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MODIFIED TSALLIS HOLOGRAPHIC DARK ENERGY

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In this work we propose Modified Tsallis Holographic Dark Energy (MTHDE) in General Relativity (GR) in the framework of Bianchi type III space-time. Einstein's field equations are solved by using a special law of variation of Hubble parameter H proposed by Berman which yields constant deceleration parameter (DP). Interestingly, for the two different constant values of deceleration parameter, we have obtained two different cosmological models. The model 1 behaves like a quintessence dark energy model whereas model 2 behaves like a cosmological constant model. A correspondence between model 1 and quintessence scalar field is established. The quintessence dynamics of the potential and scalar field are reconstructed which illustrates the accelerating phase of the Universe. Various parameters like deceleration parameter, Hubble parameter, anisotropy parameter, equation of state (EOS) parameter, etc. for both the cosmological models are thoroughly discussed. The results obtained are found to be consistent with the recent observations on the present-day Universe.
Keywords: *MTHDE: GR: Hubble parameter: deceleration parameter DP*

1. Introduction

Recent astrophysical observational data [1-6] show that our Universe is going through a phase of accelerated expansion which put new avenues in modern cosmology. A class of people are making attempts to accomodate this

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observational fact by choosing some exotic matter (known as dark energy) in the framework of general relativity. Dark energy (DE) is believed to dominate over the matter content of the Universe by 70%. In all theories and models, the cosmological constant model is the most natural and simplest candidate of DE with the equation of state (EOS) parameter $\omega = -1$ but it suffers from cosmic coincidence and fine-tuning problem [7,8]. To relieve such problems, various dark energy models have been suggested in literature such as quintessence [9], phantom [10], *k*-essence [11], tachyon [12], HDE [13], etc.

Despite of many efforts from different observational and theoretical ways, the problem of DE is still not well settled due to its unknown nature. In order to justify the source of accelerating expansion (i.e. the nature of DE) of the Universe, two different approaches have been adopted. One way is to modify the geometric part of Einstein-Hilbert action (termed as modified theories of gravity) for the discussion of expansion phenomenon [14-18]. The second approach is to propose the different forms of DE called dynamical DE models. Up to now, different dynamical DE models have been proposed in two different contexts such as quantum gravity and GR. Holographic dark energy have been proposed in the framework of quantum gravity on the basis of holographic principle [19]. The density of HDE model has the following form $\rho_{DE} = 3c^2 M_p^2 L^{-2}$ where c is a specific constant, $M_p = (8\pi G)^{-1/2}$ termed as reduced Planck mass and L represent the infrared (IR) cutoff described the size of the Universe. By considering horizon entropy of a black hole, Tsallis and Cirto assumed some quantum modification for HDE given by (Tsallis and Cirto [20]) $S_\delta = \gamma A^\delta$ with γ being an unknown constant and δ represents the non-additivity parameter chosen to have a positive value. The Bekenstein entropy is a particular case when $\delta = 1$ and $\gamma = 1/4G$ [21]. Considering the holographic hypothesis, Cohen et al. [22] proposed the relation among the system entropy S, the IR (L) and UV (Λ) cutoffs as $L^3 \Lambda^3 \le S^{3/4}$ which after combining with $S_\delta = \gamma A^\delta$ gives $\Lambda^4 \le \gamma (4\pi)^\sigma L^{2\delta - 4}$. Using this inequality, the THDE density is obtained as $\rho_T = DL^{2\delta-4}$ where D is an unknown parameter [23-25]. It is worthy to mention that for $\delta = 1$, the standard HDE is recovered. Furthermore, for $\delta = 2$, the cosmological constant model is retrieved. Using the Hubble horizon H^{-1} as the IR cutoff L, $\rho_T = DH^{-2\delta+4}$ is obtained.

Since DE occupies almost 70% of the content of the Universe today, it is rational to assume that the density of DE is a function of the Hubble parameter H and its derivative w.r.t. cosmic time [26]. In this paper, we have modified the THDE by assuming $\rho_{MT} = DH^{-2\delta+4} + E\dot{H}$. In the above expression dot (.) denotes differentiation w.r.t. cosmic time t and E is the arbitrary dimensionless parameter. The early Universe inflation can be considered as the primordial DE because DE is merely the substitute for the accelerating expansion of the Universe [27]. So, our constructed model is a good candidate to describe the inflationary stage.

Bianchi type spaces play an important role in constructing spatially homogeneous and anisotropic cosmological models to describe the behaviour of the Universe at its early stages of its evolution. The anomalies found in the cosmic microwave background (CMB) and large-scale structure (LSS) observations stimulated a growing interest in anisotropic cosmological model of the Universe. Here we confine ourselves to Bianchi type III models.

Several researchers have investigated various cosmological models in the framework of THDE. Two Tsallis Agegraphic DE (TADE) models have been proposed by using the age of the Universe and the conformal time as the IR cut-offs and study their effects on the evolution of the Universe [28]. THDE in FRW Universe with time varying deceleration parameter (DP) in the framework of FRW Universe have been investigated by [29]. Mamon [30] has studied the evolution of a fractal Universe with THDE in presence of an interacting scenario. Sadeghi et al. [31] have explored THDE by considering the complex form of the quintessence model in the framework of Brans-Dicke cosmology. Pradhan et al. [32] have discussed THDE in the modified $f(R,T)$ gravity framework with Granda-Oliveros (GO) cutoff. Mamon et al. [33] have studied THDE in presence of interacting scenario. Dubey et al. [34] have discussed the axially symmetric space-time in THDE. Korunur [35] have explored THDE in Bianchi type III space-time. Yadav [36] has worked out THDE in Brans-Dicke cosmology. Santhi and Sobhanbabu [37] have explained THDE in Saez-Ballester theory of gravitation. Dubey et al. [38] have investigated THDE using hybrid expansion law (HEL) with *k*-essence. Dubey et al. [39] have examined THDE in the non-flat Universe. Motivated by the above aforesaid works, we have modified THDE in GR in the framework of Bianchi type III space-time.

The organisation of the paper is as follows: In Section 2, we formulate the metric and field equations for MTHDE model. In Section 3, we have obtained the solutions of field equations of Bianchi type III space-time. In Section 4, we have studied the cosmological model 1 and the correspondence between model 1 and quintessence scalar field. In Section 5 we have studied the cosmological model 2. The model 1 behaves like a quintessence dark energy model whereas the model 2 behaves like a cosmological constant model. Various parameters for both the models are discussed graphically in Sections 6 and 7 respectively. The paper ends with concluding remarks in Section 8.

2. Metric and field equations

We consider the anisotropic Bianchi type III space-time

$$ds^2 = dt^2 - I^2 dx^2 - J^2 e^{-2x} dy^2 - K^2 dz^2 \tag{1}$$

where the scale factors *I*, *J* and *K* are functions of cosmic time *t* only.

The Einstein's field equations are given by

$$R_{ij} - \frac{1}{2} g_{ij} R = -\left(T'_{ij} + \overline{T}_{ij}\right), \tag{2}$$

where R_{ij} is the Ricci tensor and R is the Ricci scalar.

The energy momentum tensor T'^{i}_{j} for dark matter (DM) is

$$T'^{i}_{j} = \mathrm{diag}\left[\rho_m, 0, 0, 0\right], \tag{3}$$

where ρ_m is the energy density of DM.

The energy momentum tensor $\overline{T}^{i}_{j}$ for MTHDE is

$$\bar{T}_j^i = \text{diag}[\rho_{MT}, -p_{MT_x}, -p_{MT_y}, -p_{MT_z}] = \text{diag}[1, -\omega_x, -\omega_y, -\omega_z]\rho_{MT}$$
$$= \text{diag}[1, -\omega_{MT}, -\omega_{MT}, -\omega_{MT}]\rho_{MT}\ , \tag{4}$$

where ρ_{MT} is the energy density of MTHDE, p_{MT} is the pressure of MTHDE and $\omega_x = \omega_{MT}$, $\omega_y = \omega_{MT}$ and $\omega_z = \omega_{MT}$ are the directional equation of state (EOS) parameters on x, y and z axes respectively and $\omega_{MT}\rho_{MT} = p_{MT}$.

The Einstein's field equations (2) for the metric (1) using Eqs. (3) and (4) takes the form

$$\frac{\ddot{J}}{J} + \frac{\ddot{K}}{K} + \frac{\dot{J}\dot{K}}{JK} = -\omega_{MT}\rho_{MT} \tag{5}$$

$$\frac{\ddot{I}}{I} + \frac{\ddot{K}}{K} + \frac{\dot{I}\dot{K}}{IK} = -\omega_{MT}\rho_{MT} \tag{6}$$

$$\frac{\ddot{I}}{I} + \frac{\ddot{J}}{J} + \frac{\dot{I}\dot{J}}{IJ} - \frac{1}{I^2} = -\omega_{MT}\rho_{MT} \tag{7}$$

$$\frac{\dot{I}\dot{J}}{IJ} + \frac{\dot{J}\dot{K}}{JK} + \frac{\dot{K}\dot{I}}{KI} - \frac{1}{I^2} = \rho_m + \rho_{MT} \tag{8}$$

$$\frac{\dot{J}}{J} - \frac{\dot{I}}{I} = 0. \tag{9}$$

Eq. (9) on integration and taking integrating constant to be unity, we obtain

$$J = I. \tag{10}$$

Using Eq. (10) in Eqs. (5)-(8), we get

$$\frac{\ddot{I}}{I} + \frac{\ddot{K}}{K} + \frac{\dot{I}\dot{K}}{IK} = -\omega_{MT}\rho_{MT} \tag{11}$$

$$2\frac{\ddot{I}}{I} + \frac{\dot{I}^2}{I^2} - \frac{1}{I^2} = -\omega_{MT}\rho_{MT} \tag{12}$$

$$\frac{\dot{I}^2}{I^2}+2\frac{\dot{I}\dot{K}}{IK}-\frac{1}{I^2}=\rho_m+\rho_{MT}\,. \tag{13}$$

The energy conservation equation is

$$\dot{\rho}_m+\dot{\rho}_{MT}+\left(2\frac{\dot{I}}{I}+\frac{\dot{K}}{K}\right)(\rho_m+\rho_{MT}+p_{MT})=0\,, \tag{14}$$

where overhead dot (.) denotes differentiation w.r.t. cosmic time t.

We assume that there is no interaction between DM and MTHDE throughout the study.

3. Solutions of field equations

The average scale factor $a(t)$ and the spatial volume V are defined as

$$V=a^3=I^2K\,. \tag{15}$$

The directional Hubble's parameters H_x, H_y and H_z in the direction of x, y and z axes respectively are given by

$$H_x=H_y=\frac{\dot{I}}{I},\quad H_z=\frac{\dot{K}}{K}. \tag{16}$$

The mean Hubble's parameter H is

$$H=\frac{\dot{a}}{a}=\frac{\dot{V}}{3V}=\frac{H_x+H_y+H_z}{3}=\frac{1}{3}\left(2\frac{\dot{I}}{I}+\frac{\dot{K}}{K}\right). \tag{17}$$

The deceleration parameter q is defined as

$$q=-\frac{a\ddot{a}}{\dot{a}^2}. \tag{18}$$

The anisotropy parameter A_p is defined as

$$A_p=\frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i-H}{H}\right)^2. \tag{19}$$

Field equations (11)-(13) forms a system of three independent equations with five unknowns I, K, ω_{MT}, ρ_{MT} and ρ_m. So, we use two extra relations to solve the system of field equations completely. These are as follows:

(i) Following Chen and Jing [26] and Bharali and Das [40], we define MTHDE density ρ_{MT} as a function of Hubble parameter H and its derivative w.r.t. cosmic time t as follows

$$\rho_{MT} = DH^{-2\delta+4} + E\dot{H}, \tag{20}$$

where E is the arbitrary dimensionless parameter and the other symbols have their usual meanings.

(ii) A special law of variation for Hubble's parameter H proposed by Berman [41] is defined as

$$H = ka^{-m}, \tag{21}$$

where $k > 0$ and $m \geq 0$ are constants.

Using Eqs. (17) and (21), we have obtained two models

$$a = (mkt + k_1)^{1/m}, \quad q = m-1, \quad m < 1, \tag{22}$$

where k_1 is a constant of integration.

$$a = \exp\{k(t-k_2)\}, \quad q = -1, \quad m = 0, \tag{23}$$

where k_2 is a constant of integration.

From Eqs. (11) and (12), we get

$$\frac{\dot{K}}{K} - \frac{\dot{I}}{I} = \frac{u_0}{V}\exp\left(\int -\frac{1}{I^2}\left(\frac{\dot{K}}{K} - \frac{\dot{I}}{I}\right)^{-1} dt\right), \tag{24}$$

where u_0 is a constant of integration.

Following Adhav [42], we assume

$$\frac{\dot{K}}{K} - \frac{\dot{I}}{I} = \frac{1}{I^2}. \tag{25}$$

Using Eq. (25) in Eq. (24), we get

$$\frac{\dot{K}}{K} - \frac{\dot{I}}{I} = \frac{u_0}{V}e^{-t}. \tag{26}$$

Integrating Eq. (26), we obtain

$$K = u_1 I \exp\left[u_0 \int \frac{e^{-t}}{V} dt\right], \tag{27}$$

where u_1 is a constant of integration.

4. Model 1

When $a = (mkt + k_1)^{1/m}$, $m < 1$. Eq. (27) with $a = (mkt + k_1)^{1/m}$ implies

$$K = u_1 I \exp\left[u_0 \int \frac{e^{-t}}{(mkt + k_1)^{3/m}} dt\right] \tag{28}$$

$$V = I^2 K = a^3 = (mkt + k_1)^{3/m}. \tag{29}$$

Eqs. (28) and (29) together implies

$$I = (mkt + k_1)^{1/m} u_1^{-1/3} \exp\left[-\frac{u_0}{3} \int \frac{e^{-t}}{(mkt + k_1)^{3/m}} dt\right] \tag{30}$$

$$K = (mkt + k_1)^{1/m} u_1^{2/3} \exp\left[\frac{2u_0}{3} \int \frac{e^{-t}}{(mkt + k_1)^{3/m}} dt\right]. \tag{31}$$

Both the cosmic scale factors I and K increases as the age of the Universe increases (Fig.1, 2). The Hubble parameter H and the MTHDE density ρ_{MT} are calculated as

$$H = \frac{k}{mkt + k_1}. \tag{32}$$

The Hubble parameter H is a decreasing function of t and tends to a small value with the passage of cosmic time.

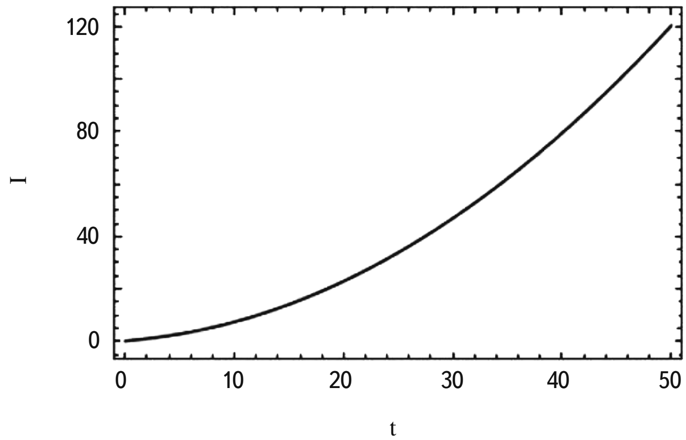

Fig.1. The plot of I versus cosmic time t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$, $u_0 = 0.03$ and $u_1 = 0.15$.

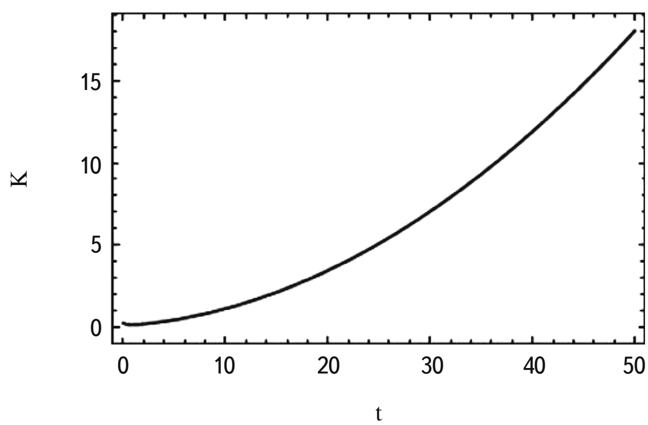

Fig.2. The variation of K against cosmic time t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$, $u_0 = 0.03$ and $u_1 = 0.15$.

$$\rho_{MT} = D\left(\frac{k}{mkt+k_1}\right)^{-2\delta+4} + E\left[\frac{-mk^2}{(mkt+k_1)^2}\right]. \tag{33}$$

Fig.3 shows that ρ_{MT} decreases and tends to a constant value as cosmic time evolves. The anisotropy parameter A_p is calculated as

$$A_p = \frac{2}{9}\left(\frac{mkt+k_1}{k}\right)^2\left[\frac{u_0^2 e^{-2t}}{(mkt+k_1)^{6/m}}\right]. \tag{34}$$

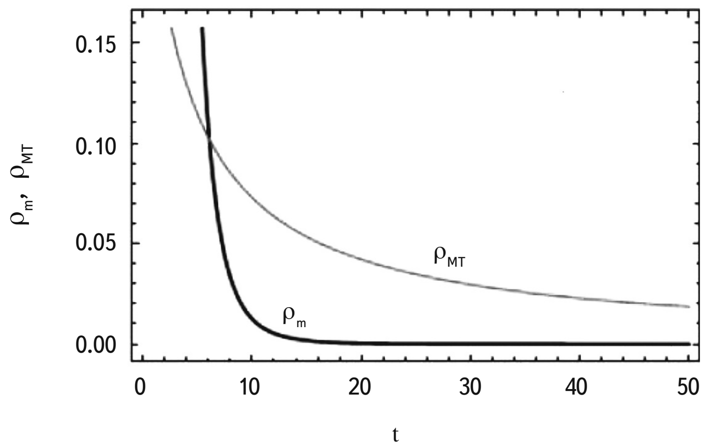

Fig.3. The variation of ρ_m and ρ_{MT} versus cosmic time t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$, $\rho_0 = 0.8$, $D = 0.5$, $\delta = 1.5$ and $E = 0.2$.

$A_p \to 0$ as observed from Fig.4. Thus, our Universe approaches isotropy at late times. The energy conservation equation for dark matter is

$$\dot{\rho}_m + 3H\rho_m = 0. \tag{35}$$

Using Eq. (32) in Eq. (35), the energy density of dark matter ρ_m is found as

$$\rho_m = \frac{\rho_0}{(mkt + k_1)^{3/m}} \tag{36}$$

ρ_0 is a constant of integration.

From Fig.3, we see that ρ_m diminishes as cosmic time evolves and ultimately approaches to zero.

The energy conservation equation for MTHDE is

$$\dot{\rho}_{MT} + 3H(\rho_{MT} + p_{MT}) = 0. \tag{37}$$

The EOS parameter of MTHDE ω_{MT} is obtained by the use of Eqs. (32), (33) and (37) as

$$\omega_{MT} = -1 - \left(\frac{mkt + k_1}{3k}\right)\left\{\frac{2D(\delta - 2)\left(\frac{k}{mkt + k_1}\right)^{-2\delta+3}\left[\frac{mk^2}{(mkt + k_1)^2}\right] + \frac{2m^2k^3E}{(mkt + k_1)^3}}{D\left(\frac{k}{mkt + k_1}\right)^{-2\delta+4} + E\left[\frac{-mk^2}{(mkt + k_1)^2}\right]}\right\}. \tag{38}$$

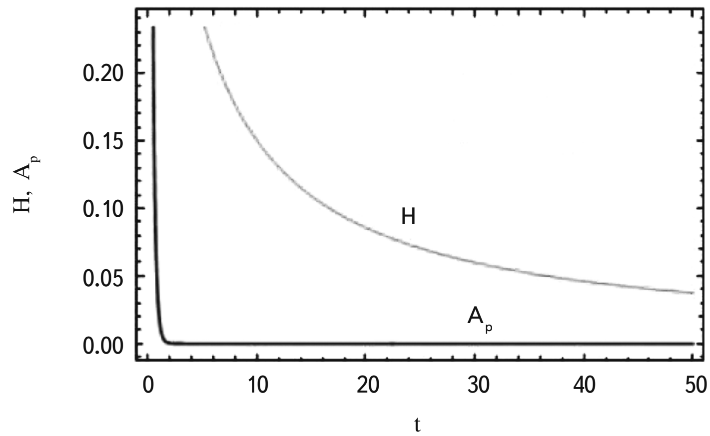

Fig.4. The evolution of H and A_p against cosmic time t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$ and $u_0 = 0.03$.

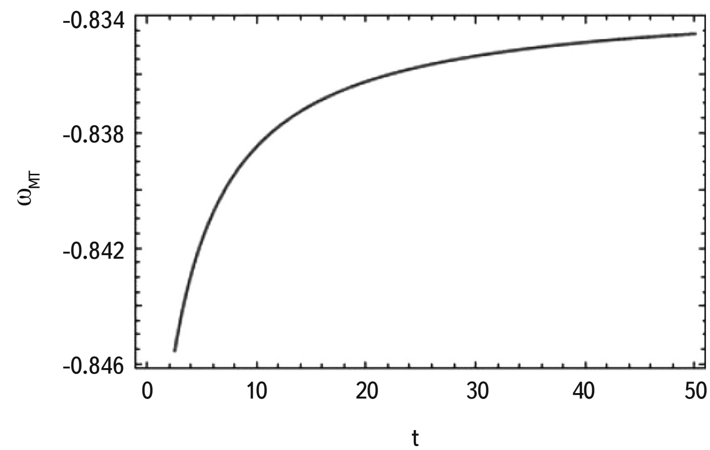

Fig.5. The plot of ω_{MT} versus t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$, $\delta = 1.5$, $D = 0.5$, $\rho_0 = 0.8$ and $E = 0.2$.

From Fig.5, it is observed that $\omega_{MT} > -1$. Thus, our model 1 behaves like a quintessence dark energy model. The present value of the EOS is calculated as $\omega_0 = -0.834$ [43-45] and this concludes that the model 1 is a quintessence dark energy model.

Correspondence between model 1 and quintessence scalar field.

The pressure and energy density for quintessence scalar field [46] are given by

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) \tag{39}$$

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{40}$$

where ϕ denotes the scalar field and $V(\phi)$ is the scalar field potential.

The EOS parameter ω_ϕ is defined as

$$\omega_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \tag{41}$$

Eqs. (33) and (40) together implies

$$D\left(\frac{k}{mkt + k_1}\right)^{-2\delta+4} + E\left[\frac{-mk^2}{(mkt + k_1)^2}\right] = \frac{\dot{\phi}^2}{2} + V(\phi). \tag{42}$$

Eqs. (38) and (41) together implies

$$\frac{\dot{\phi}^2}{2} = \left(\frac{1+\omega_{MT}}{1-\omega_{MT}}\right)V(\phi). \tag{43}$$

Using Eq. (43) in Eq. (42), we obtain the scalar field potential $V(\phi)$ as

$$V(\phi) = \left(\frac{1-\omega_{MT}}{2}\right)\left\{D\left(\frac{k}{mkt + k_1}\right)^{-2\delta+4} - \frac{Emk^2}{(mkt + k_1)^2}\right\}. \tag{44}$$

The scalar field ϕ is calculated by using Eqs. (43) and (44) and then integrating, we get

$$\phi = \phi_0 + \int\left[(1+\omega_{MT})\left\{D\left(\frac{k}{mkt + k_1}\right)^{-2\delta+4} - \frac{Emk^2}{(mkt + k_1)^2}\right\}\right]^{1/2} dt, \tag{45}$$

where ϕ_0 is the constant of integration.

Both the scalar field potential $V(\phi)$ and the scalar field ϕ diminishes and ultimately tends to a small value during the evolution of the Universe as seen from Fig.6 and 7.

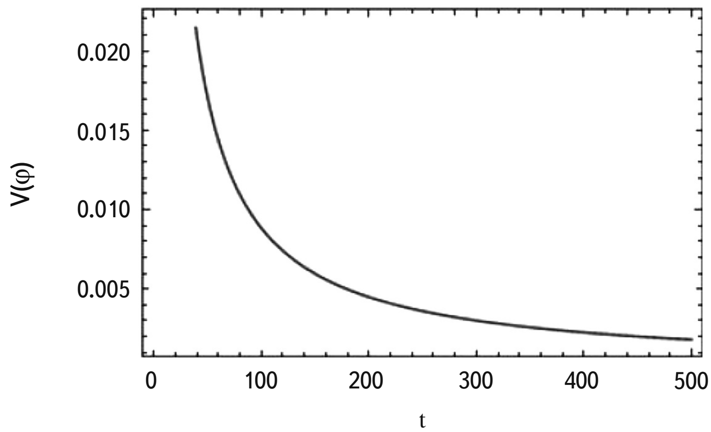

Fig.6. The plot of $V(\phi)$ versus t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$, $\delta = 1.5$, $\rho_0 = 0.8$, $D = 0.5$ and $E = 0.2$.

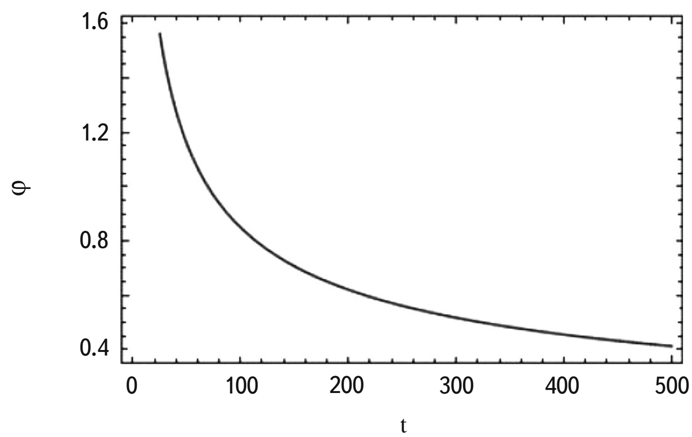

Fig.7. The evolution of ϕ against cosmic time t for $m = 0.5$, $k = 0.3$, $k_1 = 0.5$, $\delta = 1.5$, $\rho_0 = 0.8$, $D = 0.5$, $E = 0.2$ and $\phi_0 = 0.05$.

5. Model 2

When $a = \exp\{k(t-k_2)\}$, $m = 0$. Eq. (27) with $a = \exp\{k(t-k_2)\}$ implies

$$K = u_1 I \exp\left[u_0 \int \frac{e^{-t}}{\exp\{3k(t-k_2)\}} dt\right] \tag{46}$$

$$V = I^2 K = a^3 = \exp\{3k(t-k_2)\}. \tag{47}$$

Eqs. (46) and (47) together implies

$$I = u_1^{-1/3} \exp\left[\frac{-u_0}{3}\int \frac{e^{-t}}{\exp\{3k(t-k_2)\}}dt\right]\exp\{k(t-k_2)\} \tag{48}$$

$$K = u_1^{2/3} \exp\left[\frac{2u_0}{3}\int \frac{e^{-t}}{\exp\{3k(t-k_2)\}}dt\right]\exp\{k(t-k_2)\}. \tag{49}$$

Fig.8 demonstrates that the cosmic scale factors I and K increases as cosmic time evolves. The Hubble parameter H and the MTHDE density ρ_{MT} are calculated as

$$H = k \tag{50}$$

$$\rho_{MT} = Dk^{-2\delta+4}. \tag{51}$$

From Eqs. (50) and (51), we can conclude that both Hubble parameter H and MTHDE density ρ_{MT} are constant.

The energy conservation equation for dark matter is

$$\dot{\rho}_m + 3H\rho_m = 0. \tag{52}$$

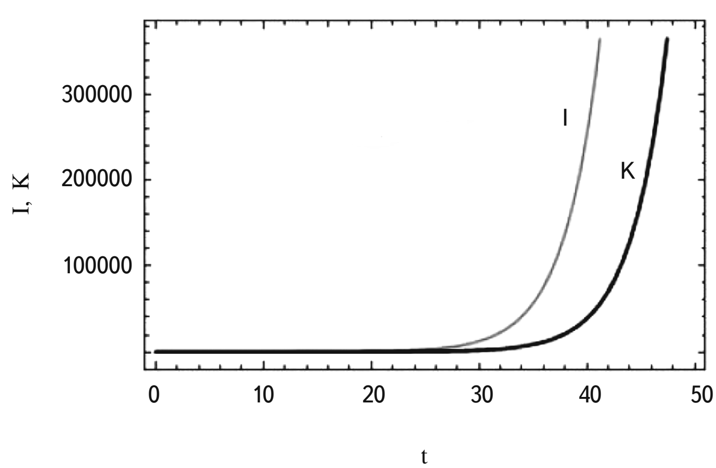

Fig.8. The plot of I and K versus cosmic time t for $k = 0.3$, $k_2 = 0.6$, $u_0 = 0.03$ and $u_1 = 0.15$.

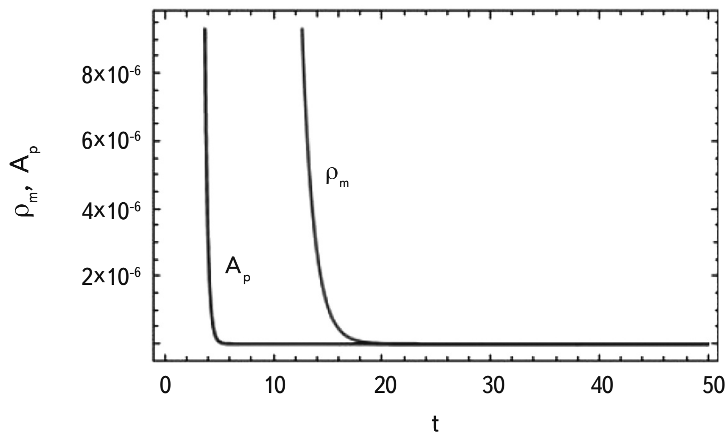

Fig.9. The graph of ρ_m and A_p versus cosmic time t for $\rho'_0 = 0.8$, $k = 0.3$ and $k_2 = 0.6$.

Using Eq. (50) in Eq. (52), ρ_m is found as

$$\rho_m = \rho'_0 e^{-3kt}, \tag{53}$$

where ρ'_0 is a constant of integration.

From Fig.9, we can conclude that $\rho_m \to 0$ as cosmic time evolves.

The energy conservation equation for MTHDE is

$$\dot{\rho}_{MT} + 3H(\rho_{MT} + p_{MT}) = 0. \tag{54}$$

Using Eqs. (50) and (51) in Eq. (54), we have obtained ω_{MT} as

$$\omega_{MT} = -1. \tag{55}$$

Thus, our Model 2 behaves like a cosmological constant model. Recent studies [5,47-50] indicate that our model 2 approaches to ΛCDM ($\omega_{MT} = -1$) served as an excellent model to describe the cosmological evolution. Hence our model 2 is in good agreement with these observations.

The anisotropy parameter A_p is obtained as

$$A_p = \frac{2}{9k^2} e^{-\{2t+6k(t-k_2)\}}. \tag{56}$$

Fig.9 indicates that as $t \to 0$, $A_p \to \infty$ and as $t \to \infty$, $A_p \to 0$. Hence, the anisotropy of our Universe dies out with the passage of cosmic time.

In all the graphs, t denotes cosmic evolution time, generally measured in giga years (1 Gyr = 10^9 y) along x axis. Along y axis, all physical quantities like the matter energy density ρ_m, MTHDE density ρ_{MT}, EOS parameter ω_{MT}, etc. are measured in geometrized units, where the speed of light $c = 1$ and the gravitational constant $G = 1$.

6. Graphical discussions of model 1

I and K are increasing functions of t as observed from Fig.1 and 2.

Both H and A_p are decreasing functions of t as observed from the above figure. H tends to a small value whereas $A_p \to 0$ at the later age of the Universe.

Both ρ_m and ρ_{MT} decreases with the passage of t. ρ_m approaches to zero whereas ρ_{MT} approaches to small value at the later epoch.

From the above figure, we can conclude that $\omega_{MT} > -1$ at the late times. This indicates that our model 1 behaves like a quintessence dark energy model.

The scalar field potential $V(\phi)$ decreases and ultimately approaches to a small value as cosmic time evolves. ϕ tends to a small value at the later age of the Universe as observed from the above figure.

7. Graphical discussions of model 2

I and K increases with the passage of cosmic time as observed from Fig.8.

Both ρ_m and A_p are decreasing functions of t and tends to zero at the later age of the Universe.

8. Conclusions

In this paper we have studied a Bianchi type III Universe filled with dark matter and MTHDE in General Relativity. To determine the solutions of the field equations completely, we make use of a special law of variation of Hubble parameter H proposed by Berman that yields constant DP. Interestingly, we have obtained two different cosmological models for two different constant values of DP. The EOS parameter of MTHDE also behaves like quintessence DE for model 1. Using these results, we have established a correspondence between MTHDE model with the quintessence scalar field. Quintessence potential and the dynamics of the quintessence scalar field are reconstructed for this anisotropic accelerating model of the Universe. Furthermore, it is observed from Eq. (55) that for large cosmic time the EOS parameter of the MTHDE for model 2 becomes -1. Therefore, in the late time evolution of the Universe, our model 2 behaves like a cosmological constant model. Also, the deceleration parameter appears with negative sign which implies accelerating expansion of the Universe. Perlmutter et al. [3] and Riess et al. [1,51,52] proved that the deceleration parameter of the Universe is in the range $-1 \leq q \leq 0$, and the present-day Universe is undergoing an accelerated expansion. From Fig.4 and 9, we see that the anisotropy parameter $A_p \rightarrow 0$ as $t \rightarrow \infty$. Hence, for sufficiently large time, our MTHDE models predict that the anisotropic nature vanishes and it will become isotropic at late times. This implies that our MTHDE models become isotropic at late times even though the space-time is anisotropic. Our results show that the Universe is anisotropic in the early stage and at the late time dynamics anisotropy of the Universe damps out and the present day Universe becomes isotropic as suggested by different observational data. We have found that the results are consisent with current cosmological observational data. The models presented in this paper could give an appropriate description of the evolution of the Universe.

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