Axiomathes (2019) 29:297–310
https://doi.org/10.1007/s10516-018-9397-z

ORIGINAL PAPER

Fungibility in Quantum Sets

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Received: 2 April 2018 / Accepted: 23 August 2018 / Published online: 27 August 2018

Abstract

It can be intuitively understood that sets and their elements in mathematics reflect the atomistic way of thinking in physics: Sets correspond to physical properties, and their elements correspond to particles that have these properties. At the same time, quantum statistics and quantum field theory strongly support the view that quantum particles are not individuals. Some of the problems faced in modern physics may be caused by such discrepancy between set theory and physical theory. The question then arises: Is it possible to reconstruct the concept of set as a collection of objects that model quantum particles rather than as a mere collection of individuals? David Deutsch has argued that identical entities can be diverse in their attributes, and that this nature, what he calls fungibility, must lie at the heart of quantum physics. In line with this idea, a set theory with fungible elements is established, and the collection of such sets is shown to be endowed with an ortholattice structure, which is better known as quantum logic.

Keywords Fungibility · Individuality · Multiverse · Set theory · Ortholattice · Quantum logic

1 Background

1.1 Individuality and Identity

In classical Maxwell–Boltzmann statistics, a permutation of indistinguishable particles is counted as a new arrangement. For example, letting A and B be particles, and 1 and 2 states, we find that the following two arrangements

i. "A is in 1 and B is in 2"
ii. "A is in 2 and B is in 1"

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are different even if the particles A and B are identical, and the states 1 and 2 are empirically indistinguishable. Hence, classical particles are regarded as individuals.[1] Contrastingly, in quantum Bose–Einstein and Fermi–Dirac statistics, a permutation of indistinguishable particles is not counted as a new arrangement. This means that quantum particles have no individuality.[2] The idea that classical particles are individuals but quantum particles are not is called *Received View*.

The problem then arises as to how to formalize the non-individual objects. Manin (1977) mentioned the possibility of developing a totally new language, which is adequate for describing elementary particles. da Costa et al. (1992), French and Krause (2006) proposed quasi-set theory, in which the principle of identity does not hold in general. Dalla Chiara and Toraldo di Francia (1993) developed quaset theory to serve the same purpose, in which neither the axiom of extensionality nor PII is valid.

In this paper, we will take a different approach focusing on the concept of fungibility advocated by Deutsch (1997, 2010, 2011a, b). He takes the stand that identical objects can become different, and this viewpoint is, in his words, what "Leibniz never thought of".

1.2 Quantum Fungibility

We will explain the concept of quantum fungibility proposed by Deutsch (2011a, b) who supports the multiverse theory in quantum physics.

The term *fungibility* means interchangeability of specific entities. For example, a dollar bill is fungible since it can be mutually traded for another dollar bill. Money is fungible not only legally but also physically: When \$1 is deposited to a bank account and then \$1 is withdrawn from the account subsequently, it is meaningless to ask whether the withdrawn \$1 is the one that was deposited recently or the one that was in the account originally. In this sense, \$1 in a bank account can be called a *configurational entity*.

In quantum mechanics, as Deutsch points out, elementary particles are also configurational entities. Since they are higher-energy configurations of the vacuum, it is meaningless to ask which of the configurations with identical attributes is the one that was emerging recently or the one that was there originally. In fact, these particles are completely identical but can become different. Such *diversity within fungibility* is the heart of quantum physics and is present in various forms.[3]

Let us give another example. A particle has several attributes such as position, mass, and charge. Among these, we consider here *spin* carried by an electron, which is a binary physical system that has values in $\{+1, -1\}$. Suppose that we perform a

[1] Since two indistinguishable individuals could be distinct, it may appear that Leibniz's Principle of the Identity of Indiscernibles (PII) is violated. It is possible, however, to defend PII by invoking the principle of impenetrability, which is tacitly supposed in classical physics.

[2] Still, if PII were abandoned, individuality could be retained.

[3] In Deutsch's own words, "⋯ when a random outcome (in this sense) is about to happen, it is a situation of diversity within fungibility: the diversity is in the variable 'what outcome they are *going* to see"' (Deutsch 2011a).

measurement of spin using a measuring apparatus. By this measurement, a component of spin in a given direction, say z, is determined to be $+1$ or -1. Next, we perform another measurement of the spin in a direction orthogonal to z, say x. The measured value is either $+1$ or -1, which is determined independently of the first measurement. Finally, we measure the z spin again and find that a 50% chance of measuring $+1$ and a 50% chance of measuring -1. This means that the measurement of x spin destroyed the result of the first measurement of z spin. In other words, we cannot obtain precise information about the components of spin in different directions simultaneously. It is what is called *Heisenberg's uncertainty principle*. This time the fungible entities are not money, but universes: Infinitely many universes, which were originally identical in every respect, differentiate such that in half of them the last measurement values are $+1$ and in the other half -1.

To recapitulate: Fungible entities are identical in every respect but can become different. Since PII is the claim that indistinguishable entities must be identical, there is no contradiction between PII and the fungibility.[4]

2 Formalization

In modern mathematics, sets are building blocks of complex mathematical constructs. Our aim is to redefine the concept of set not as a collection of structureless elements, but as that of fungible entities abstracted from quantum particles. The outline is as follows. Recall that an ordinary set can be specified as the interpretation of a formula in the language of first-order logic. For example, the interpretation of a formula $A(x)$, denoted by $[A(x)]$, is the set $\{x \mid A(x)\}$. We extend this definition of a set by using a richer framework of possible world semantics for first-order modal logic, in which possible worlds correspond to multiple universes, and accessibility between worlds correspond to fungibility between universes. Thus, *diversity within fungibility* is incorporated into set theory.

2.1 Language

To begin with, we specify a language, in which we express formulas.

i. Constant symbols

   $c, d, e, \dots$

ii. Individual variables

   $x, y, z, \dots$

iii. Predicate symbols

   $P, Q, R, \dots$

[4] The converse of PII, the Indiscernibility of Identicals, conflicts with the fungibility.

iv. Logical connectives

$\wedge$ (conjunction)
$\neg$ (negation)

v. Quantifier

$\forall$ (universal quantification)

The definitions of a term and a first-order formula are as usual. For example, c and y are terms and can be arguments of formulas. $P(c)$, $P \wedge \neg Q$, and $\forall x.R(x)$ are formulas and express some propositions. Metavariables $A, B, C, \cdots$ are used to range over formulas. $\vee$ (disjunction) and $\exists$ (existential quantifier) are defined as abbreviations : $A \vee B \equiv \neg(\neg A \wedge \neg B)$ and $\exists x.A \equiv \neg\forall x.\neg A$. The language we adopt here is minimal: Predicate symbols except those of arity 1 and function symbols are not considered.

2.2 Structure

Following the usual procedure in logic, the truth value of a first-order formula is determined by a structure. A *structure* is a pair $\langle U, V\rangle$, where U is a nonempty set and V is a function that maps a constant symbol to an element in U and a predicate symbol a subset of U. To be precise, it is necessary to give a structure of the form $\langle U(w), V(w, -)\rangle$ for each world w so that a formula is interpreted in multiple worlds.

Definition of W and R. Let W be an ordinary set, whose elements are referred to as *worlds*, and R a reflexive and symmetric relation on W, which is referred to as *fungibility*.[5] In each fungible world, there exists a fungible element identical to that in the actual world.[6]

Definition of $U(w)$ *and* $r(w, w')$. For $w \in W$, $U(w)$ is defined to be a set such that all $U(w)$'s are isomorphic.[7] For $w, w' \in W$ such that $R(w, w')$, $r(w, w')$ is defined to be a function that maps $u \in U(w)$ to $r(w, w')(u) \in U(w')$ satisfying the following conditions:

i. $r(w, w)$ is the identity function on $U(w)$, denoted by $\mathrm{id}_{U(w)}$, that is, $r(w, w)(u) = u$ for any $u \in U(w)$.
ii. $r(w, w') \circ r(w', w) = r(w, w) = \mathrm{id}_{U(w)}$.[8]

Definition of $V(w, -)$. For $w \in W$, $V(w, -)$ is a function that maps a constant symbol c to an element $V(w, c) \in U(w)$ and a predicate symbol P a subset $V(w, P) \subseteq U(w)$ satisfying the following conditions:

5 $R(w, w')$ if and only if w is fungible with w'.

6 This immediately gives rise to the problem of *transworld identity*, that is, the problem of how an individual in w can be said to be identical to an individual in w'. Our approach to this problem is to stipulate, following what Lewis did in his counterpart theory (Lewis 1986), that the fungibility be a similarity relation.

7 $U(w)$ and $U(w')$ are isomorphic if and only if there exists a bijection $\varphi : U(w) \longrightarrow U(w')$.

8 This means that all $r(w, w')$'s are bijections.

i. $r(w', w)(V(w', c)) = V(w, c)$
ii. $\bigcap_{w' \in W} r(w', w)(\bigcup_{w'' \in W} r(w'', w')(V(w'', P))) = V(w, P)$[9]

The last condition may need some explanation. In the usual first-order logic, an interpretation of a predicate symbol is merely the set of elements that share an attribute. The situation we have in mind is, however, that these elements may have fungibility: even identical elements will possibly differ in their attributes. For example, the measurement value of the z spin of an electron depends on which world we are in. Therefore, it seems less obvious how a predicate symbol should be interpreted in our setting. In fact, it is a negated predicate that must always be taken into account. For any predicate symbol P, the negated predicate $\neg P$ should be interpreted to mean that there is no probability of having the attribute represented by P. Hence it is natural to define that an element c has the attribute represented by $\neg P$ in the actual world w if and only if for any world w' that is fungible with w, c does not have the attribute represented by P in w'. Furthermore, this statement should still hold if P is replaced by $\neg P$, and $\neg\neg P$ is identified with P: an element c has the attribute represented by P in the actual world w if and only if for any world w' that is fungible with w, c does not have the attribute represented by $\neg P$ in w'. That is, an element c has the attribute represented by P in the actual world w if and only if for any world w' that is fungible with w, there exists some world w'' that is fungible with w' such that c has the attribute represented by P in w''. This is what we needed.

We then extend $V(w, -)$ to a function defined on all formulas in the following way.

- $[P(x)](w) = V(w, P)$ where P is a predicate symbol.
- $[A \wedge B](w) = [A](w) \cap [B](w)$
- $[\neg A](w) = \bigcap_{w' \in W} r(w', w)([A](w')^c)$

In the definition of $[\neg A](w)$, it is necessary to refer to all fungible worlds so that the following proposition holds.

Proposition 1 $\bigcap_{w' \in W} r(w', w)(\bigcup_{w'' \in W} r(w'', w)([A](w''))) = [A](w)$ *for any formula* A.

Proof First, we show that

$$[A](w) \subseteq \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w)\big([A](w'')\big)\right).$$

[9] We write $\bigcap_{w' \in W} r(w', w)(X(w'))$ as shorthand for $\bigcap_{w' \in W}(\{r(w', w)(y) \mid y \in X(w')\})$ and write $\bigcup_{w' \in W} r(w', w)(X(w'))$ as shorthand for $\bigcup_{w' \in W}(\{r(w', w)(y) \mid y \in X(w')\})$. For $w', w \in W$ such that $R(w', w)$ does not hold, $r(w', w)(X(w'))$ is regarded as $\emptyset$.

Suppose that $R(w', w)$ holds. Then $R(w, w')$ also holds due to the symmetry. Therefore, $\bigcup_{w'' \in W} r(w'', w')([A](w''))$ contains all elements of $\{r(w, w')(y) \mid y \in [A](w)\}$. That is,

$$\{r(w, w')(y) \mid y \in [A](w)\} \subseteq \bigcup_{w'' \in W} r(w'', w')([A](w'')).$$

By employing the monotonicity of $\bigcap_{w' \in W} r(w', w)$ with respect to $\subseteq$, we have that

$$\bigcap_{w' \in W} r(w', w)\{r(w, w')(y) \mid y \in [A](w)\} \subseteq \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w''))\right),$$

where the LHS is simply equal to $[A](w)$.

Next, we show that

$$\bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w''))\right) \subseteq [A](w)$$

by induction on the construction of A.

Case P This is immediate from the definition of $V(w, P)$.

Case $A \wedge B$ Suppose

$$x \in \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A \wedge B](w''))\right),$$

which means that

$$x \in \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'') \cap [B](w''))\right).$$

That is,

$$x \in \bigcap_{w' \in W} \{r(w', w)(y) \mid y \in \bigcup_{w'' \in W} \big(\{r(w'', w')(z) \mid z \in [A](w'')\} \cap \{r(w'', w')(z) \mid z \in [B](w'')\}\big)\}.$$

By applying the infinite distributive law, we have that

$$x \in \bigcap_{w' \in W} \{r(w', w)(y) \mid y \in \bigcup_{w'' \in W} \{r(w'', w')(z) \mid z \in [A](w'')\} \cap \bigcup_{w'' \in W} \{r(w'', w')(z) \mid z \in [B](w'')\}\},$$

which can be divided into

$$x \in \bigcap_{w' \in W} \{r(w', w)(y) \mid y \in \bigcup_{w'' \in W} \{r(w'', w')(z) \mid z \in [A](w'')\}\}$$

and

$$x \in \bigcap_{w' \in W} \{r(w', w)(y) \mid y \in \bigcup_{w'' \in W} \{r(w'', w')(z) \mid z \in [B](w'')\}\}.$$

Using the induction hypothesis, we can simplify these to

$$x \in [A](w) \text{ and } x \in [B](w).$$

That is,

$$x \in [A](w) \cap [B](w),$$

which means that

$$x \in [A \wedge B](w).$$

Since x was arbitrary, we conclude that

$$\bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')\big([A \wedge B](w'')\big)\right) \subseteq [A \wedge B](w).$$

Case $\neg A$ We have already shown that

$$[A](w') \subseteq \bigcap_{w'' \in W} r(w'', w')\left(\bigcup_{w''' \in W} r(w''', w'')([A](w'''))\right).$$

By employing the monotonicity of $\bigcup_{w'\in W} r(w',w)$ with regard to $\subseteq$, we have that

$$\bigcup_{w'\in W} r(w',w)([A](w')) \subseteq \bigcup_{w'\in W} r(w',w)\left(\bigcap_{w''\in W} r(w'',w')\left(\bigcup_{w'''\in W} r(w''',w'')([A](w'''))\right)\right).$$

We apply the set complement operation to obtain

$$\left(\bigcup_{w'\in W} r(w',w)\left(\bigcap_{w''\in W} r(w'',w')\left(\bigcup_{w'''\in W} r(w''',w'')([A](w'''))\right)\right)\right)^c \subseteq \left(\bigcup_{w'\in W} r(w',w)([A](w'))\right)^c.$$

By de Morgan's law, we have that[10]

$$\bigcap_{w'\in W} r(w',w)\left(\bigcup_{w''\in W} r(w'',w')\left(\bigcap_{w'''\in W} r(w''',w'')([A](w''')^c)\right)\right) \subseteq \bigcap_{w'\in W} r(w',w)([A](w')^c),$$

which means that

$$\bigcap_{w'\in W} r(w',w)\left(\bigcup_{w''\in W} r(w'',w')([\neg A](w''))\right) \subseteq [\neg A](w).$$

□

3 Algebra of Quantum Sets

In the preceding section, we have introduced a multiple-world semantics for first-order formulas. Within this framework, we will now give a formal definition of quantum sets. In addition, we will show that the collection of all quantum sets is not, in general, a Boolean algebra but an ortholattice.

[10] This step is not straightforward. For example, the RHS is worked out as follows: $(\bigcup_{w'\in W} r(w',w)([A](w')))^c = (\bigcup_{w'\in W}(\{r(w',w)(y) \mid y \in [A](w')\}))^c = \bigcap_{w'\in W}(\{r(w',w)(y) \mid y \in [A](w')\}^c) =$ $= \bigcap_{w'\in W}\{r(w',w)(y) \mid y \in [A](w')^c\} = \bigcap_{w'\in W} r(w',w)([A](w')^c)$. Here the second last equality follows from the fact that $r(w',w)$ is a bijection. We will omit such details in the rest of the paper.

3.1 Quantum Sets on $\langle W, R \rangle$

Using the same symbols as in Sect. 2.2, we refer to a pair $\langle U, r \rangle$ as a *quantum universal set on* $\langle W, R \rangle$. In the usual set theory, a universal set is a set that contains all objects related to a specific problem.[11] For example, when we talk about calculus, we consider the set of all real numbers $\mathbb{R}$ as a universal set. Note that an ordinary universal set, say $\mathbb{R}$, can also be seen as the quantum universal set $\langle \mathbb{R}, r \rangle$ on $\langle \{a\}, R \rangle$, where r, a and R are arbitrary. Thus, the definition of a quantum universal set on $\langle W, R \rangle$ can be thought of as an extension of that of an ordinary universal set.

Next, we define an *element* of $\langle U, r \rangle$ as a map $[c]$ that assigns a world w to an interpretation $V(w, c)$ of a constant symbol c, that is, $[c] : W \longrightarrow \{V(w, c) \mid w \in W\}$ such that $w \mapsto V(w.c)$, and a *subset* of $\langle U, r \rangle$ a map $[A]$ that assigns a world w to an interpretation $[A](w)$ of a formula A of arity 1, that is, $[A] : W \longrightarrow \{[A](w) \mid w \in W\}$ such that $w \mapsto [A](w)$. Henceforth, we will fix a quantum universal set $\langle U, r \rangle$ on $\langle W, R \rangle$, and simply call a subset of $\langle U, r \rangle$ a *quantum set on* $\langle W, R \rangle$.

Proposition 2 *The collection of all quantum sets on $\langle W, R \rangle$ is not, in general, a Boolean algebra but an ortholattice.*

Proof We define quantum set operations as follows. Let Π be the collection of all quantum sets on $\langle W, R \rangle$, and $\subseteq_{\langle W,R \rangle}$ the relation on Π such that $[A] \subseteq_{\langle W,R \rangle} [B]$ if and only if $[A](w) \subseteq [B](w)$ for each $w \in W$.[12]

Claim Π is a partially ordered set with respect to $\subseteq_{\langle W,R \rangle}$.

Indeed, it is immediate to verify that $\subseteq_{\langle W,R \rangle}$ satisfies the reflexivity, transitivity and antisymmetry conditions.

Claim Π is a lattice with respect to $\subseteq_{\langle W,R \rangle}$.

Indeed, we can find the infimum and the supremum of any pair elements of Π as follows. For the infimum, we set $\inf\{[A], [B]\} = [A \wedge B]$, noting that we can prove the following properties:

i. $[A \wedge B] \subseteq_{(W,R)} [A]$ and $[A \wedge B] \subseteq_{(W,R)} [B]$.
ii. For any $X \in \Pi$, if $X \subseteq_{(W,R)} [A]$ and $X \subseteq_{(W,R)} [B]$, then $X \subseteq_{(W,R)} [A \wedge B]$.

These follow from the fact that for any $w \in W$, $[A \wedge B](w)(= [A](w) \cap [B](w))$ is the infimum of $\{[A](w), [B](w)\}$ with respect to $\subseteq$. For the supremum, we set $\sup\{[A], [B]\} = [\neg(\neg A \wedge \neg B)]$, noting that we can prove the following properties:

[11] We confine ourselves to dealing with 'small' sets here in order to avoid Russell's paradox.

[12] Here $\subseteq$ denotes the usual inclusion relation.

i. $[A] \subseteq_{(W,R)} [\neg(\neg A \wedge \neg B)]$ and $[B] \subseteq_{(W,R)} [\neg(\neg A \wedge \neg B)]$.
ii. For any $X \in \Pi$, if $[A] \subseteq_{(W,R)} X$ and $[B] \subseteq_{(W,R)} X$, then $[\neg(\neg A \wedge \neg B)] \subseteq_{(W,R)} X$.

Here, the first part of (i) is proved as follows.[13] Using Proposition 1, we have that, for any $w \in W$,

$$[A](w) \subseteq \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'')) \right).$$

We also have that, for any $w'' \in W$ such that $R(w, w')$ and $R(w', w'')$,

$$[A](w'') \subseteq [A](w'') \cup [B](w'').$$

From this and the monotonicity of $\bigcup_{w'' \in W} r(w'', w')$ and $\bigcap_{w' \in W} r(w', w)$ with respect to $\subseteq$, we have that

$$\bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'')) \right) \subseteq \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'') \cup [B](w'')) \right).$$

Therefore, by the transitivity of $\subseteq$, we have that

$$[A](w) \subseteq \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'') \cup [B](w'')) \right),$$

which means that

$$[A](w) \subseteq [\neg(\neg A \wedge \neg B)](w).$$

For (ii), letting

$$[A](w'') \subseteq X(w'') \quad \text{and} \quad [B](w'') \subseteq X(w'')$$

for an arbitrary $w'' \in W$, we have that

$$[A](w'') \cup [B](w'') \subseteq X(w'').$$

From this and the monotonicity of $\bigcup_{w'' \in W} r(w'', w')$ and $\bigcap_{w' \in W} r(w', w)$ with respect to $\subseteq$, we have that

$$\bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'') \cup [B](w'')) \right) \subseteq \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')(X(w'')) \right).$$

[13] The second part can be proved exactly in the same way.

Using Proposition 1, we can simplify this to

$$\bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w'') \cup [B](w''))\right) \subseteq X(w),$$

which means that

$$[\neg(\neg A \wedge \neg B)](w) \subseteq X(w).$$

Claim Π is an ortholattice with respect to $\subseteq_{\langle W,R \rangle}$.

Let 0 be the map $\emptyset : W \longrightarrow \{\emptyset(w) \mid w \in W\}$ such that $w \mapsto \emptyset(w)$, and 1 the map $U : W \longrightarrow \{U(w) \mid w \in W\}$ such that $w \mapsto U(w)$. Then, we see that 0 is the minimum element of Π, and 1 the maximum element of Π. For the orthocomplementation of $[A]$, we set $[A]^{\perp} := [\neg A]$, noting that we can prove the following properties:

i. $([A]^{\perp})^{\perp} = [A]$.
ii. If $[A] \subseteq_{(W,R)} [B]$, then $[B]^{\perp} \subseteq_{(W,R)} [A]^{\perp}$.
iii. $\inf([A], [A]^{\perp}) = 0$ and $\sup([A], [A]^{\perp}) = 1$.

Indeed, for (i), we have that, for any $w \in W$,

$$\begin{aligned} ([A]^{\perp})^{\perp}(w) &= \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w''))^c \right)^c \\ &= \bigcap_{w' \in W} r(w', w)\left(\bigcup_{w'' \in W} r(w'', w')([A](w''))\right) \\ &= A(w) \end{aligned}$$

where the last equality follows from Proposition 1. For (ii), letting

$$[A](w) \subseteq [B](w)$$

for an arbitrary $w \in W$, we have that

$$[B](w)^c \subseteq [A](w)^c.$$

From this and the monotonicity of $\bigcap_{w' \in W} r(w', w)$ with respect to $\subseteq$, we have that

$$\bigcap_{w' \in W} r(w', w)([B](w')^c) \subseteq \bigcap_{w' \in W} r(w', w)([A](w')^c),$$

which means that

$$[\neg B](w) \subseteq [\neg A](w).$$

For (iii), we have that, for any $w \in W$,

$$\begin{aligned}[A \wedge \neg A](w) =&[A](w) \cap \bigcap_{w' \in W} r(w', w)([A](w')^c) \\ \subseteq&[A](w) \cap r(w, w)([A](w)^c) \\ =&[A](w) \cap [A](w)^c \\ =&0(w)\end{aligned}$$

Since $0(w)$ is the minimum element of $U(w)$ with respect to $\subseteq$, we also have that

$$[A \wedge \neg A](w) \subseteq X^{\perp}(w)$$

for an arbitrary X. We then apply (ii) to obtain

$$(X^{\perp})^{\perp}(w) \subseteq [A \wedge \neg A]^{\perp}(w).$$

Using (i), we can rewrite this as

$$X(w) \subseteq [\neg(\neg A \wedge \neg(\neg A))](w).$$

Since X was arbitrary, we conclude that

$$[\neg(\neg A \wedge \neg(\neg A))](w) = 1(w).$$

□

3.2 Distributivity Breaking and Fungibility

The fact that Π is not, in general, a Boolean algebra is closely related to the fungibility. We will show this by constructing a toy model in which the distributivity of inf over sup can fail. Let W be $\{w_0, w_1, w_2, w_3\}$ and R the reflexive and symmetric closure of $\{(w_0, w_1), (w_1, w_2), (w_2, w_3)\}$, that is,

$$\overset{\curvearrowright}{w_0} \longleftrightarrow \overset{\curvearrowright}{w_1} \longleftrightarrow \overset{\curvearrowright}{w_2} \longleftrightarrow \overset{\curvearrowright}{w_3}.$$

Further, our language is taken to consist only of the constant symbols c, d and the predicate symbols P, Q. We then give a semantic valuation of the predicate symbols as follows.

$$\begin{aligned}V(w_0, P) &= \{[c](w_0)\}, \quad V(w_0, Q) = \{[d](w_0)\}, \\ V(w_1, P) &= \{[c](w_1)\}, \quad V(w_1, Q) = \{[d](w_1)\}, \\ V(w_2, P) &= \{[d](w_2)\}, \quad V(w_2, Q) = \{[c](w_2)\}, \\ V(w_3, P) &= \{[d](w_3)\}, \quad V(w_3, Q) = \{[c](w_3)\}.\end{aligned}$$

Under this valuation, we have that

$$[c](w_1) \notin [Q](w_1) \ \text{ and } [c](w') \in [Q](w'). \tag{1}$$

for some w' (namely w_2) such that $R(w_1, w')$. Regarding w_1 as the actual world we are in, c an electron and $Q[x]$ the predicate asserting that the value of z spin of an electron x is $+1$, we mean by (1) that the actual measurement value of z spin of c is not $+1$, but there could be a chance of being measured as $+1$. We can rewrite (1) as

$$[c](w_1) \in [Q](w_1)^c \ \text{ and } \ [c](w_1) \in \bigcup_{w' \in W} r(w', w_1)([Q](w')).$$

That is,

$$[c](w_1) \in [Q](w_1)^c \ \text{ and } \ [c](w_1) \in [Q]^{\perp}(w_1)^c$$

where we have used the following equalities:

$$\begin{aligned}\bigcup_{w' \in W} r(w', w_1)([Q](w')) &= \left(\bigcap_{w' \in W} r(w', w_1)([Q](w'))^c \right)^c \\ &= [Q]^{\perp}(w_1)^c.\end{aligned}$$

Therefore, we have that

$$[P](w_1) \cap [Q](w_1) = \emptyset \ \text{ and } \ [P](w_1) \cap [Q]^{\perp}(w_1) = \emptyset.$$

Using similar arguments, we can observe that

$$\begin{aligned}&[P](w_0) \cap [Q](w_0) = \emptyset, \quad && [P](w_0) \cap [Q]^{\perp}(w_0) = \{[c](w_0)\}, \\ &[P](w_2) \cap [Q](w_2) = \emptyset, \quad && [P](w_2) \cap [Q]^{\perp}(w_2) = \emptyset, \\ &[P](w_3) \cap [Q](w_3) = \emptyset, \quad && [P](w_3) \cap [Q]^{\perp}(w_3) = \{[d](w_3)\}.\end{aligned}$$

With these values, we have that

$$\begin{aligned}&[\neg(\neg(P \wedge Q) \wedge \neg(P \wedge \neg Q))](w_1) \\ &= \bigcap_{w' \in W} r(w', w_1) \left(\bigcap_{w'' \in W} r(w'', w')([P \wedge Q](w''))^c \cap \bigcap_{w'' \in W} r(w'', w')([P \wedge \neg Q](w''))^c \right)^c \\ &= \bigcap_{w' \in W} r(w', w_1) \left(\bigcup_{w'' \in W} r(w'', w')(([P](w'') \cap [Q](w'')) \cup ([P](w'') \cap [Q]^{\perp}(w''))) \right) \\ &= \emptyset,\end{aligned}$$

while we also have that

$$\begin{aligned}[P \wedge \neg(\neg Q \wedge \neg(\neg Q))](w_1) &= [P](w_1) \cap 1(w_1) \\ &= [P](w_1) \\ &= \{[c](w_1)\} (\neq \emptyset).\end{aligned}$$

Therefore, we conclude that

$$[P \wedge \neg(\neg Q \wedge \neg(\neg Q))](w_1) \not\subseteq [\neg(\neg(P \wedge Q) \wedge \neg(P \wedge \neg Q))](w_1)$$

which implies that

$$\inf\{[P], \sup\{[Q], [Q]^{\perp}\}\} \not\subseteq_{(W,R)} \sup\{\inf\{[P], [Q]\}, \inf\{[P], [Q]^{\perp}\}\}.$$

□

4 Concluding Remarks

This paper has presented a theory of sets with fungible elements and has shown that the collection of all such sets forms an ortholattice. Since the ortholattice structure is also known as quantum logic, it is reasonable to call these sets quantum sets (Goldblatt 1974; Dalla Chiara et al. 2002) .[14]

Deutsch has mentioned that what is lacking in studies of the multiverse is an explicit mathematical description (Deutsch 2010). It would be interesting to see in the future what kind of mathematics, which is expressive enough to describe the multiverse, can be built on this theory of quantum sets.

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[14] On the other hand, there has been much controversy whether such quantum logic based on the lattice of closed subspaces in a Hilbert space is really entitled to be referred to as the basis of quantum physics (Jammer 1974).