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ANALYTIC SOLUTION OF THE NONLINEAR RADIATIVE DIFFUSION PROBLEM IN A ONE-DIMENSIONAL PURELY SCATTERING MEDIUM. II

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This paper is the second part of the author's previous work. Its purpose is to illustrate the efficiency of using the previously introduced auxiliary functions known as linear images (LI) of the reflection-transmission function with simple examples of nonlinear problems for a one-dimensional purely scattering medium. First, an explicit analytic solution is obtained for the nonlinear "direct" problem of determining the fields of the emerging radiation from a "composite" medium consisting of a reflecting surface and a layer of finite thickness with known reflecting-transmitting properties. Then solutions are obtained for the nonlinear "inverse" problems of determining: (a) the external exciting fields based on data from the radiation emerging from the medium, (b) the fields at an "inaccessible" boundary of the medium based on observations of the light regime at one of its boundaries, (c) the intensities of the radiation traveling in one direction based on measurements of the fields in the opposite direction, (d) some characteristics of the medium based on measurements of the intensities of radiation incident on it from outside and emerging through one of its boundaries, and (e) fields inside the medium based on measurements of the bidirectional radiation fields at just one boundary of the medium. Finally, it is shown that in the nonlinear problem of illumination of an semi-infinite medium the phenomenon of "bleaching" of the medium does not occur; the light regime coincides with the solution of the linear problem, both outside and inside the medium.

Keywords: *radiative transfer: nonlinear problem*

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1. Introduction

In the first part of this article [1], an explicit solution was obtained for the simple nonlinear problem of reflection-transmission of radiation by a one-dimensional, purely scattering layer of finite thickness consisting of two-level atoms. It reduced to solving Ambartsumyan's functional equation of complete invariance for the auxiliary functions introduced by us and known as linear images (LI) of reflection-transmission. Here it was also possible to determine the radiation field inside the medium.

In this article it is shown that LI can also be used to study some more complicated nonlinear, including inverse, radiative transfer problems with wider applications. In the direct statement this is the problem of transfer in media with underlying surfaces and the inverse problems involve recovering the radiation fields and optical characteristics of a medium in cases usually met in practice where the observational data are insufficiently complete. Explicit analytic solutions of several of these kinds of nonlinear problems are introduced in this paper.

2. Problems with an underlying surface

In the linear theory of radiative transfer, problems concerning atmospheres with underlying scattering and absorbing surfaces are of great applied significance. In this case, the unknown radiation field is formed by a multiple interaction of two objects with previously known reflecting-transmitting capabilities (e.g., see Refs. 2 and 3). Astrophysical applications most often examine problems in which the surfaces lie below the atmosphere (in the case of planets they may, for example, solid, liquid, or plant cover). However, there are also problems in which these surfaces lie above the atmosphere, as with a cloud cover created by the greenhouse effect in planetary atmospheres, as well as a coverage effect for an individual star, or radiation reflection effects in the case of close pairs of stars. Thus, use of our previously obtained analytic solution for the simple nonlinear problem to analyze more complicated problems including underlying surfaces is of special interest.

Consider a system consisting of two interacting objects nominally regarded as an underlying surface and an atmosphere. The question involves determining the role of the underlying surface with its nonlinear optical properties and, in the general case, the contribution of the nonlinearity of both components is of interest.

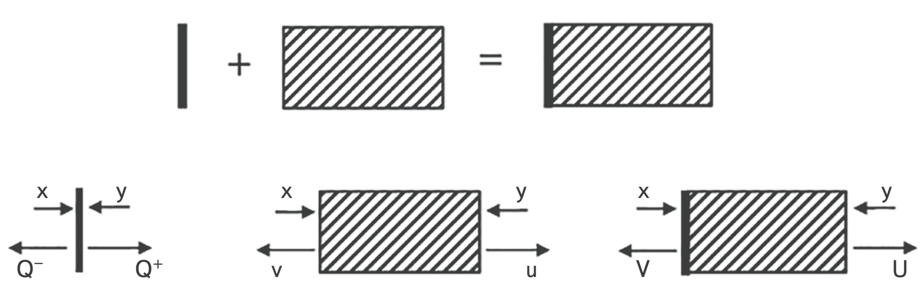

Fig. 1. Illustrating the combination of an underlying surface and a scattering-absorbing medium (top). Notation for physical quantities describing the component parts and the combined medium separately (bottom).

Let pure isotropic scattering take place in an isotropic one-dimensional medium of geometric thickness L adjacent on the left boundary (see Fig. 1) to a surface with known reflection and transmission properties $Q^{\pm}$. The latter depend nonlinearly on beams incident from the left and right (x,y). It is also assumed that the surface has two-sided emissivities $\varepsilon^{\pm}$ that do not depend on the external excitation. In other words,

$$Q^{\pm} \equiv Q^{\pm}\left(x, y; \varepsilon^{\pm}\right) = \tilde{Q}^{\pm}(x, y) + \varepsilon^{\pm} \tag{1}$$

are the intensities of the radiation directed to the right "+" and left "-" when radiation of intensity x and y is incident on the surface from the left and right, respectively. We also introduce the quantities $U \equiv U(x, y) \equiv U\left(x, y; \varepsilon^{\pm}; L\right)$ and $V \equiv V(x, y) \equiv V\left(x, y; \varepsilon^{\pm}; L\right)$ which are the intensities of the radiation emerging, respectively, through the right and left boundaries of the surface+atmosphere system when it is illuminated from the left and right by external radiation of intensity x and y. Here the reflection and transmission properties of the individual component parts of this system are assumed to be specified: $Q^{\pm}\left(x, y; \varepsilon^{\pm}\right)$ for the underlying surface and $u \equiv u(x, y) \equiv u(x, y; L)$ and $\upsilon \equiv \upsilon(x, y) \equiv \upsilon(x, y; L)$ for the atmosphere of geometric thickness L when the adjoining surface is absent.

The top part of Fig. 1 shows how the surface and atmosphere are combined into a single system and the bottom indicates the radiative reflection-transmission characteristics for each of the components, as well as the medium obtained by combining them.

The problem is to find U and V from the known $Q^{\pm}$, u, and υ. We denote the intensities of the radiation propagating to the right and left at the contact boundary of the atmosphere with the surface by p and s, respectively.

The formulas for nonlinear combination of the layers [4] for the unknowns in this case can be written as

$$U(x, y) = u(p, y), \quad V(x, y) = Q^{-}\left(x, s; \varepsilon^{-}\right), \tag{2}$$

$$\begin{cases} s = \upsilon(p, y) \\ p = Q^{+}\left(x, s; \varepsilon^{+}\right), \end{cases} \tag{3}$$

where u and v are written explicitly as [1]

$$u = y + (x - y)T, \quad \upsilon = x - (x - y)T, \tag{4}$$

while the LI $T \equiv T(x, y; L) \equiv T(x + y; L) \equiv T(z; L) \equiv T(z)$ is given by

$$T(z) = q\frac{1 + bz}{1 + qbz}, \tag{5}$$

where $q \equiv q(L)$ is the transmissivity of the medium of geometric thickness L in the one-dimensional, linear, and conservative radiative transfer problem.

The quantity s in Eq. (3) with Eqs. (4) and (5) taken into account is found using the transcendental equation

$$s = \frac{(1-q)Q^{+} + \left[1 + \left(Q^{+} + y\right)b\right]qy}{1 + \left(Q^{+} + y\right)bq}, \tag{6}$$

which, with the notation

$$A(s) \equiv (1-q) - (s-y)bq, \quad B \equiv 1 + bqy, \quad C \equiv (1 + by)qy, \tag{7}$$

and using Eq. (1), takes the simple form

$$A(s)\tilde{Q}^{+}(x, s) - \tilde{B}s + \tilde{C} = 0, \tag{8}$$

where

$$\tilde{B} \equiv 1 + bq\left(\varepsilon^{+} + y\right), \quad \tilde{C} \equiv (1-q)\varepsilon^{+} + \left[1 + b\left(\varepsilon^{+} + y\right)\right]qy. \tag{9}$$

A solution of the transcendental equation (8) for s can, for example, be constructed with an appropriate choice of iteration scheme subject to the conditions

$$Q^{+}\left(x, s; \varepsilon^{+}\right) \leq x + s + \varepsilon^{+} \quad \text{or} \quad \tilde{Q}^{\pm}(x, y) \leq x + y, \tag{10}$$

reflecting the absence of photon multiplication processes in the surface. After determining s, the unknown V is found directly from the second of Eqs. (2). The other unknown U is determined from the first of Eqs. (2) subject to Eqs. (4) and (5),

$$U(x, y) = \frac{(1-q)y + [1 + (p+y)b]pq}{1 + (p+y)bq}, \tag{11}$$

where p and $p+y$ are found, with the aid of Eqs. (3), (1), and (8):

$$p = \frac{Bs - C}{A(s)} = \frac{s - [1 - (s-y)b]qy}{1 - q - (s-y)bq}, \quad p + y = \frac{s + (1-2q)y}{1 - q - (s-y)bq}. \tag{12}$$

Equation (3) also makes it possible to rewrite Eq. (11) directly in the form

$$U(x,y)=\frac{(1-q)y+(1+by)qQ^{+}+bq(Q^{+})^{2}}{1+\ (Q^{+}+y)bq},\quad Q^{+}\equiv Q^{+}(x,s;\varepsilon^{+}). \tag{13}$$

In this way the problem we have stated is solved in the general case, when the repetitively interacting components of the system have nonlinear properties.

There is also some interest in particular cases of this problem, where one or another of the components of the composite system has linear properties.

2.1. Linear surface. In this case it is evident that

$$Q^{+}(x,s;\varepsilon^{+})=t^{+}x+r^{+}s+\varepsilon^{+},\quad Q^{-}(x,s;\varepsilon^{-})=r^{-}x+t^{-}s+\varepsilon^{-}, \tag{14}$$

where $t^{\pm}$ and $r^{\pm}$ denote the two-sided reflectivity and transmission of the underlying surface in the linear case. From Eqs. (8) and (9) we have

$$A_0 s^2+B_0 s-C_0=0, \tag{15}$$

where we have introduced the notation

$$A_0\equiv r^{+}bq,\quad B_0\equiv b_0+b_1x+b_2y,\quad C_0=a_0+a_1x+a_2y+a_3xy+a_4y^2, \tag{16}$$

$$b_0\equiv 1-(1-q)r^{+}+bq\varepsilon^{+},\quad b_1\equiv bqt^{+},\quad b_2\equiv(1-r^{+})bq, \tag{17}$$

$$a_0\equiv(1-q)\varepsilon^{+},\quad a_1\equiv(1-q)t^{+},\quad a_2\equiv(1+b\varepsilon^{+})q,\quad a_3\equiv bqt^{+},\quad a_4\equiv bq. \tag{18}$$

A physical solution of Eq. (15) corresponds to a positive sign in front of the square root. After determining s from Eq. (15), the radiation emerging through the left boundary of the medium is found using Eq. (2) subject to Eq. (14),

$$V(x,y)=r^{-}x+t^{-}s+\varepsilon^{-}, \tag{19}$$

and the radiation $U(x,\ y)$ emerging through the right boundary, as before, is found from Eqs. (11)-(13).

2.2. Linear atmosphere. In this case it is sufficient [1] to set $b\equiv 0$ in any of Eqs. (6)-(9) in order to reduce the question to solving the equation

$$s=(1-q)Q^{+}\left(x,s;\varepsilon^{+}\right)+qy \quad \text{or} \quad s=(1-q)\left[\tilde{Q}^{+}(x,s)+\varepsilon^{+}\right]+qy\,. \tag{20}$$

The resulting equation has an extremely transparent physical significance – an internal field propagating toward the surface consists of two components: from the fraction of radiation of the surface itself, Q^{+}, which has been reflected from the linear medium and a part y of the radiation incident on the right hand boundary which passed through the atmosphere. After solving Eq. (20) the intensity of the radiation emerging through the right boundary is given by Eq. (13) or (11) on setting $b\equiv 0$,

$$U(x,y)=qQ^{+}\left(x,s;\varepsilon^{+}\right)+(1-q)y=\frac{qs+(1-2q)y}{1-q}, \tag{21}$$

and $V(x, y)$ by Eq. (2).

2.3. Both components linear. This is the simplest case; the problem becomes linear, so the solution is given here only for reference. It can be obtained in various ways; in particular, from Eq. (20) subject to Eqs. (1) and (14), it is easy to obtain

$$s=\frac{(1-q)\left(\varepsilon^{+}+t^{+}x\right)+qy}{1-(1-q)r^{+}}. \tag{22}$$

3. Nonlinear transfer problems in an inverse statement

From the standpoint of astrophysical interpretation of observations, inverse problems for recovery of the true characteristics of objects from observed quantities are much broader and of greater variety (e.g. Refs. 3 and 5-9). Furthermore, the analysis of diffuse radiation fields using the mathematically exact formulation of the inverse problem is occasionally much more effective and more interesting than direct modeling of multiple interactions of radiation with matter. In fact, the initial radiative properties of the object, phenomenon, or physical situation being studied more often have to be recovered precisely by practical measurement of the intensities of the observed radiation fields. However, despite their practical importance, inverse nonlinear radiant energy transfer problems are still little studied because of their mathematical complexity. Thus, it is appropriate to try the analytic expression of the direct problem obtained in Ref. 1 and use it for analyzing problems in the inverse formulation as well.

As an illustration, in the following we introduce some simple examples that are typical for models of different experimental-practical situations when the observed quantities determine quantities inaccessible to measurement or some local characteristics or other of the medium itself.

3.1. Determination of the external exciting fields based on data on the radiation emerging from the medium. Here we seek to determine the intensity x and y of beams entering the medium based on the observed values of u and υ of the radiation emerging from the medium. From Eq. (4) we obtain a system of two equations:

$$\begin{cases} x+y=u+\upsilon \\ x-y=\dfrac{u-\upsilon}{2T(x+y)-1}. \end{cases} \tag{23}$$

Given the first of Eqs. (23), the solution of this system is easily written down directly in the form

$$x=\frac{(u+\upsilon)T(u+\upsilon)-\upsilon}{2T(u+\upsilon)-1}, \quad y=\frac{(u+\upsilon)T(u+\upsilon)-u}{2T(u+\upsilon)-1}, \tag{24}$$

where the LI is specified by Eq. (5). Equations (24) yield an explicit solution of this problem.

3.2. Determining the intensity of radiation at a boundary of the medium that is not accessible to measurement based on observations of the light regime at the other boundary. If measurements can be made at only one boundary of the medium (e.g., the left boundary), then the radiant intensities measured at this boundary (incident x and reflected v) can be used to determine the corresponding values (entering y and emerging u intensities) on the opposite (right) boundary of the medium. For this, it is sufficient to solve the system (23) for y and u given Eq. (5):

$$bqy^2+[1+(x-\upsilon)b]qy=(1+b\upsilon)qx-(x-\upsilon), \tag{25}$$

$$bqu^2+[1-(x-\upsilon)b]qu=(1+b\upsilon)qx-(x-\upsilon)(1-q). \tag{26}$$

In these quadratic equations, the physical solution corresponds to a positive sign in front of the square root. Here, if only one of Eqs. (25) and (26) is solved, then the solution of the other can be found directly from Eq. (4).

3.3. Determining the radiant intensities in one of the directions based on measurements of the intensity of the oppositely directed radiation. Let, as above, the two boundaries of the medium be simultaneously illuminated by powerful beams of radiation of intensity x and y, respectively, with known values in one and the same direction (e.g., from left to right x and u). It is required to determine the analogous intensities of the radiation corresponding to the reverse direction (i.e., from right to left y and υ):

$$y = \frac{(1+bqx)u-(1+bx)qx}{1-q+(x-u)bq} \quad \text{or} \quad y = \frac{u-[1+(x-u)b]xq}{1-[1-(x-u)b]q}, \tag{27}$$

$$\upsilon = \frac{(1-q)x-(x-u)(1+ub)q}{(1-q)+(x-u)bq}. \tag{28}$$

3.4. Determine some characteristics of the medium based on measurements of the intensities of the incident radiation and of the radiation emerging through one of its boundaries. Let the intensities of the radiation incident on the medium (x,y) be measured, along with that of the radiation (e.g., υ) emerging through one of the boundaries. It is required to determine both the optical thickness $\tau(0,0)$ of this medium in the unperturbed state (i.e., when $x=0$, $y=0$) and to find how the optical thickness $\tau(x,y)$ varies with the intensity (x,y) of the external exciting radiation, i.e., to determine the level of bleaching of the medium as a function of the external exciting radiation and, finally, find the local characteristic b of the substance in the medium by an "experimental" method,

With the second of Eqs. (4) and including Eq. (5) it is easy to obtain

$$\frac{1}{q} = \frac{x-y-\left[y^2+xy-(x+y)\upsilon\right]b}{x-\upsilon}. \tag{29}$$

Here (see Ref. 10)

$$q = \frac{1}{1+\frac{\tau(0)}{2}}, \tag{30}$$

where all the previously used notation [1] is retained:

$$\tau(x,y,L) = \tau(x+y,L) \equiv \tau(z,L) \equiv \tau(z), \quad \tau(0) = \int_0^L k_0(l)\,dl = \frac{h\upsilon}{2}B_{12}N,$$
$$k_0(l) = n(l)\frac{h\upsilon}{2}B_{12}, \quad N \equiv \int_0^L n(l)\,dl. \tag{31}$$

From Eqs. (29) and (30) we finally obtain the optical thickness of the unperturbed medium in terms of υ, x, y, and the value b of the microcharacteristic of the substance in the scattering-absorbing medium,

$$\tau(0) = 2\frac{\upsilon-y-\left(y^2+xy-z\upsilon\right)b}{x-\upsilon} = 2\frac{\upsilon-y}{x-\upsilon}(1+bz). \tag{32}$$

With Eq. (31) we also find the number of neutral atoms

$$N=\frac{4}{h\upsilon B_{12}}\frac{\upsilon-y-\left(y^{2}+xy-z\upsilon\right)b}{x-\upsilon}=\frac{4}{h\upsilon B_{12}}\frac{\upsilon-y}{x-\upsilon}\left(1+bz\right). \quad (33)$$

It is appropriate to emphasize that Eqs. (32) and (33) determine the parameters of the unperturbed medium, despite the fact that the measurements applied specifically to an actual excited medium. Evidently, the right hand sides of Eqs. (32) and (33) are invariant with respect to the power (x,y) of the external exciting radiation. Besides the initial optical thickness $\tau(0)$ of the medium, these same observations can also be used to determine its actual optical thickness $\tau(z)$ as a function of the power of the exciting fields x and y:

$$\tau(z)=\int_{0}^{L}k(z,l)dl=\frac{\tau(0)}{1+bz}=2\frac{\upsilon-y}{x-\upsilon}. \quad (34)$$

b is one of the local characteristics of the material in the medium and is directly related to atomic parameters, i.e.,

$$b=\frac{B_{12}+B_{21}}{2A_{21}}=\frac{c^{2}}{4h\upsilon_{12}^{3}}\left(1+\frac{g_{2}}{g_{1}}\right). \quad (35)$$

Up to now this value has been assumed known in advance. If we now pose the problem of determining it by an independent "experimental" method, then data from yet another observation will be required. In fact, using the results of the two existing observations (x_1, y_1, υ_1) and (x_2, y_2, υ_2), the invariant (32) easily yields its value:

$$\frac{\upsilon_{1}-y_{1}}{x_{1}-\upsilon_{1}}\left(1+bz_{1}\right)=\frac{\upsilon_{2}-y_{2}}{x_{2}-\upsilon_{2}}\left(1+bz_{2}\right), \quad (36)$$

$$b=\frac{\left(\upsilon_{1}-y_{1}\right)\left(x_{2}-\upsilon_{2}\right)-\left(\upsilon_{2}-y_{2}\right)\left(x_{1}-\upsilon_{1}\right)}{\left(x_{1}-\upsilon_{1}\right)\left(\upsilon_{2}-y_{2}\right)z_{2}-\left(x_{2}-\upsilon_{2}\right)\left(\upsilon_{1}-y_{1}\right)z_{1}}. \quad (37)$$

We note, finally, that the formulas given above become substantially simpler when the medium is illuminated only on one side, i.e., for example when $y\equiv 0$ and $x\neq 0$. Here we encounter the classical problem of determining the optical parameters of the medium by measuring the incident x and reflected (from the medium) υ radiation:

$$\tau(0)=2\frac{\upsilon_{0}}{x-\upsilon_{0}}\left(1+xb\right),\quad N=\frac{4}{h\upsilon B_{12}}\frac{\upsilon_{0}}{x-\upsilon_{0}}\left(1+xb\right), \quad (38)$$

$$\tau(x)=2\frac{v_0}{x-v_0}, \quad b=\frac{(x_2-v_2)v_1-(x_1-v_1)v_2}{(x_1-v_1)x_2\,v_2-(x_2-v_2)x_1\,v_1}, \tag{39}$$

where $v_0 \equiv v(x,0)$.

3.5. Determining the radiation field inside the medium based on measurements of the two-sided fields at one of the boundaries of the medium. In the first part of this paper [1], the problem of determining the field inside the medium was solved in two ways: as a direct problem of adding two separate layers with previously known reflection-transmission properties and as an inverse problem formulated as a Cauchy problem for the radiative transfer equation. Here it will be examined in the form of the classical inverse "remote ranging" problem, where "measurement" data on the intensities of entering and emerging radiation are used to determine the inner light regime of the medium without using the radiative transfer equation at all. We begin with the formulas for nonlinear combination of layers first obtained by Ambartsumyan [11,12].

Let a medium of geometric thickness L be illuminated by radiation with intensities x and y on its left and right boundaries, respectively. It is necessary to determine the intensities of the rightward (+) and leftward (-) directed radiation $I^{\pm}(l;x,y;L)$ at depth l of this medium based on the known reflected radiation $v(x,y;L)\equiv I^{-}(0;x,y;L)$. Making an imaginary cut at an arbitrary depth $0<l<L$, it is easy to write (Fig. 2)

$$v(x,y;L)=v\left(x,I^{-}(l;x,y;L);l\right). \tag{40}$$

In fact, Eq. (40) shows [11,12] (see Ref. 1, as well) that the field $v(x,y;L)$ emerging from the layer [0, L] for an external driver (x,y) is invariant with respect to cutting off the section [l, L] from the original medium with retention on the remaining part [0, l] an intensities $x, I^{-}(l;x,y;L);l$ as an external influences.

It follows from Eq. (40) and Fig. 2 that one and the same pair of measured values ($v;x$) corresponds to an arbitrary number of values of l and L (with $0 < l < L$), for which the solution of the inverse problem 3.2 for recovery of the entering and emerging intensities y and u for a layer of thickness L will simultaneously also yield a solution of the more general problem of recovering the internal radiation fields $I^{\pm}(l;x,y;L)$ at an arbitrary depth l of a layer L. In fact, with the replacements $y\to I^{-}(l;x,y;L)$ and $u(x,y;L)\to I^{+}(l;x,y;L)$, as well as $q(L)\to q(l)$ and the notation change $v(x,y;L)\equiv V(L)$, Eqs. (25) and (26) can be rewritten in the form

$$bq(l)\left(I^{-}\right)^2+\{1+[x-V(L)]b\}q(l)I^{-}-[1+bV(L)]q(l)x-[x-V(L)]=0, \tag{41}$$

$$bq(l)\left(I^{+}\right)^2+\{1-[x-V(L)]b\}q(l)I^{+}-[1+bV(L)]q(l)x-[x-V(L)][1-q(l)]=0. \tag{42}$$

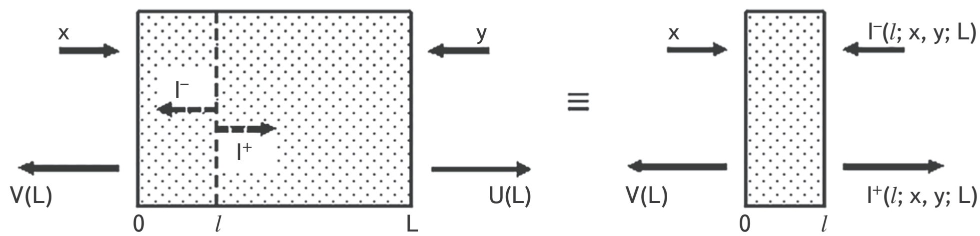

Fig. 2. Illustrating the equivalence of the problems concerning the emerging radiation and the radiation field inside the medium.

These last equations can easily be derived directly. For example, to derive Eq. (41) it is sufficient to use the second of Eqs. (4) and include Eq. (5) in the right hand side of Eq. (40). Here if one of the quantities $I^{\pm}(l;x,y;L)$ is already determined from the corresponding Eq. (41) or (42), then the other can be found without invoking the corresponding equation, using just one of the explicit expressions for nonlinear combination of layers:

$$I^{+}(l;x,y;L)=u\left(x,I^{-}(l;x,y;L);l\right) \text{ or } I^{-}(l;x,y;L)=\upsilon\left(I^{+}(l;x,y;L),y;L-l\right). \tag{43}$$

Explicit forms of the latter are easily written with the aid of Eq. (4):

$$I^{+}(l;x,y;L)=I^{-}(l;x,y;L)+\left[x-I^{-}(l;x,y;L)\right]T_{l}\left(x+I^{-}(l;x,y;L)\right), \tag{44}$$

$$I^{-}(l;x,y;L)=I^{+}(l;x,y;L)-\left[I^{+}(l;x,y;L)-y\right]T_{L-l}\left(y+I^{+}(l;x,y;L)\right), \tag{45}$$

where the LI T_l and T_{L-l} are given by Eq. (5); here the subscript also indicates the layer thickness to which the given LI applies. Thus, Eqs. (41) and (44) or, for that matter, (42) and (45), determine the field inside the medium as the solution of the inverse problem; the same unknowns were determined in Ref. 1 by formulating the direct problem by means of Eqs. (44) and (45).

3.6. The internal field in a semi-infinite medium. We now consider the special case of "problem 3.5" where the optical thickness of the purely scattering medium is infinite, letting

$$y=0, \quad \tau\equiv\int_0^L k_0(l)dl=\infty, \quad I^{\pm}(l;x,0;L)\Big|_{\tau=\infty}\equiv I_{\infty}^{\pm}(l,x)\equiv I_{\infty}^{\pm}. \tag{46}$$

It follows from Eqs. (4), (5), and (30) that for pure scattering:

$$V(L)\big|_{\tau=\infty} = x\,, \tag{47}$$

i.e., from a purely scattering semi-infinite medium in the nonlinear case, by analogy with the linear case, all the radiation that entered it from outside through its outer boundary is completely reflected. Given Eq. (47) and the fact (cf. Eq. (30)) that

$$q(l) = \frac{2}{2+\int_0^l k_0(l')dl'} \neq 0\,, \tag{48}$$

Eqs. (41) and (42) become

$$b\left(I_\infty^-\right)^2 + I_\infty^- - [1+bx]x = 0\,, \tag{49}$$

$$b\left(I_\infty^+\right)^2 + I_\infty^+ - [1+bx]x = 0\,. \tag{50}$$

Equations (49) and (50), which describe the fields “ahead of” and “behind” the propagating radiation are identical; furthermore, they do not depend on the depth parameter, that is,

$$I_\infty^+(l,x) = I_\infty^-(l,x) = \text{const}(x), \quad 0 < l < \infty\,, \tag{51}$$

i.e., at each depth of the semi-infinite medium the intensities of the radiation moving “ahead of” and “behind” are equal to one another and the field itself is uniform at all depths. From Eq. (51) with $I_\infty^\pm(0,x) = x$ (cf. Eq. (47)) or directly from solution of Eqs. (49) and (50) we shall have

$$\text{const}(x) = x\,, \quad \text{i.e} \quad I_\infty^\pm(l,x) = x\,. \tag{52}$$

Thus, in the nonlinear problem of external illumination of a semi-infinite medium consisting of two-level atoms, for pure scattering both the radiation emerging from the medium and the field inside it are the same at all depths and equal to the radiation entering the system. This light regime also occurred in the linear problem (e.g., see Eqs. (49-50) with $b \equiv 0$). Furthermore, even in the limiting case of infinitely powerful illumination of the medium from outside this nonlinear problem still continues to be linear. This result is by no means trivial, since three infinities operate in it: an infinite interaction time, infinitely powerful external illumination, and infinite optical thickness of the medium. As a result of their interaction the optical properties of the medium do not change, it does not bleach at all, and a regime with a linear homogeneous light field is established everywhere. That is, here the optical thickness is important, as opposed to the case of a medium with finite optical thickness, where under the same conditions the power of the radiation predominates, bleaching the medium and in the limit bringing it to full transparency. In another context the nonlinear problem of a uniform semi-infinite medium consisting of two-level atoms has been examined previously [11-13], and the more general case of a three-level atom in Ref. 14.

4. Conclusion

It has been shown in this paper that knowledge of the LI, the auxiliary function introduced in Refs. 15 and 16, for reflection-transmission in the one-dimensional nonlinear problem of a two-level atom with pure scattering can be used to obtain explicit solutions of some more complicated direct and inverse problems.

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