in Goodman's sense. More precisely, proof-understanding can be identified with a translation of a premise into an exemplification, by which one express "experiments" of different forms. This leads us to the following definition:

> One understands a proof better if it contains an aesthetic symbol use. This is the case if some partial steps of the proof can be interpreted as exemplifications of a general idea (schema). The explanatory content grows proportionally to the number of aesthetic proof steps.

Contrary to Goodman's explicit statement that.

> "presence or absence of one or more of [the symptoms] does not qualify or disqualify anything as aesthetic; nor does the extent to which these features are present measure the extent to which an object or experience is aesthetic" (Goodman 1978, 68),

I support the hypothesis that different acts of exemplification precisely define degree of aesthetics in mathematics.

[10] A diagram in the sense of Peirce, "is in the first place a Token, or singular object used as a Sign" N.E. IV. 315, note. 1 (circa 1906). Cf. even N.E. IV, X (S.d.) and N.E. II. 968 (c. 1873) where Peirce criticizes the "distrust of intuition" of Weierstraß or N.E. III. 101 where he interprets Klein's intuition as "observation of diagrams".

[11] "Consider what effects that might conceivably have practical bearing you conceive the object of your conception to have. Then your conception of those effects is the whole of your conception of the object" (C.P., 5.422; Peirce 1878/79, p. 48 and C.P. 5.402).

5 Mathematical Examples

In the literature, diagrammatic proofs in the ordinary sense[12] and visual thinking in geometry are often used as examples of aesthetic proofs, because they use properties to which one attribute an aesthetic value. However, in these studies the focus is less on the question of the presuppositions necessary to attribute properties an aesthetic value or function than on the question of their evidential force. In contrast, the problem in analysis and algebra is exactly dual: here it is not the power of proof that is in question, but the aesthetic character.

The frequency with which proofs of the irrationality of $\sqrt{2}$ is cited to illustrate the beauty of mathematics[13] seems reason enough to examine if these proofs or some of them are better understood by the symptom presented.[14]

How to achieve a good understanding of the proof of the theorem *The square root of 2 is not rational*?

There are numerous proofs, including graphical ones,[15] but I will select only three. Their short discussion will show that there exist different symptoms of an aesthetic language use in some of the proofs.[16]

Proof 1 (involving just one prime number: *2*).

A rational number is a fraction *p/q*, where *p* and *q* are integers; we will assume that *p* and *q* have no common factor, since, if they do, it can be eliminated. To say that "$\sqrt{2}$ is not rational" is simply to say that $\sqrt{2}$ cannot be written in the form (*p/q*).

We *suppose* it could be written in this form and deduce a contradiction:

$$(A)\ \sqrt{2} = p/q$$

Multiplying both sides of (A) with q and squaring both sides gives.

$$(B)\ 2q^2 = p^2$$

Concerning formula (B), Papert (1978, 5) remarks:

> "All subjects who have become more than very superficially involved in the problem show unmistakable signs of excitement and pleasure when the hit on the last equation [B]".

My interpretation explains the reason for the excitement: The sentences (*A*) and (*B*) are two representations of the same equation. But the internal relationship between the singular terms *p, q* and *2* can be read in two ways: $2q^2$ and p^2 are both instantiations of the relation "=" which denote them, and they are *exemplifications*

[12] in contrast to Peirc's semiotic view of diagrams.

[13] Cf. e.g. Hardy (1940), Papert (1978) or Spies (2013, 232–243).

[14] In Heinzmann (1997, 141–143), I gave a topological example for exemplification.

[15] Cf. e.g. Van Bendegem (2008).

[16] Cf. on the following remarks Heinzmann (2015 and 2016).

of the predicate "even number", because numbers divisible by *2* are *even* by definition. Equations (*A*) and (*B*) are logical equivalent but (*B*) "makes in addition an exemplifying assertion that is necessary for the proof to "succeed" (Wolkenbauer 2009, 6)[17]. The "ambiguity between the meanings not only makes the proof possible […but] it enriches our aesthetic experience of mathematical reasoning. [This] difference between denotation and exemplification is something that cannot be resolved by the syntax of the expressions alone" (Wolkenhauer 2009, 6). Now, as William Byers argues, ambiguity is a crucial mechanism in mathematics that "transforms the mathematical landscape from the static to one that is dynamic" (Byers 2010, 26). The described dynamic switch from denotation to exemplification is exactly such a creative ambiguity possessing an aesthetic interpretation. However, you may say, exemplification is not very central in the demonstration, which continues as follows:

(C) If p^2 is even, p itself has to be even.[18]

(D) If p is even, then $p=2c$ for some integer value c. Substituting $2c$ for p in (B) gives.

$2q^2=(2c)^2=4c^2$ and, by dividing by 2:

$q^2=2c.^2$

So q^2 is even and according to (C) q is itself even. This contradicts the assumption that p/q is reduced to lowest terms.

Proof 2 (involving the set of all prime numbers).

Proof by prime power expansion:

(A) $\sqrt{2}=p/q$. Then
(B) $2q^2=p^2$.

(C') Assume that every positive integer (except the number 1) can be represented in exactly one way apart from rearrangement as a product of one or more *primes* (this result is known as *The Fundamental Theorem of Arithmetic*). Factor both, p and q, into a product of primes. p^2 and q^2 are factored into a product of the very same primes as *p and q,* each taken twice. Therefore, the prime number 2 has an even exponent in the prime factorization p^2 and q^2; consequently, it has an odd one in the prime factorization of $2q^2$. Contradiction to the assumed Fundamental Theorem.

The proof of *the fundamental theorem of arithmetic* uses, for example, *Euclid's lemma* (if a prime divides the product of two numbers, it must divide at least one of those numbers), and *mathematical induction:*

$$\mathrm{IC}:\ [\mathrm{E(I)} \wedge \forall \mathrm{n}(\mathrm{N(n)} \wedge \mathrm{E(n)} \rightarrow \mathrm{E(nI)})] \rightarrow \forall \mathrm{n}(\mathrm{N(n)} \rightarrow \mathrm{E(n)}).$$

[17] (*A*) is not an exemplification of the predicate "rational number" because it doesn't metaphorically possess the quality of rational number.

[18] Proof by contraposition (using classical logic): If p is odd, then p^2 is; $p=2\,k+1$; $p^2=(2\,k+1)^2=4k^2+4\,k+1$.

Without analyzing the different lemma's of the proof, it seems difficult to consider instances of the induction schema as exemplifications of the schema, which is ineradicably more complex: For an arbitrary number *n*, we can certainly indicate an operation that leads to a proof of *E(n)*, but it is impossible to indicate a uniform form of proof, because the length of the proof, that is, the number of applications, depends on *n*, in such a way that a singular proof, for a certain argument, cannot be considered, given its internal structure, as an exemplification of the scheme of proof: it's just an instance.

It remains to notice that the working mahematician even may object that we should proof a theorem by using a powerful result only if the strength and breadth of the latter are really necessary.

Proof 3 A short proof of the irrationality of √*2* can be obtained from the Rational root theorem, that is, if *p(x)* is a monic polynomial with integer coefficients, then any rational root of *p(x)* is necessarily an integer.

So, one starts with the formula.

$$\sqrt{2} = x$$

and obtains:

$$2 = x^2 or\, p(x) = x^2 - 2$$

It follows that root √*2* of *p(x)* is either an integer or irrational. Because √*2* is not an integer, √*2* must therefore be irrational.

This proof seems to presuppose a fundamental use of exemplification:

The schema of polynomial equations with integer coefficients.

$$a_n x^n + a_{n-1} x^{n-1} + \ldots a_0 = 0$$

are instantiated by a monic polynomial equation.

$$x^2 - 2$$

that exemplifies the general polynomial, and finally submitted to special valuations. Mathematically, Bézout's identity can be used to show that the proof of the Rational root theorem itself is based on a series of exemplifications (see details in Heinzmann (2016, 160–161)). According to the given symptom, this proof is therefore more aesthetic than the other two. Or put it differently: this proof exemplifies metaphorically better beauty (Goodman/Elgin 1988, chap I.6).

In *conclusion*, to show aesthetic elements in a proof presupposes that we can identify the symbolic functioning in the reasoning; we need to be able to distinguish those parts of the proof that contain exemplifications. This is the analytical explanation of the problem raised by Poincaré in relation to the fact that it is not enough to understand the logical operations of a demonstration step by step in order to truly understand why it is a proof. The understanding why a demonstration is correct requires that we know how to make it work in accordance with an aesthetic symptom, since our attention must not be focused solely on its denotative function but also, and above all, on what it exemplifies. That is the translation in Goodman's spirit of Poincaré's injunction: "*le mathématicien doit travailler en artiste*" (Poincaré1905, 282).

One main challenge remains open: in trivial cases, the rigor of a mathematical proof consists in its translation in a gapless logical chain of deductions. In less trivial proofs, this translation become a modal character. But the working mathematician considers a proof as rigorous without convincing himself that a translation in gapless logical chain is possible. My guess is that he limits himself to understanding the proof without having to carry out the logical correctness of all the proof steps. If this is correct, the above intuitive-aesthetic considerations, and thus the explanatory proofs, could have a much more general scope than just distinguishing good proofs.

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