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ANALYTIC SOLUTION OF THE NONLINEAR RADIATIVE DIFFUSION PROBLEM IN A ONE-DIMENSIONAL PURELY SCATTERING MEDIUM. I

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A simple example of a nonlinear problem of reflection and transmission of radiation by a one-dimensional scattering and absorbing isotropic medium of finite geometric thickness with conservative and isotropic scattering is examined. The effectiveness of the linear images (LI) method developed previously by the author for solving nonlinear radiative transfer problems is demonstrated. An explicit expression is obtained for the desired LI. It is used to obtain a solution of the reflection and transmission problem for two-sided illumination of a medium of finite thickness in explicit closed form and the radiation field inside it is determined for two mutually opposite statements of the problem: "direct" and "inverse." Numerical results are given illustrating the difference between the solutions of these problems in the linear and nonlinear cases.

Keywords: *radiative transfer: nonlinear problem*

1. Introduction and statement of the problem

Radiative energy transfer problems in astrophysical media (stellar and planetary atmospheres, interstellar clouds, cosmic gas-dust complexes, etc.) play an important role in the interpretation of observed spectra of cosmic objects. The physical processes of multiple interactions of radiation with matter lead to a mutual change in primary

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characteristics such as those of radiation propagating in a given medium, as well as the properties of a scattering and absorbing medium, itself. In linear transport problems only the first of these two factors is considered, with the optical properties of the medium assumed to be known previously and invariant. This simplifying assumption is not made in nonlinear problems, so the established characteristics of the radiation field and the scattering medium are now determined jointly, in a self-consistent way. Because of the mathematical complexity, these problems are studied mainly just by numerical methods, and in the best case, numerical-analytic methods (e.g., see the references in Ref. 1). It is important to note, however, that in some special cases, solutions of these kinds of problems can still be obtained analytically in a closed, explicit form. Examples of these are of some interest since a comparative analysis based on them can provide a direct analytic clarification of the basic differences between the physical pictures of linear and nonlinear models of radiative transfer.

In this study, which consists of two parts, analytic solutions are presented for a simple nonlinear problem of the illumination of a one-dimensional and isotropic medium of finite geometric thickness consisting of two-level atoms with pure and isotropic scattering in the case of two-sided illumination of the medium from outside. Here it becomes possible to analyze a number of related nonlinear transfer problems in their "direct" and "inverse" statements. In particular, the radiative field and optical characteristics of the medium are described as functions of the power of the external exciting radiation.

Consider a one-dimensional medium of geometric thickness L consisting of two-level atoms, which is exposed to intense beams of external exciting radiation on both sides, $l = 0$ and $l = L$, with intensities x and y, respectively. It is required to find the intensities $u(x, y, L)$ and $v(x, y, L)$ of the radiation emerging from the medium through, respectively, the right and left boundaries, as well as the intensities $I^{-}(l, x, y, L)$ and $I^{+}(l, x, y, L)$ of the radiation proceeding at an arbitrary depth l toward the right (+) and left (-) boundaries.

2. Initial equations

The nonlinear radiative transfer problem in a one-dimensional anisotropic medium of finite geometrical thickness has been examined in general with the aid of the invariance principle [1-3] (see Refs. 4 and 5, as well) and the method of adding layers [6-8]. Then, for further simplification of the solution of this nonlinear problem, the method of linear transforms (LI) was introduced [9]. (For a first communication about this see Ref. 10. In Ref. 11 the method is extended to the problem of determining the radiation field inside a medium.) Linear images are auxiliary functions that describe the process of multiple scattering of a single photon, or a beam of them of unit intensity, incident on one of two boundaries of a medium, while the medium itself is illuminated from two sides. It turns out that the solution of the nonlinear problem at hand is expressed in terms of elementary linear combinations of these auxiliary functions.

For u and v in the special case of an isotropic medium and conservative scattering the following explicit expressions were obtained in Ref. 9 (see Ref. 12, as well):

$$u = (x-y)T + y\,, \quad v = -(x-y)T + x \tag{1}$$

and functional equation of Ambartsumian's complete invariance (ACI)

$$\left[k(x+v)\frac{\partial}{\partial x} + k(y+u)\frac{\partial}{\partial y}\right]T = -TM(x, y), \tag{2}$$

which is a symmetric, quasilinear differential equation, where $T \equiv T_L \equiv T(x, y) \equiv T(x, y; L)$ is the LI for transmission, while

$$M(x, y) = M(y, x) \equiv \frac{k(x+v) - k(y+u)}{x-y}, \tag{3}$$

where $k(\xi)$ is the absorption coefficient for the given medium in an elementary interaction of the radiation with the matter. The isotropy of the medium and the conservative character of the scattering show up in the respective equations

$$v(x, y) \equiv u(y, x), \tag{4}$$

$$u + v = x + y\,, \quad R + T = 1 \tag{5}$$

where $R \equiv R(x, y) \equiv R(x, y; L)$ is the LI for reflection. The initial conditions for Eq. (2) are specified in the form

$$T(x, 0; L) = \sigma(x) \;\text{ or }\; T(0, y; L) = \sigma(y), \tag{6}$$

where $\sigma \equiv \sigma(\varsigma) \equiv \sigma(\varsigma; L)$ is the solution of the more particular nonlinear problem of passage of a single photon through a medium with one-sided illumination [9]

$$\left[\frac{\partial}{\partial L} + \varsigma\sigma\frac{k(2\varsigma - \varsigma\sigma)}{2}\frac{\partial}{\partial\varsigma}\right]\sigma = -\sigma^2\,\frac{k(2\varsigma - \varsigma\sigma)}{2}, \tag{7}$$

$$\sigma\big|_{L=0} = 1 \;\text{ or }\; \sigma\big|_{\varsigma=0} = q\,, \tag{8}$$

and $q \equiv q_L \equiv q(L) = \sigma(0; L) = T(0, 0; L)$ is the analogous solution for the linear problem that satisfies the well known invariant imbedding equation ($0 \le l \le L$)

$$\frac{\partial}{\partial l} q = -q^2 \frac{k_0}{2}, \quad q\big|_{l=0} = 1, \quad k_0 \equiv k(0) \equiv k(0,l) \equiv k_0(l) = n(l)\frac{hv}{2} B_{12}, \tag{9}$$

which has the solution [4]

$$q_L \equiv q(\tau_0) = \left(1 + \frac{\tau_0}{2}\right)^{-1}. \tag{10}$$

In Eq. (10) the concept of limiting (i.e., corresponding to the linear case) optical thickness [5] is used, with

$$\tau_0 \equiv \int_0^L k_0(l) dl = \frac{hv}{2} B_{12} \int_0^L n(l) dl. \tag{11}$$

It follows from this discussion that to obtain the complete solution of the problem of determining $u, v,$ and $I^{\pm}$ stated above, it is first necessary to find the transmission LI $T(x, y; L)$ from the ACI equation (2)-(3) with initial conditions (6) derived from Eqs. (7)-(8). Some of the results obtained below were given previously without proof in Refs. 13 and 14.

3. The solution of the ACI equation for the transmission LI

The absorption coefficient with conservative scattering [9] takes the form

$$k(\xi) \equiv k(\xi, l) = \frac{k(0)}{1 + b\xi}, \quad \xi \equiv x + y, \tag{12}$$

where

$$b \equiv \frac{B_{12} + B_{21}}{2A_{21}} = \frac{c^2}{4hv^3}\left(1 + \frac{g_2}{g_1}\right), \tag{13}$$

(here the standard notation [15] is used).

For the transition to the linear case it is sufficient to expand the denominator in Eq. (12) in a Taylor series subject to the condition $b\xi << 1$ and retain only the first term. The result shows that this transition corresponds formally to the substitution $b \equiv 0$.

The quasilinear equation (2)-(4) in the characteristics has the form

$$\frac{\partial x}{k(x+v)} = \frac{\partial y}{k(y+u)} = -\frac{\partial T}{TM(x,y)}. \tag{14}$$

When the conditions

$$1+b\xi_1 \neq 0, \quad 1+b\xi_2 \neq 0, \quad \text{where} \quad \xi_1 \equiv x+v, \quad \xi_2 \equiv y+u, \tag{15}$$

are met, substituting Eq. (12) in Eqs. (14) and (3) after the necessary contractions yields

$$\partial x = -\frac{(1+b\xi_2)(x-y)}{b(\xi_2-\xi_1)}\frac{\partial T}{T}, \tag{16}$$

$$\partial y = -\frac{(1+b\xi_2)(x-y)}{b(\xi_2-\xi_1)}\frac{\partial T}{T}. \tag{17}$$

On the other hand, calculating the expressions in parentheses that contain ξ with Eq. (1) yields the equations

$$\begin{gathered} 1+b\xi_1 = 1+2bx-b(x-y)T, \quad 1+b\xi_2 = 1+2by+b(x-y)T, \\ \xi_2-\xi_1 = 2(x-y)(T-1). \end{gathered} \tag{18}$$

With the last equations, Eqs. (16) and (17) are written in simpler form as

$$\partial x = [1+2by+b(x-y)T]\frac{\partial T}{2b(1-T)T}, \tag{19}$$

$$\partial y = [1+2bx-b(x-y)T]\frac{\partial T}{2b(1-T)T}, \tag{20}$$

which, when added, reduce the problem to integrating the equation

$$\frac{b\partial\xi}{1+b\xi} = \frac{\partial T}{(1-T)T}. \tag{21}$$

This equation manifests an extremely important property of the unknown quantity T. It turns out that the transmission LI is a function of only the variable ξ (neglecting, of course, its dependence on the thickness of the medium L), which is the sum of the intensities illuminating the medium on two sides

$$T(x,y) \equiv T(x+y) \equiv T(\xi). \tag{22}$$

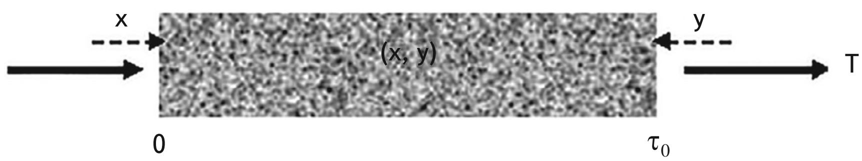

Fig. 1. For physical interpretation of the LI.

It is easy to understand that this is a direct consequence of the isotropy of the medium and the physical significance of the LI. Integrating the left and right sides of Eq. (21) with respect to ξ and T over mutually consistent limits $[0,\xi]$ and $[q,T(\xi)]$, we obtain

$$\ln(1+b\xi)=\ln\frac{(1-q)T(\xi)}{[1-T(\xi)]q}, \tag{23}$$

which yields an explicit expression for the transmission LI,

$$T(\xi)=q\frac{1+b\xi}{1+qb\xi}=\frac{2(1+b\xi)}{\tau_0+2(1+b\xi)}. \tag{24}$$

Equation (10) has also been used in this last equality. It is easy to confirm by direct substitution that the solution (24) converts the initial quasilinear Eq. (2) into an identity.

The LI $T(\xi,\tau_0)$ has a clear physical significance (Fig. 1). It represents the probability of transmission of a single photon by a layer of limiting optical thickness τ_0 when the two sides of the medium are illuminated by radiation with intensities (x, y) (see Fig. 1). In the linear case, as expected, Eq. (24) implies $T_0\equiv T\big|_{b=0}=q$.

In the pioneering work of Ambartsumian [16.8] it is shown that during the transition from the linear transmission problem to the nonlinear case, a new effect arises, expressed as a reduction in the original limiting optical density as the intensity of the incident radiation increases. The reason for this is the transition of some fraction of the atoms in the medium into an excited state, so that the latter cease to absorb photons and the medium appears to be "bleached." This is intuitively clear from the plots of the transmission coefficient shown in Fig. 2.

Figure 2 illustrates the bleaching of the medium for different optical thicknesses given by Eq. (24). (Here an in the following, in the figures it is nominally assumed that b = 1.) With increasing intensity of the incident radiation the fraction of the radiation transmitted by the medium increases nonlinearly. Beginning with the linear (for $\xi=0$) value of q, the fraction of transmitted radiation tends monotonically to unity, so that the medium should become fully transparent, as $\lim_{\xi\to\infty}T(\xi)=1$. Here when the initial limiting optical thickness of the medium is greater, the bleaching process proceeds more slowly.

The reflection TI can also be found explicitly using Eq. (5),

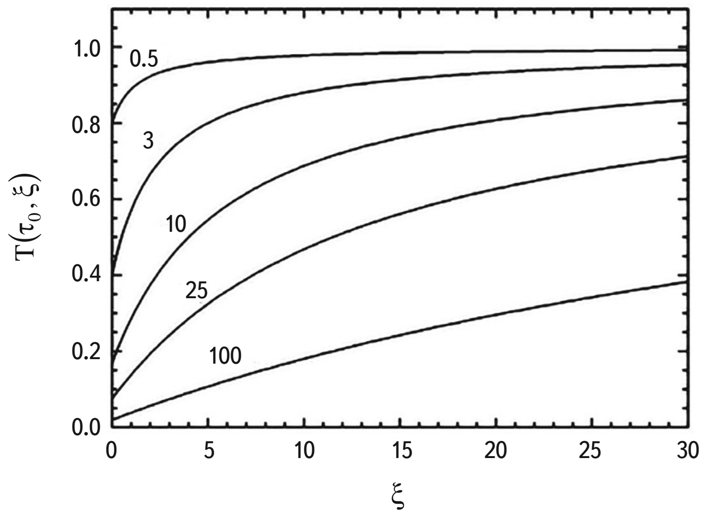

Fig. 2. Transmission of the medium as a function of the intensity of the exciting radiation.

$$R(z) = \frac{1-q}{1+b\xi q} = \frac{\tau_0}{\tau_0 + 2(1+b\xi)}. \tag{25}$$

4. Solution of the nonlinear reflection-transmission problem in explicit form

Equations (1) give the solution of the original nonlinear reflection-transmission problem for two-sided illumination of the medium in terms of an auxiliary function T. Substituting Eq. (24) in Eq. (1) and using Eq. (10), we obtain explicit forms for this solution:

$$\begin{aligned} u &= \frac{(1+b\xi)qx + (1-q)y}{1+b\xi q} = \frac{2(1+b\xi)x + \tau_0 y}{\tau_0 + 2(1+b\xi)}, \\ v &= \frac{(1-q)x + (1+b\xi)qy}{1+b\xi q} = \frac{\tau_0 x + 2(1+b\xi)y}{\tau_0 + 2(1+b\xi)}. \end{aligned} \tag{26}$$

For $b = 0$, Eqs. (26) describe the linear approximation (see Eqs. (24) and (1), as well):

$$u_0 \equiv qx + (1-q)y = \frac{2x + \tau_0 y}{\tau_0 + 2}, \quad v_0 \equiv (1-q)x + qy = \frac{\tau_0 x + 2y}{\tau_0 + 2}. \tag{27}$$

Here, as above, the subscript "0" denotes the solutions of the corresponding linear problems $u|_{b=0} = u_0$ and

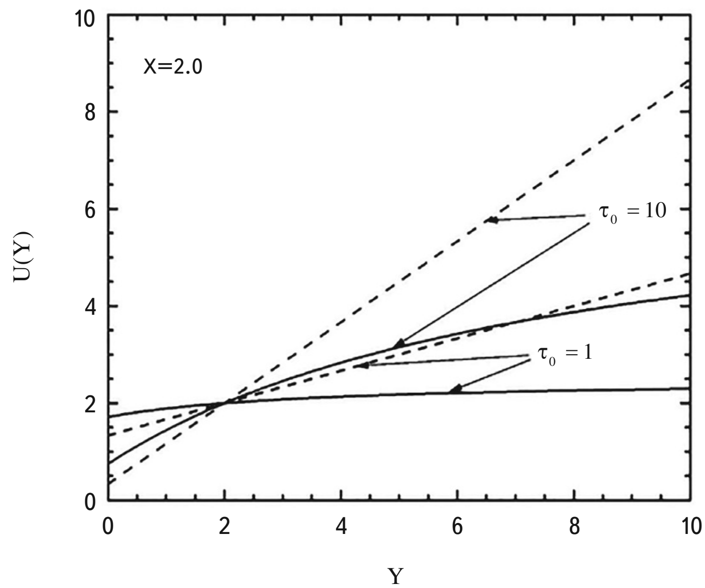

Fig. 3. The radiation transmitted by the medium as a function of the intensity of the external radiation on the right hand side.

$v|_{b=0} = v_0$. By comparing Eqs. (27) and (26) it is easy to show the direct relationship between the radiation fields in the linear and nonlinear cases:

$$u = \frac{u_0 + b\xi qx}{1 + b\xi q} = \frac{(\tau_0 + 2)u_0 + 2b\xi x}{(\tau_0 + 2) + 2b\xi}, \quad v = \frac{v_0 + b\xi qy}{1 + b\xi q} = \frac{(\tau_0 + 2)v_0 + 2b\xi y}{(\tau_0 + 2) + 2b\xi} \cdot (28)$$

Figure 3 shows the dependence of the transmitted radiation $u(x, y, \tau_0)$ on the increase in the intensity y of the outer illumination on the right side for different optical thicknesses ($\tau_0 = 1$ and $\tau_0 = 10$) with $x = 2$.

A smooth curve corresponds to the nonlinear case and a dashed curve is a solution for the linear problem. In the nonlinear case, with increasing power of the external exciting field the rise in intensity of the diffuse radiation transmitted by the medium slows down strongly compared to the linear case. In addition, the difference between the two curves increases monotonically. The difference between these solutions is greater when the medium is thicker.

Thus, using the concept of the LI in this special case of an isotropic and purely scattering medium makes it possible both to distinguish the linear structure of the solution to the nonlinear problem being studied here, as happened in the general case [9], but also to show that the unknown functions for the two energy variables $u(x, y)$ and $v(x, y)$ are expressed in terms of the auxiliary function $T(\xi)$ (i.e., in terms of their LI), which depends on only one variable $\xi \equiv x + y$. Here $T(\xi)$, in turn, is determined analytically from Eq. (24) with the aid of the ACI functional equation (2). It is important that in this case, because of Eqs. (6) and (22) it is no longer necessary to examine the nonlinear problem (7)-(8) in which the medium is illuminated on one side. To find $T(\xi)$ it is sufficient to restrict ourselves to preliminary solution of just the linear transmission problem (9). In the structure of the analytic solution (28) for the nonlinear problem, the complete solution of the linear case (27) stands out explicitly, and this is an explicit indication that the two problems have the same asymptotic behavior for $b = 0$.

5. The problem of an internal luminous regime for the medium

We now examine the more general problem of determining the radiation field $I^{\pm}$ inside the medium. We solve it in two different ways: in the form of the "direct" and "inverse" problems. In the first case, the medium is assumed to be composite, with the solutions to the reflection and transmission problems for the components previously known and the field at the contact boundary where they are joined to be determined. In the second case, the intensities of the incident and reflected radiation, which are known only at one boundary of the medium, are used to find the radiation field inside the medium.

5.1. The direct problem. Let the solutions $u(x, y; L)$, $v(x, y; L)$ and $u(x, y; L-l)$, $v(x, y; L-l)$ of the partial radiative reflection-transmission problems for two separate media [0, *l*] and [*l*, *L*], i.e., two separate sublayers from an original composite medium [0, *L*] with thicknesses *l* and *L* – *l*, respectively, be known. It is required to determine the field $I^{\pm}(l; x, y; L)$ at geometric depth *l* inside the composite medium [0, *L*] (with thickness $L = l + (L - l)$ obtained by combining its two component parts [0, *l*] and [*l*, *L*] mentioned above, when it is illuminated from outside by intensities (*x*,*y*). A "+" sign corresponds to the direction of increasing depth. The method of adding layers directly yields the following system [1,2,7-9] directly:

$$\begin{cases} p = u(x, s; l) \\ s = v(p, y; L-l), \end{cases} \tag{29}$$

where

$$p \equiv I^{+}(l; x, y; L), \quad s \equiv I^{-}(l; x, y; L). \tag{30}$$

With Eqs. (1) and (24), the system (29) can be rewritten in the form

$$\begin{cases} (1 + q_l bx)(p - s) + (sp)q_l b = -q_l s + (1 + bx)q_l x \\ (1 + q_{L-l} by)(p - s) - (ps)q_{L-l} b = q_{L-l} p - (1 + by)q_{L-l} y \, . \end{cases} \tag{31}$$

Multiplying the first of Eqs. (31) by q_{L-l} and the second by q_l and then adding them easily yields

$$p - s = \frac{f_2}{f_1}, \tag{32}$$

where the following notation has been introduced ($\xi^{\pm} \equiv x \pm y$):

$$f_1 \equiv q_l + q_{L-l} - \left(1 - b\xi^{+}\right) q_l q_{L-l}, \tag{33}$$

$$f_2 \equiv \xi^{-} \left(1 + b\xi^{+}\right) q_l q_{L-l}. \tag{34}$$

On substituting Eq. (32) in the first of Eqs. (31), after some simple calculations we arrive at the following quadratic equation

$$As^2 + Bs - C = 0, \tag{35}$$

where

$$A \equiv bq_l B \equiv \left(1 + b\frac{f_2}{f_1}\right) q_l, \quad C \equiv \left(1 + bx\right) q_l x - \left(1 + bq_l x\right) \frac{f_2}{f_1}. \tag{36}$$

With the aid of Eqs. (30), (32), and (35) we finally obtain

$$I^{+}\left(l; x, y; L\right) = \frac{f_2}{f_1} + I^{-}\left(l; x, y; L\right), \quad I^{-}\left(l; x, y; L\right) = \frac{1}{2bq_l}\left(-B + \sqrt{B^2 + 4bq_l C}\right). \tag{37}$$

It is obvious that the physical solution to Eq. (35) corresponds to a positive sign in front of the root, such that the limiting transition to the linear case (b = 0) is satisfied.

5.2. Determining the internal radiation field as an inverse problem. The transfer equation for the quantity $I^{\pm} \equiv I^{\pm}(l) \equiv I^{\pm}(l; x, y; L)$ has the form

$$\frac{dI^{\pm}}{dl} = \pm\alpha\left(I^{+}, I^{-}\right), \tag{38}$$

where $\alpha(x, y)$ is the "collision integral" for this Boltzmann equation for a photon gas. $\alpha(x, y)$ is the nonlinear "radiative response" of an elementary layer to the external, two-sided interaction (x,y). In the classical statement of the transfer theory, the solution of Eq. (38) is a two-point boundary value problem with boundary conditions specified at both ($l=0$, $l=L$) boundaries of the medium:

$$I^+(0;x,y;L)=x\,,\quad I^-(L;x,y;L)=y\,. \tag{39}$$

In the boundary value problem of Eqs. (38) and (39), the determination of the emerging radiation and the internal field are treated jointly. It is well known that the boundary value problems are rather complicated for analysis, so they are reduced to initial value problems, i.e., Cauchy problems, where possible. Since the solution of the problem with emerging radiation is already known (Eq. (28)), it can easily be used to go directly from the two-point boundary value problem to the initial value problem. For this it is sufficient at one boundary of the medium to specify the values of the incident and emerging radiation together, i.e., instead of beginning with Eq. (39), to start with

$$I^+\Big|_{l=0}=x\,,\quad I^-\Big|_{l=0}=v(x,y;L)\quad\text{or}\quad I^+\Big|_{l=L}=y\,,\quad I^-\Big|_{l=L}=u(x,y;L). \tag{40}$$

Then the boundary value problem (38)-(39) is replaced by one of two separate, equivalent Cauchy problems (38), (40).

This problem is solved very simply. Since the collision integral is known explicitly in the isotropic and conservative case we are examining [9],

$$\alpha(x,y)=\mp(x-y)k(\xi)/2\,, \tag{41}$$

the transfer equation (38) can be rewritten in the form

$$\frac{dI^+}{dl}=\frac{dI^-}{dl}=-\left(I^+-I^-\right)k\left(I^++I^-\right)/2\,, \tag{42}$$

where $k(\xi)$ is given by Eq. (12). Introducing the notation

$$I^++I^-\equiv H\,,\quad I^+-I^-\equiv S\,, \tag{43}$$

and using Eq. (12), we obtain

$$\frac{dH}{dl}=-Sk_0/(1+bH),\quad \frac{dS}{dl}=0\,, \tag{44}$$

where it is evident that $H\equiv H(l)\equiv H(l,x,y,L)$ and $S\equiv S(l)\equiv S(l,x,y,L)$. Integrating the first of Eqs. (44) using the first pair of conditions (40), we obtain

$$\int_{H(0)}^{H(l)} (1+bH(l'))dH(l') = H(l) - H(0) + \frac{b}{2}\left[H^2(l) - H^2(0)\right] = -\int_0^l k_0(l')S(l')dl'. \tag{45}$$

From the second of Eqs. (44) and also from Eqs. (43) and (40), we will have, respectively,

$$S(l) = S(0), \quad H(0) = x + v(x, y; l), \quad S(0) = x - v(x, y; L). \tag{46}$$

Equations (46) and (43), in turn, yield

$$I^+(l) - I^-(l) = x - v(x, y; l). \tag{47}$$

It is easy to see that, given Eq. (28), the resulting Eq. (47) is the same, to within the notation (30), as Eq. (32) or (37) from the direct problem. Equation (45) with Eq. (9) leads to the quadratic equation

$$\frac{b}{2}H^2(l) + H(l) - D(l) = 0, \tag{48}$$

where

$$D(l) \equiv \frac{b}{2}H^2(0) + H(0) - S(0)\int_0^l \frac{h\nu}{2} B_{12} n(l')dl'. \tag{49}$$

Given the notation (45), conditions (40), and expressions (28), Eq. (48) can be rewritten as

$$A\left(I^-\right)^2 + BI^- - C = 0, \tag{50}$$

where

$$A \equiv 2b, \quad B \equiv 2[1 + bS(0)], \quad C \equiv D(l) - \frac{b}{2}S^2(0) - S(0). \tag{51}$$

It is easy to show that the quadratic equations (35) and (50) are identical.

Figure 4 is a plot of the intensity $I^{\pm}(\tau, x, y, \tau_0)\Big|_{\substack{\tau=0.5 \\ x=y \\ \tau_0=1}} \equiv I^{\pm}(x) \equiv I(x)$ of the internal radiation field at optical depth $\tau = 0.5$ of a layer of unitary optical thickness ($\tau_0 = 1$) as a function of the external radiation power *x*, when the medium is illuminated at both boundaries by beams of equal power $x = y$. The dashed curve applies to the linear and the smooth curve, to the nonlinear case.

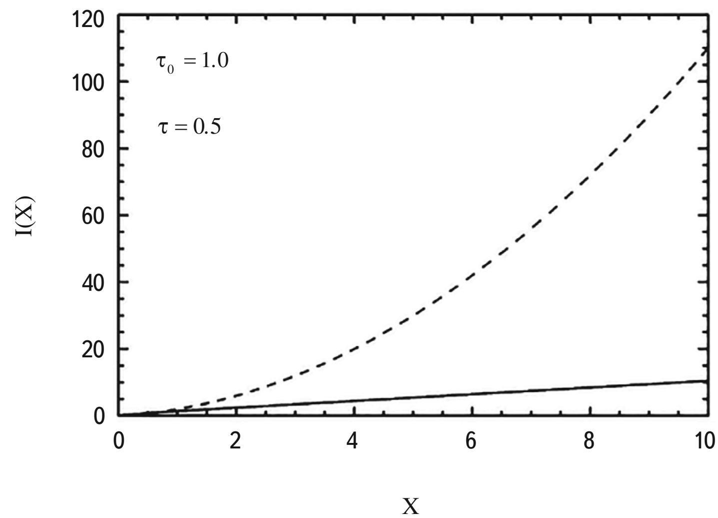

Fig. 4. The intensity variation inside the medium.

Figure 4 shows that with increasing power of the exciting radiation, the discrepancy between the solutions for the linear and nonlinear problems becomes very substantial.

6. Conclusion

Therefore, with the example of an analytic solution of a simple nonlinear radiative transfer problem in an isotropic, one-dimensional, conservative and isotropically scattering medium illuminated from both sides by intense radiation beam, we have demonstrated the effectiveness of the previously introduced [10,9,12] concept of linear images (LI). Here an explicit form of this quantity was first found, and then it was used to determine the intensities of the radiation emerging from the medium. Then an explicit solution of the general problem of determining the radiation field inside the medium was constructed. The order of determining the unknowns reduces to the scheme ($\xi \equiv x+y$), i.e.,

$$T(\xi, \tau_0) \to u(x, y, \tau_0) \to I^{\pm}(\tau, x, y, \tau_0). \quad (52)$$

Two contrary approaches were used to solve the problem. First the method of adding layers was used to examine a composite medium with specified reflection-transmission characteristics of its two constituent sublayers. An inverse problem (Cauchy problem for the transfer problem) was then formulated to recover the radiative field inside the medium based on observational data related to one of the two outer boundaries of the medium.

The simple example discussed above shows that discovering the linear structure of the solution of a nonlinear radiative transfer theory problem by introducing the concept of a linear images of the desired solution effectively simplifies the process of searching for the solution and in some simple cases makes it possible to obtain the solution analytically. One of the dangers of analyzing real fields of diffuse radiation obtained in linear approximations is that they may yield excessive estimates of both the intensities of the diffuse radiation field and the optical parameters of the medium, even when the external exciting radiation is comparatively low.

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