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TIME DEPENDENT RADIATIVE TRANSFER PROBLEMS IN A ONE-DIMENSIONAL MEDIUM

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The solution of several one-dimensional problems in nonstationary radiative transfer at frequencies in a spectral line is given. An approach based on searching for the unknowns in the form of Neumann series expansions is applied. The evolution of the line profile formed during reflection from a semi-infinite atmosphere is studied both with coherent and with fully incoherent scattering in the medium. The time dependence of the profiles formed at the boundaries of a finite atmosphere is also examined. In the two problems it is assumed that the atmosphere is illuminated by radiation either in the form of a $\delta(t)$ *-pulse or by radiation with a unit intensity pulse. The solution takes into account both possible causes of time loss by photons during diffusion in the medium: the time spent by an atom in an excited state and the time lost by photons in passing between two successive scattering events. It is shown that with this general statement of the problem, the resultant probability density distribution function of the emerging radiation is given by a convolution of the distributions corresponding to the two components of the photon time expenditure.*

Keywords: *nonstationary radiative transfer: Erlang-n distribution: probability density distribution: cumulative distribution: spectral line profiles*

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1. Introduction

Different phenomena studied in astrophysics have time variations, which are an additional and extremely important basis for interpreting observational data. The nature of these variations differs in a great diversity from the standpoint of the physics of a phenomenon, as well as the rate at which it takes place. Vivid examples may include flare effects in stars on different scales from small bursts of stellar matter to immense phenomena associated with the loss of upper layers of stars accompanied by the release of immense energy (Novae, Supernovae). We may also point out relaxation phenomena, when the equilibrium state in a radiating medium is established over a more or less prolonged time. Strictly speaking, all observed phenomena are to a greater or lesser extent subject to time variation, so it becomes necessary to develop a theory of time dependent radiative transfer theory.

This theory is used to examine problems taking the nonstationarity of different characteristics such as radiation falling on a medium and the possible changes of these or other parameters of a radiating medium such as its optical thickness and ability of absorb and scatter radiant energy incident on it. Here we are interested in a class of problems where the intensity of radiation incident on a medium depends on time and the time spent by an atom in an excited state and the time spent by photons during multiple scattering processes in it is taken into account in the calculations. The latter customarily consists of two components: the time an atom is in an excited state and the time spent by photons in flight between two successive scattering events. Both are random quantities distributed exponentially with average values denoted by t_1 and t_2, respectively. t_2 is determined by the density of absorbing atoms and ions n and by the atomic absorption coefficient coefficient k, with $k_2 = 1/nkc$, where c is the speed of light. We note that this quantity depends significantly on physical conditions in the medium. For example, in a highly ionized medium, where it is necessary to deal with scattering processes on free electrons, in a two-electron approximation we will have $t_2 = 1/\left(1 + n^+/n_1\right)n_1 kc$, where n^+ is the number of ionized atoms and n_1 is the number of atoms in the ground state per unit volume [1]. As for the intensity of radiation incident on a medium, the cases of greatest interest are $\delta(t)$, the image (shape) pulse, and the shape intensity specified by the unit step $H(t)$, known as the Heaviside function.

The work of developing the theory in this direction was begun in the papers of Milne [1] and Chandrasekhar [2,3], where only the time spent by an atom in an excited state was taken into account. While the first of these papers took the characteristic time in a given state as the average lifetime of an atom in a given energy level, a more precise formulation was given in the second. A major contribution to the development of developing the theory of nonstationary radiative transfer theory was by the Leningrad school of astrophysicists. Sobolev [4,5] examined the problem of nonstationary irradiation of a medium both on the basis of the equations of radiative transfer and applying the probabilistic approach which he developed. Minin [6,7] developed a method which made it possible to use the Laplace transform to reduce these problems to solving their stationary analogs. The numerical solution to the nonstationary problem by a Laplace transform was given in Ref. 8. The Ambartsumyan invariance principle was first used in Refs. 9 and 10 to solve the problem of diffuse reflection and transmission of a medium with finite optical thickness. A similar problem for a nonuniform medium and isotropic scattering is dealt with in a series of papers by Matsumoto (e.g., Refs. 11 and 12) in which both of the above mentioned cases of illuminating a medium were examined.

The methods based on the Laplace transform come into great difficulty when dealing with these results. Thus, an alternative approach [13,14] has been proposed for solving time dependent radiative transfer problems. The idea developed by these authors was based on constructing a Neumann series in the nonstationary problem and the standard problems corresponding to it. This method was used in our previous paper [15] for a one-dimensional medium employing a recurrence relation obtained by us in Ref. 16. Given the mathematical complexity of nonstationary radiative transfer problems, one-dimensional models have been most often considered or solutions to solving problems with one of two mutual contradictory assumptions, $t_1 << t_2$ or $t_1 >> t_2$.

Three problems with arbitrary values of the parameters t_1 and t_2 are examined in this paper in a one-dimensional approximation for the two above-mentioned cases of illumination of a medium. Its purpose is to show that for a correct statement of the solution of this common problem no fundamental difficulties appear compared to the often encountered particular models.

2. Evolution of spectrum line profiles formed by diffuse reflection from a semi-infinite atmosphere

For intuitiveness in the one-dimensional approximation we examine the simplest problem of diffuse reflection of light from a semi-infinite absorbing and coherently scattering uniform atmosphere illuminated at time $t = 0$ by continuum light of unit intensity either in the form of a pulse $\delta(t)$, or in the form of a step $H(t)$. The profile of the absorption coefficient in the spectrum line is assumed Dopplerian $\alpha(x) = \left(1/\sqrt{\pi}\right)\exp\left(-x^2\right)$, where x is the dimensionless frequency measured by the shift from the line center in Doppler widths. The effect of absorption in the continuum spectrum is specified by the quantity β, which represents the ratio of the absorption coefficients in the continuum and and the line center. For the probability of reradiation of a photon in an elementary scattering event, we shall use the generally accepted notation λ. In the stationary problem, applying the Ambartsumyan invariance principle for determining the reflection coefficient $\rho(x)$ (with a probabilistic sense) yields [17]

$$\rho(x) = \frac{1}{\lambda}\left(2 - \lambda - 2\sqrt{1-\lambda}\right). \tag{1}$$

If the role of scattering in the continuum is taken into account, then in accordance with Refs. 15 and 17, it is sufficient in place of Eq. (1) to substitute

$$\tilde{\lambda}(x) = \frac{\lambda\alpha(x) + \gamma}{\alpha(x) + \beta + \gamma}, \tag{2}$$

for λ, where γ is the ratio of the continuum scattering component to the absorption coefficient at the center of the spectral line. The expansion of the function $\rho(x)$ in a Neumann series is written in the form

$$\rho(x)=\sum_{n=1}^{\infty}\rho_n\tilde{\lambda}^n(x). \tag{3}$$

The coefficients ρ_1 and ρ_2 in the expansion (3) are easily found by adding an infinitely thin layer to the medium with a subsequent limiting transition and equal, respectively, to 0.25 and 0.125. In these two cases, a photon is reflected once from the medium itself. The other coefficients associated with two-photon reflection from the medium are determined recurrently using a formula obtained in Ref. 15 by applying the invariance principle,

$$\rho_n=\frac{1}{2}\left(\rho_{n-1}+\frac{1}{2}\sum_{k=1}^{n-1}\rho_k\rho_{n-k-1}\right). \tag{4}$$

This formula allows a simple interpretation: the first term in parentheses describes processes associated with single reflection of light from a medium, and the second term, double reflection. Double-scattering events are statistically independent and, therefore, given by a sum that is a discrete analog of the convolution. In Ref. 18 the author gives the values of the first 40 coefficients in the expansion (4). We note that for $n > 4$, the values of ρ_n are asymptotically sufficiently well described by the three-parameter exponential $\rho_n \sim \exp\left(a+bn+cn^2\right)$, where $a=-1.90267$, $b=-0.25674$, and $c=0.0036$.

We examined the temporal picture of diffuse reflection of light from a semi-infinite coherent scattering atmosphere in the above-mentioned Ref. 18 in connection with a study of the role of scattering in the continuum. A method for determining the total time spent by photons during diffusion in a medium was described there for the first time. Since this method will also be used to solve other problems, we dwell briefly on it here.

Thus, let a scattering and absorbing semi-infinite atmosphere be illuminated from outside with time-varying radiation. Instead of time t we introduce two dimensionless quantities $u=t/t_1$ and $\omega=t/t_2$ into the discussion. The time spent by photons during diffusion in the medium, as noted above, is made up of the time spent in an absorbing state and the time of flight between two successive scattering events. Each of these two components, in turn, is the sum of a number of independent exponential distributions of random quantities realized during multiple scattering events in the medium. The distribution function of sum of a number n of such quantities is specified by the Erlang-n distribution (a special case of the gamma distribution)

$$\mathrm{Er}\left(\omega,k,\Lambda\right)=e^{-\Lambda\omega}\frac{\Lambda^k\omega^{n-1}}{\left(k-1\right)!}, \tag{5}$$

which depends on two parameters: the shape k and rate Λ [19]. The distribution is stable, so that the sum and product of these distributions is again a distribution of this type with corresponding values of the parameters. In our problem the parameter Λ takes on the significance of a quantity reciprocal to the average time of one or another of these elementary processes. Given this, the probability distribution function (PDD) of the time lost by a photon in an absorbing state for n-fold scattering will have the form

$$f_1(u,n)=\frac{u^{n-1}}{(n-1)!}e^{-u}\,. \tag{6}$$

An analogous function for the time spent by a photon in free flight between scattering events is given by

$$f_2(\omega,n+1)=\frac{\omega^{n}}{n!}e^{-\omega}\,, \tag{7}$$

where it is noted that the number of such flights exceeds the number of scattering events per unit since the calculation also takes the path covered by a photon when it falls into the medium into account. The two functions given separately with Eqs. (3) and (4) make it possible to determine the evolution of the intensity reflected from the medium for each of two cases of time loss. However, here we are interested in the general case, where both reasons for the loss of time are taken into account. Random quantities corresponding to the two time loss processes examined here are evidently statistically independent, so that the PDD for the total amount of time lost by a photon while it is in the medium will be determined by the convolution of the above two distributions, i.e.,

$$F_n(z)=\int_0^z f_2(\omega,n+1)f_1(z-\omega,n)\,d\omega\,. \tag{8}$$

This integral is calculated explicitly and has the form

$$F_n(z)=\frac{e^{-z}}{n!(n-1)!}\int_0^z \omega^n(z-\omega)^{n-1}\,d\omega=\frac{z^{2n-1}}{(2n-1)!}e^{-z}\,. \tag{9}$$

We now introduce into consideration the function $\overline{\rho}(x,z)$, so that $\overline{\rho}(x,z)dz$ represents the probability of reflection from a semi-infinite medium of a photon of frequency x in the time interval ($z, z+dz$). By analogy with Eq. (3), the Neumann series for the reflection function is written in the form

$$\overline{\rho}(x,z)=e^{-z}\sum_{n=1}^{\infty}\rho_n\tilde{\lambda}^n(x)\frac{z^n}{(2n-1)!}\,. \tag{10}$$

This result describes the evolution of the line profile formed by reflection from an atmosphere illuminated by radiation of form d(t), an image pulse. Based on Eq. (8) it is easy to determine the so-called cumulative distribution function (CDF) describing the process of establishing a stationary regime in a medium up to a time z_0 when it is illuminated by radiation in the form of a unit step $H(t)$,

$$P(x, z_0) = e^{-z_0} \sum_{n=1}^{\infty} \rho_n \tilde{\lambda}^n \sum_{k=0}^{\infty} \frac{z^{2n+k}}{(2n+k)!}. \tag{11}$$

The physical significance of the time variables $z = t/\bar{t}$ and $z_0 = t_0/\bar{t}$ yields the relationship of $\bar{t}$ to t_1 and t_2: $\bar{t} = t_1 t_2/(t_1 + t_2)$. We made numerical calculations based on Eqs. (10) and (11) in Ref. 18 in connection with a study of the influence of scattering in the continuum on the evolution of the profile of a line profile formed by reflection from a semi-infinite atmosphere. Thus, as an illustration and further discussion we limit ourselves just to one typical case related to a comparatively strong line.

The distributions shown in Fig. 1 describe the process of the evolution of a line profile formed by reflection from a radiative medium of unit intensity in the continuum spectrum which varies with time as $\delta(t)$ (left) and H(t) (right). The curves in the figures make it possible to conclude that the probability density distribution (PDD) constructed with simultaneous accounting for the two types of losses of time in scattering events expressed by the product of the corresponding distributions yield the correct conclusions which differ qualitatively from those where the dimensionless time of the form $t/(t_1 + t_2)$ is used [6,7,16]. It is clear from the figure that for these values of the scattering parameters an emission line is formed where the wings of the spectrum line are established much earlier than its core.

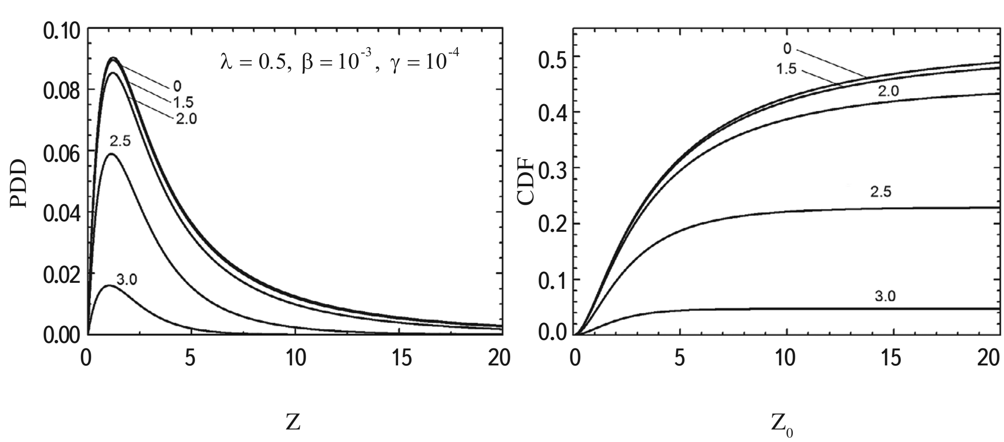

Fig. 1. Probability density distribution and cumulative distribution function for different frequencies within a line, noted above the curves, for the indicated values of the parameters describing the diffusion of the radiation in a semi-infinite atmosphere.

3. Finite medium

The above discussions can easily be generalized to the case of a medium with a finite optical thickness τ_0. The radiant intensities of interest to us that emerge from the medium are expressed in terms of the reflection coefficients $\rho(x, \tau_0)$ and $q(x, \tau_0)$, which have a probabilistic significance. Under the assumption that for physical properties like those in the preceding example, using the invariance principle in the stationary problem leads to the following system of differential equations:

$$\frac{1}{2\nu(x)}\frac{d\rho}{d\tau_0}+\rho(x,\tau_0)=\frac{\tilde{\lambda}(x)}{4}\left[1+\rho(x,\tau_0)\right]^2, \tag{12}$$

$$\frac{1}{\nu(x)}\frac{dq}{d\tau_0}=-\left[1-\frac{\tilde{\lambda}(x)}{2}\left(1-\rho(x,\tau_0)\right)\right]q(x,\tau_0). \tag{13}$$

Equations (12) and (13) are written under the condition that only the boundary $\tau = 0$ of the medium is irradiated. Neumann series of the unknown quantities can be written in the form

$$\rho(x,\tau_0)=\sum_{n=1}^{\infty}\rho_n(x,\tau_0)\tilde{\lambda}^n(x), \quad q(x,\tau_0)=\sum_{n=0}^{\infty}q_n(x,\tau_0)\tilde{\lambda}^n(x). \tag{14}$$

The relative complexity of this problem lies in the fact that determining both of the pairs of coefficients $\rho_n(x, \tau_0)$ and $q_n(x, \tau_0)$ reduces to calculating integrals which, although they are calculated analytically in explicit form, still make the problem time consuming. As in the previous example, the first two terms in the expansions (14) are not associated with multiple scattering and are found more simply as

$$\rho_1(x,\tau_0)=\frac{1}{4}\left(1-e^{-2\nu(x)\tau_0}\right), \quad \rho_2(x,\tau_0)=\frac{1}{8}\left(1-\left(1+2\nu(x)\tau_0\right)e^{-2\nu(x)\tau_0}\right), \tag{15}$$

$$q_0(x,\tau_0)=e^{-\nu(x)\tau_0}\, q_1(x,\tau_0)=\frac{\tilde{\lambda}(x)}{2}\tau_0\, e^{-\nu(x)\tau_0}, \tag{16}$$

The remaining coefficients are found by successive calculation of the integrals

$$\rho_n(x,\tau_0)=2\nu(x)\int_0^{\tau_0}\Phi_n(x,t)e^{-2\nu(x)(\tau-t)}dt, \quad q_n(x,\tau_0)=\nu(x)\int_0^{\tau_0}\Psi_n(x,t)e^{-2\nu(x)(\tau-t)}dt, \tag{17}$$

where

$$\Phi_n(x,\tau_0)=\frac{1}{2}\left[\rho_{n-1}(x,\tau_0)+\frac{1}{2}\sum_{k=1}^{n-2}\rho_k(x,\tau_0)\rho_{n-k-1}(x,\tau_0)\right], \tag{18}$$

$$\Psi_n(x,\tau_0)=\frac{1}{2}\left[q_{n-1}(x,\tau_0)+\frac{1}{2}\sum_{k=1}^{n-1}q_k(x,\tau_0)q_{n-k-1}(x,\tau_0)\right]. \tag{19}$$

Figure 2 illustrates the behavior of the coefficients $\rho_n(x,\tau_0)$ and $q_n(x,\tau_0)$ in a Neumann series as a function of optical thickness and frequency in a spectral line. The relatively sharper drop in the coefficients $\rho_n(x,\tau_0)$ with increasing n than in the case of a semi-infinite atmosphere discussed above is noteworthy. As $\tau_0\to\infty$, evidently, we have $\rho(x,\tau_0)\to\rho(x)$.

The convergence of the coefficients also speeds up on going from the center of the line to its wings. Thus,

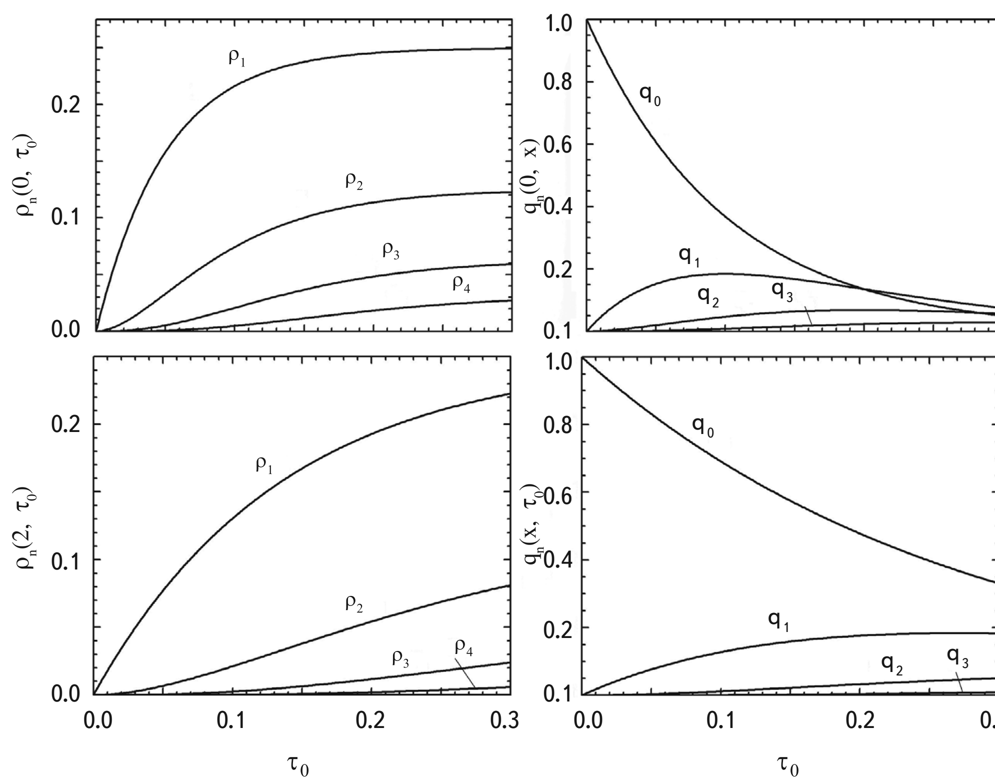

Fig. 2. The coefficients $\rho_n(x,\tau_0)$ and $q_n(x,\tau_0)$ as functions of the optical thickness of the medium at the line center (top panels) and at an intermediate frequency x=2 (bottom panels).

we conclude that, despite the comparative complexity of determining these coefficients, in practice to ensure satisfactory accuracy of these results it turns out to be sufficient to limit oneself to finding a small number of them, especially when speaking of relatively optically thinner lines, as well as of the wings of the lines.

Proceeding to the temporal description of radiative transfer in a medium of finite optical thickness, we introduce consideration of the functions $\overline{\rho}(x,\tau_0,z)$ and $\overline{q}(x,\tau_0,z)$, which determine the intensities of the radiation reflected and transmitted by the medium as functions of the time z. Arguments analogous to those in the previous paragraph allow us to write

$$\overline{\rho}(x,\tau_0,z)=e^{-z}\sum_{n=1}^{\infty}\rho_n(x,\tau_0)\widetilde{\lambda}^n(x)\frac{z^n}{(2n-1)!},\text{n} \tag{20}$$

$$\overline{q}(x,\tau_0,z)=\Lambda(z-\overline{z})\left[q_0(x,\tau_0)\overline{z}+e^{-z}\sum_{n=1}^{\infty}\rho_n(x,\tau_0)\widetilde{\lambda}^n(x)\frac{z^n}{(2n-1)!}\right], \tag{21}$$

where $\overline{z}=\lim\limits_{t_1\to\infty}(\tau_0\,\overline{t})=\lim\limits_{t_1\to\infty}\tau_0\,t_2/(1+t_2/t_1)=\tau_0\,t_2$ is the dimensionless time for passage of the radiation through a finite medium without scattering.

Figures 3 and 4 show the functions PDD and CDF for reflection and transmission by a medium with finite optical thickness and allow us to follow the evolution of the profiles of spectrum lines formed during reflection and transmission by a medium. Based on the particular example shown here, one may conclude that in this particular case the absorption line formed by transmission is established overall more rapidly than the emission line formed by reflection from the medium. The wings of either line are formed on the average more rapidly than their cores. It is

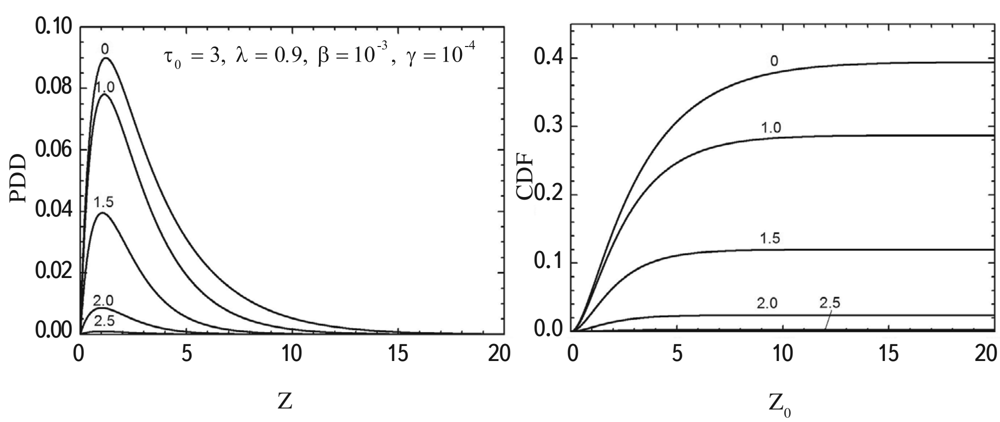

Fig. 3. The PDD for reflection from a medium of optical thickness $\tau_0 = 3$ (left) and the CDF (right) for these values of the multiple scattering parameters at different frequencies (indicated above the curves) within a spectral line.

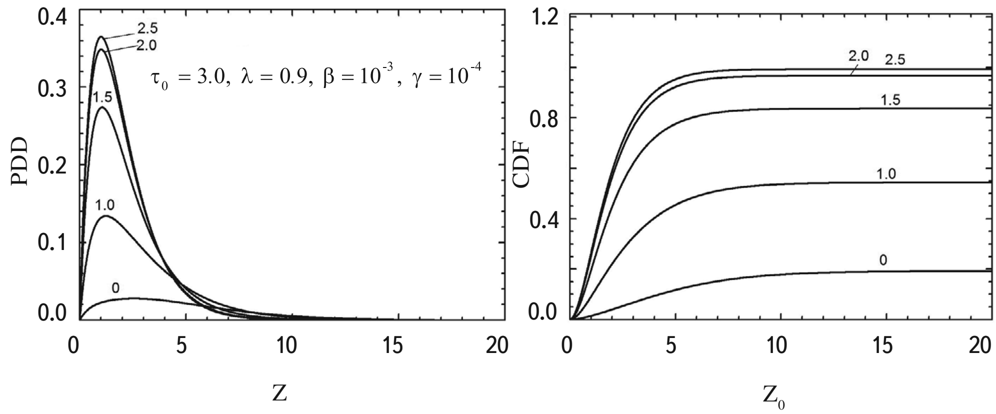

Fig. 4. The PDD for transmission of radiation by a medium of optical thickness $\tau_0 = 3$ (left) and the CDF (right) for the same parameters and frequencies. It is assumed that the time readout is taken immediately after the emergence of photons from the medium.

clear from Figs. 2 and 3 that in a semi-infinite medium, as might be expected, the reflection line evolves longer than in the case of a finite medium.

The solution technique described here in two paragraphs makes it possible to study the effect of one or another local optical property on the process of forming a spectral line, which is extremely important for the study of different kinds of nonstationary phenomena in astrophysics.

4. Fully incoherent scattering

We now consider the preceding one-dimensional problem taking the frequency redistribution of radiation into account, beginning, as above, with the stationary problem.

The functional equation for the reflection function $\rho(x', x)$, which now depends both on the frequency of a photon incident on the medium and on the frequency of a photon reflected from it, is easily found by applying the invariance principle (see, for example, Ref. 20):

$$\frac{2}{\lambda}\left[\nu(x)+\nu(x')\right]\rho(x',x) = r(x',x) + \int_{-\infty}^{\infty} r(x',x'')\rho(x'',x)dx'' + \int_{-\infty}^{\infty} \rho(x',x'')r(x'',x)dx'' + \int_{-\infty}^{\infty} \rho(x,x'')dx'' \int_{-\infty}^{\infty} r(x'',x''')\rho(x''',x)dx''', \tag{22}$$

where $r(x', x)$ is the directionally averaged redistribution function in frequencies and $\nu(x) = \alpha(x) + \beta$. In this case, the Neumann series is written in the form

$$\rho(x', x) = \sum_{n=1}^{\infty} \lambda^n \rho_n(x', x). \quad (23)$$

In this paragraph, for simplicity, the role of scattering in the continuum is neglected, so that the dependence of λ on frequency vanishes.

Arguments analogous to those in the previous paragraph make it possible to construct the coefficients $\rho_n(x', x)$ that we need. In particular, the coefficients ρ_1 and ρ_2 associated with reflection from the medium itself are more simply constructed and have the form

$$\rho_1(x', x) = \frac{1}{2} \frac{r(x', x)}{\nu(x') + \nu(x)}, \quad (24)$$

$$\rho_2(x', x) = \frac{1}{2[\nu(x') + \nu(x)]} \left[\int_{-\infty}^{\infty} r(x', x'') \rho_1(x'', x) dx'' + \int_{-\infty}^{\infty} \rho_1(x', x'') r(x'', x) dx'' \right]. \quad (25)$$

The remaining coefficients are found recurrently using

$$2[\nu(x') + \nu(x)] \rho_n(x', x) = \int_{-\infty}^{\infty} r(x', x'') \rho_{n-1}(x'', x) dx'' + \int_{-\infty}^{\infty} \rho_{n-1}(x', x'') r(x'', x) dx'' + \sum_{k=1}^{n-2} \int_{-\infty}^{\infty} \rho_k(x, x'') dx'' \int_{-\infty}^{\infty} r(x'', x''') \overline{\rho}_{n-k-1}(x''', x) dx'''. \quad (26)$$

The way described here is suitable for incoherent scattering with an arbitrary redistribution function with respect to frequency. The calculations in this paper are done for the approximation of fully incoherent scattering which is often used in applications. In this case $(x', x) = \alpha_0(x')\alpha_0(x)$, where $\alpha_0(x) = \pi^{-1/4}\alpha(x)$. Then, instead of Eqs. (24)-(26) we shall have

$$\rho_1(x', x) = \frac{1}{2} \frac{\alpha_0(x')\alpha_0(x)}{\nu(x') + \nu(x)}, \quad \rho_1(x', x) = \frac{\alpha_0(x')\varphi_1(x) + \alpha_0(x)\varphi_1(x')}{\nu(x') + \nu(x)}, \quad (27)$$

$$\rho_n(x',x)=\frac{1}{2[\nu(x')+\nu(x)]}\left[\alpha_0(x')\varphi_{n-1}(x)+\alpha_0(x)\varphi_{n-1}(x')+\sum_{k=1}^{n-2}\varphi_k(x)\varphi_{n-k-1}(x')\right], \tag{28}$$

where

$$\varphi_n(x)=\int_{-\infty}^{\infty}\rho_n(x',x)\alpha_0(x')dx'. \tag{29}$$

and the symmetry of the reflection function with respect to its arguments has been taken into account.

The transition to the time dependent problem we have discussed is made, as above, by introducing a time dependent reflection function $\overline{\rho}(x',x,z)$, for which Eq. (10) is written in the form

$$\overline{\rho}(x',x,z)=e^{-z}\sum_{n=1}^{\infty}\rho_n(x',x)\lambda^n\frac{z^n}{(2n-1)!}. \tag{30}$$

Figure 5 illustrates the evolution of the values of the line profile at different distances x from its center for the case where scattering in the medium is fully incoherent. The far wings of the line, roughly at $x \geq 2.5$, vary insignificantly with time, so the corresponding curves are absent in the figures. In comparing the process of formation of a spectral line owing to reflection for the two types of scattering in the medium that we have examined, attention should be paid to the different behavior of its core and wings over time. The core of the line ($x \leq 1.5$), formed during coherent scattering, is almost flat and its different parts evolve essentially the same way. At the same time, the line formed with complete frequency redistribution is stronger and broader compared to the preceding example. As for the wings of the line, they change negligibly with time.

5. Conclusion

In this paper we have limited ourselves to examining three simple one-dimensional problems in order to demonstrate the advantages of the method based on expanding the unknowns in a Neumann series, when easily executable recurrence relations derived by us in Ref. 16 can be used. Here both possibilities for the loss of time by a photon as it migrates in the medium were assumed in the calculation: the time spent by an atom in an excited state and the time spent by a photon on the path between scattering events. It seems that a correct accounting for both types of losses can be made with a convolution of the corresponding two Erlang distributions. The PDD and CDF obtained with a correct statement of the problem ensure a physically easily tractable description of the evolution

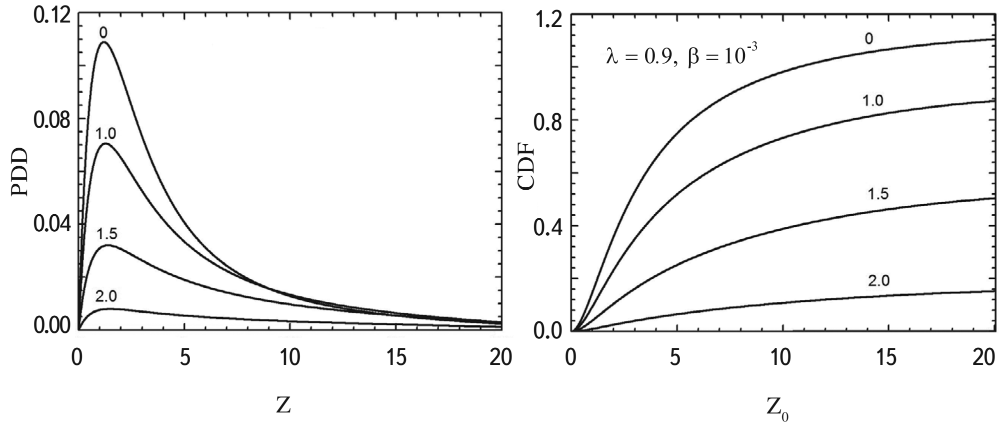

Fig. 5. The same distributions as in Fig. 1 when the spectral line is formed by reflection from a semi-infinite atmosphere in which scattering takes place with complete frequency

of the profiles of the lines that are formed. It is also important to note that the approach taken in this paper is easily realized for solving the problems in their more general statement.

REFERENCES

1. E. A. Milne, J. London Math. Soc. **1**, 40 (1926).
2. S. Chandrasekhar, Proc. Roy. Soc. London, A **192**, 508 (1948).
3. S. Chandrasekhar, Radiative Transfer, Oxford: Clarendon press (1950), p. 354.
4. V. V. Sobolev, Astron. zh. **27**, 81 (1950).
5. V. V. Sobolev, Radiative Energy Transfer in the Atmospheres of Stars and Planets [in Russian], Gostekhizdat, Moscow (1956).
6. I. N. Minin, Vestnik LGU **13**, 137 (1959).
7. I. N. Minin, DAN SSSR **154**, 1059 (1964).
8. R. Bellman, H. Kagivada, and S. Ueno, J. Math. Anal. Appl. **9**, 424 (1964).
9. S. Ueno, J. Math. Anal. Appl. **4**, 1 (1962).
10. R. Bellman, H. Kagivada, and S. Ueno, J. Math. Anal. Appl. **15**, 171 (1966).
11. M. Matsumoto, Publ. Astron. Soc. Japan **19**, 163 (1967).
12. M. Matsumoto, Publ. Astron. Soc. Japan **19**, 434 (1967).
13. M. Matsumoto, J. Math. Anal. Appl. **51**, 403 (1976).
14. B. D. Ganapol, J. Quant. Spectrosc. Radiat. Transfer **25**, 159 (1981).
15. A. G. Nikogosyan, Astrophysics **64**, 345 (2021).
16. A. G. Nikogosyan, Astrophysics **21**, 527 (1984).

17. G. L. Israelian and A. G. Nikoghossian, J. Quant. Spectrosc. Radiat. Transfer **56**, 509 (1996).

18. A. G. Nikoghossian, ComBAO **68**, No. 1, 2 (2021).

19. P. G. Moschopuolos, Ann. Inst. Stat. Math. **37**, 541 (1985).

20. A. G. Nikogosyan and G. A. Artyunyan, DAN SSSR **229**, 583 (1976).