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THE EFFECT OF NEUTRINO MASS IN COSMOLOGY

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The influence of the small neutrino mass on the properties of the cosmological neutrino gas and on the evolution of the universe is estimated. The characteristics of the equilibrium neutrino gas are first calculated and the temperatures up to which to which they can be regarded as ultrarelativistic for different neutrino masses are found. The evolution of cosmological neutrinos of any type are traced for different neutrino masses. The red shifts neutrinos remain ultrarelativistic are determined. The effective temperature of the neutrino gas and the effect of neutrino mass on the rate of expansion of the universe are determined.

Keywords: *Cosmological neutrinos: neutrino masses: evolution of the universe.*

1. Introduction

The existence of the particles subsequently referred as neutrinos by Enrico Fermi was proposed by Wolfgang Pauli in 1930 (he refered to them as neutrons). This was done in a personal letter to participants at a conference in Tuebingen [1] and in 1931 in a talk at a session of the American Physical Society in Pasadena. The purpose of this proposal was to ensure the conservation of energy in beta decay of various nuclei. An analogous problem arises during decay of a neutron into a proton and an electron. The mean lifetime of a free, motionless neutron is

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$\bar{t}_N = 14.7$ min (the half decay time is $t_{1/2} = \ln(2)\bar{t}_N = 10.2$ min). The electron emerging in this process carries energy, but it turned out to be smaller than the difference between the rest energy of the neutron and proton, and the distribution of the energies was random. The introduction of the neutrino solved yet another inconsistency in that the nonconservation of momentum, specifically spin, was eliminated. It was difficult to observe this kind of particle because of its very weak interactions with other particles and nuclei (the reaction cross sections were too low), so its existence followed from the conservation of energy, momentum, and angular momentum in experiments with decay reactions. Pauli's hypothesis was later fully verified and neutrinos were detected on an accelerator in 1953 in a neutrino capture reaction, as opposed to its emission [2]. A final conclusion about the existence of the neutrino and antineutrino was reached by the same authors later [3] and they received the Nobel Prize in physics for the experimental discovery of the neutrino. The history of this prediction and the confirmation of its validity are described in an article by Pauli, himself [4]. The subsequent work of Pauli in the theory of β-decay and the general problem of neutrinos is given in a review [5].

At present, solar neutrinos and neutrinos in cosmic rays are observed in several neutrino observatories, in particular the Ice Cube, with a side of 1 km, in Antarctica [6].

The properties of neutrinos have been studied in some detail. Three types (flavors) of neutrinos and the corresponding antineutrinos exist: electronic, muon, and tau neutrinos. For a long time it was assumed that the mass of the neutrino is exactly zero, like that of the photon. Experiments with so-called neutrino oscillations, when they change from one type to another, show, however, that neutrinos have finite masses [7-9] (T. Kajita and A. B. McDonald, Nobel prize in physics, 2015). Compared to the masses of other particles, they are very small and it is very difficult to determine them. The possible consequences for astronomy of the nonzero mass of neutrinos are discussed in [10]. The masses of neutrinos, however, are substantially lower than assumed previously. Based on observations of the oscillations the differences of the squares of the masses has been determined and other methods have been used to determine the sum of the masses of all the types. These later estimates give $m_{\nu e}^2 - m_{\nu\mu}^2 \approx \left|m_{\nu\mu}^2 - m_{\nu\tau}^2\right| < 0.6 \div 0.7$ (eV)^2 [11], while the sum of the neutrino masses does not exceed ≈ 1 eV [12] (all in energy units).

The existence of cosmological neutrinos follows from the theory of the hot universe, according to which its early evolution was at a high temperature in a material in a state of thermodynamic equilibrium where balancing weak interaction reactions should take place with the emission and absorption of neutrinos (so their chemical potential is zero). As the temperature falls owing the cosmological expansion, neutrinos separate from the rest of the matter and propagate freely. It may apparently possible to detect these neutrinos at some time, but apparently not soon. Perhaps it will be possible to observe indirect manifestations of the presence of neutrinos.

There is another difficulty with the theory of neutrinos: if the mass of a neutrino is detected, then its flavor will be undetermined and *vice versa*. It can be said that neutrinos are not isolated particles but a superposition of particles with a state specified by a 3×3 matrix.

There is some evidence that neutrinos of another, fourth type exist, known as sterile since they do not participate in any interactions, including weak interactions, except gravitational. These neutrinos have a greater mass than the three types mentioned above and may form at least part of dark matter in the universe, so they are assumed to be such a component [13].

Calculations have recently been published on the evolution of neutrinos during cosmological expansion if their initial composition is specified [14]. The effect of neutrinos on leptogenesis [15], on the formation of the large-scale structure of the universe [16,17], on the L_α forest [18], on the anisotropy and polarization of the relict radiation [19], on the time variation of red shifts [20], etc. [11] (this last article is in an archive that also contains reviews of previous work), have been discussed.

The purpose of this paper is simpler. It seems that there is interest in evaluating the influence of neutrinos on the course of the evolution of the universe and also in finding more exact solutions for the temperature of the neutrino gas than under the assumption of zero neutrino mass. Here it is sufficient to assume that agreement between mass and flavor is attained under thermodynamic equilibrium, i.e., all the neutrinos and antineutrinos can be treated independently.

We begin by discussing the theory of equilibrium neutrinos in the assumed approximation.

2. Equilibrium neutrino gas

2.1. Equilibrium distribution of neutrinos. The concentration n_ν, mass density ρ_ν, and pressure P_ν of fermions with spin 1/2, mass m_ν, and zero chemical potential, as all types of neutrinos have, are given in a state of thermodynamic equilibrium by the Fermi-Dirac formulas [21],

$$n_\nu = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,\mathrm{d}p}{e^{\varepsilon(p)/k_B T_\nu}+1}, \tag{1}$$

$$\rho_\nu = \frac{8\pi}{h^3 c^2}\int_0^\infty \frac{\varepsilon(p)p^2\,\mathrm{d}p}{e^{\varepsilon(p)/k_B T_\nu}+1}, \tag{2}$$

$$P_\nu = \frac{8\pi}{3h^3}\int_0^\infty \frac{\upsilon p^3\,\mathrm{d}p}{e^{\varepsilon(p)/k_B T_\nu}+1}. \tag{3}$$

Here we use the standard notation, recalling only that $\varepsilon(p) = c\sqrt{m_\nu^2 c^2 + p^2}$ is the total energy (including the rest energy) of a particle with momentum p, and $\upsilon = c^2\,p/\varepsilon(p)$ is its velocity. For zero chemical potential the entropy density, i.e., the entropy per unit volume, is given in terms of these two quantities:

$$
s_\nu = \frac{S_\nu}{V} = \frac{\rho_\nu c^2 + P_\nu}{T_\nu} = \frac{8\pi}{h^3}\frac{1}{T_\nu}\int_0^\infty \left(\varepsilon(p) + \frac{\upsilon p}{3}\right)\frac{p^2\,\mathrm{d}\,p}{e^{\varepsilon(p)/k_B T_\nu}+1}\,. \tag{4}
$$

Substituting the variable of integration $z = pc/(k_B T)$, we rewrite the distributions as

$$
n_\nu = a_n T_\nu^3 G_n(y), \quad G_n(y) = \frac{2}{3\zeta(3)}\int_0^\infty \frac{z^2\,\mathrm{d}\,z}{e^{\sqrt{z^2+y^2}}+1}\;, \quad a_n = 12\pi\zeta(3)\left(\frac{k_B}{hc}\right)^3 = \frac{15.2}{(\mathrm{cmK})^3} \tag{5}
$$

$$
\begin{gathered}
\rho_\nu = a_\rho T_\nu^4 G_\rho(y), \quad G_\rho(y) = \frac{120}{7\pi^4}\int_0^\infty \frac{z^2\,\mathrm{d}\,z\sqrt{z^2+y^2}}{e^{\sqrt{z^2+y^2}}+1}\,, \\
a_\rho = \frac{7}{15}\pi^5\frac{k_B^4}{h^3 c^5} = \frac{7}{8}\frac{\grave{a}_{SB}}{c^2} = 7.37\cdot 10^{-36}\,\frac{\mathrm{g}}{\mathrm{cms^3K^4}}
\end{gathered} \tag{6}
$$

$$
\begin{gathered}
P_\nu = a_P T_\nu^4 G_P(y), \quad G_P(y) = \frac{120}{7\pi^4}\int_0^\infty \frac{z^4}{\sqrt{z^2+y^2}}\frac{\mathrm{d}\,z}{e^{\sqrt{z^2+y^2}}+1}\,, \\
a_P = \frac{c^2}{3}a_\rho = \frac{7}{24}\grave{a}_{SB} = 2.21\cdot 10^{-15}\,\frac{\mathrm{g}}{\mathrm{cms^2K^4}}\,.
\end{gathered} \tag{7}
$$

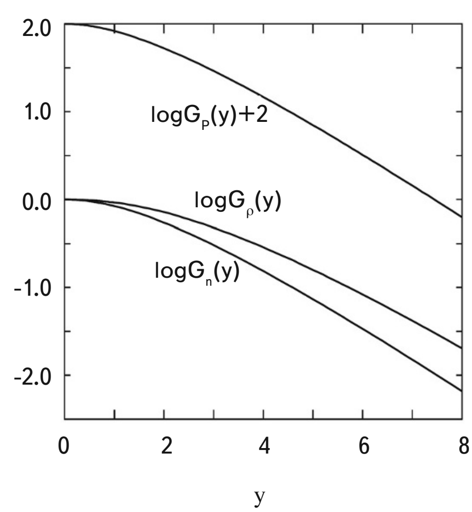

Fig. 1. The functions $\log G_n(y)$, $\log G_\rho(y)$, and $\log G_P(y)$. 2 is added to the last of them to separate it from the others.

Here and in the following $\zeta(x)$ is the Riemann zeta function and $y = m_\nu c^2 / k_B T_\nu$ is the dimensionless inverse temperature of the neutrino gas. The coefficients in the formulas for ρ_ν and P_ν are expressed in terms of the Stefan constant $\grave{a}_{SB} = \left(8\pi^5/15\right)\left(k_B^4/h^3c^3\right) = 7.5657\cdot 10^{-15}$ g/(cms^{2}K^4).

Plots of the functions $G_n(y)$, $G_\rho(y)$, and $G_P(y)$ are shown in Fig. 1.

2.2. Limiting cases. Let us examine the extreme cases of the distributions.

2.2.1. Ultrarelativistic case. When $y = 0$ the three factors in Eqs. (5)-(7) go to unity, i.e., $G_n(0) = G_\rho(0) = G_P(0) = 1$, so that for $y << 1$ the limits are found simply after replacing these factors in this way.

The distribution includes y^2, but only first order terms can be obtained in an expansion with respect to y^2, since the second derivatives with respect to y^2 diverge logarithmically at $y = 0$. The expansions have the simple form

$$G_n(y) \sim 1 - \frac{\ln 2}{3\zeta(3)} y^2 + \frac{y^3}{18\zeta(3)}, \quad G_\rho(y) \sim 1 - \frac{5}{7}\frac{y^2}{\pi^2}, \quad G_P(y) \sim 1 - \frac{15}{7}\frac{y^2}{\pi^2}. \tag{8}$$

The value of $\zeta(3) = 1.20205690316$ in the first equation. The coefficients of y and y^3 in the last two functions equal zero, but in the first one term y^3 is retained. We also give the formula for the entropy density,

$$s_\nu \sim a_s T_\nu^3 \left(1 - \frac{15}{14}\frac{y^2}{\pi^2}\right), \quad a_s = \frac{28}{45}\pi^5 \frac{k_B^4}{h^3c^3} = 8.82\cdot 10^{-15} \frac{\text{g}}{\text{cms}^2\text{K}^4}. \tag{9}$$

2.2.2. Nonrelativistic limit. In the opposite case of $y >> 1$, Expansions on MacDonald functions can be used:

$$G_n(y) = \frac{2}{3\zeta(3)} y^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} K_2((n+1)y), \tag{10}$$

$$G_\rho(y) = \frac{120}{7\pi^4} y^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left[\frac{3}{(n+1)y} K_2((n+1)y) + K_1((n+1)y)\right], \tag{11}$$

$$G_P(y) = \frac{360}{7\pi^4} y^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} K_2((n+1)y). \tag{12}$$

These yield the asymptotic behavior

$$G_n(y) \sim \frac{\sqrt{2\pi}}{3\zeta(3)} y^{3/2} e^{-y}\left(1+\frac{15}{8y}\right), \quad G_\rho(y) \sim \frac{60}{7\pi^4}\sqrt{2\pi}\, y^{5/2} e^{-y}\left(1+\frac{27}{8y}\right),$$
$$G_P(y) \sim \frac{180}{7\pi^4}\sqrt{2\pi}\, y^{3/2} e^{-y}\,. \tag{13}$$

2.3. Average particle energy and the degree of "relativity." Finding the average energy of a neutrino is of some interest, i.e.,

$$\bar{\varepsilon}_\nu(y) = \frac{\rho_\nu}{n_\nu} c^2 = a_{\rho n} k_B T_\nu \frac{G_\rho(y)}{G_n(y)}, \quad a_{\rho n} = \frac{7\pi^4}{180\zeta(3)} = 3.1514\,. \tag{14}$$

For small y the average energy is much greater than the rest energy $\bar{\varepsilon}_\nu(y) \sim a_{\rho n} k_B T_\nu = a_{\rho n}\, m_\nu c^2 / y$. Otherwise, if $y \to \infty$, the energy approaches the neutrino rest energy $m_\nu c^2$.

In view of the above estimates, it is natural to introduce a dimensionless quantity characterizing the closeness of the state of the neutrino gas to the limits:

$$r = y\frac{\rho_\nu / m_\nu - n_\nu}{n_\nu} = \frac{a_{\rho n} G_\rho(y) - y G_n(y)}{G_n(y)} \sim \begin{cases} a_{\rho n} - y & \text{for} \quad y << 1\,, \\ 3/2 & \text{for} \quad y >> 1\,. \end{cases} \tag{15}$$

Figure 2 shows the variation in r with dimensionless temperature y and for three values of the neutrino mass as a function of temperature.

On specifying the limiting deviation of the parameter from the asymptotic value $a_{\rho n} - r < \varepsilon$, we obtain the following simple estimates: $y < \varepsilon$ and $T_\nu > m_\nu c^2 / k_B\, \varepsilon$.

We now turn to cosmological models that include the neutrino.

3. Cosmological model with massless neutrinos

3.1. Parameters of the model. According to the cosmological model currently taken to be the most adequate real universe and referred to as the Standard or ΛCDM model, space is flat. The main components of this model are dusty matter, including visible (baryon) and dark matter, and so-called dark energy, which corresponds to the cosmological term of Einstein.

Another two components can be included in this model, specifically radiation and ultrarelativistic neutrinos. The radiation is referred to a relict, since it remains from epochs of the evolution of the universe when it was hotter. After the recombination epoch, radiation separated from the other matter and as it propagates freely and remains

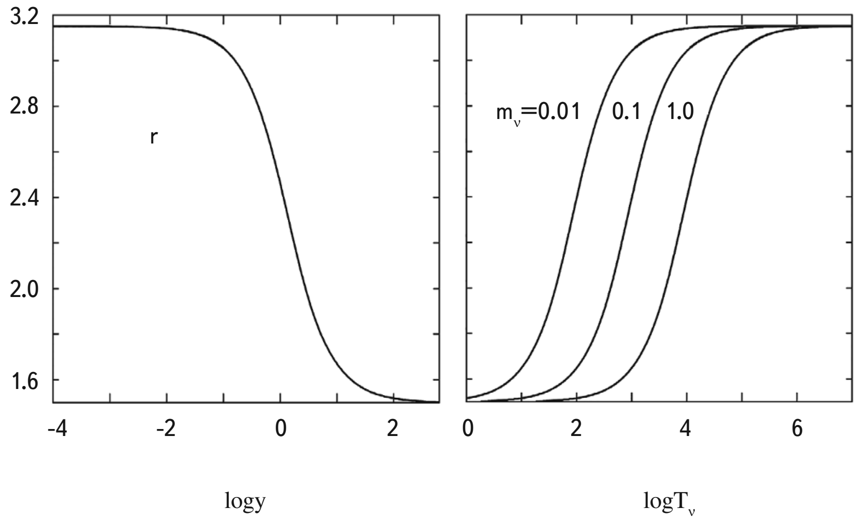

Fig. 2. The degree of relativity as a function of log$y$ and logT_ν (for three values of the particle mass in eV).

thermal (black body), it cools. The modern value of the temperature of the relict radiation, $T_0 = 2.726$, is the most accurate value known in cosmology. This temperature corresponds to the energy density given by the Stefan-Boltzmann formula, $\alpha_{SB}T_0^4 = 4.17\cdot 10^{-13}$ erg/cm^3, and the corresponding mass density is $\rho_r^0 = \alpha_{SB}T_0^4/c^2 = 4.65\cdot 10^{-34}$ g/cm^3.

The Hubble constant is known with less accuracy (a few percent); we assume that $H_0 = 70$ km$/(\mathrm{sMpc}) = 2.27\cdot 10^{-18}$ s^{-1}. Then the modern critical density is $\rho_c^0 = 3H_0^2/8\pi G = 9.21\cdot 10^{-30}$ g/cm^3, so the fraction of radiation in it is $\Omega_r^0 = \rho_r^0/\rho_c^0 = 5.06\cdot 10^{-5}$.

The fraction of dark energy [22] is 0.721±0.035. Here we assume that $\Omega_\Lambda^0 = 0.72$, so $\rho_\Lambda^0 = 6.63\cdot 10^{-30}$ g/cm^3.

The density of dusty matter is defined as the addition up to the total density to the sum of the densities of dark energy, radiation, and neutrinos. The latter has to be found.

The relationship between the neutrino temperature and radiation was obtained [23] when the neutrino mass was assumed to be zero, so they must be ultrarelativistic, like photons. We reproduce the discussion of these authors.

3.2. Temperature and density of massless neutrinos. According to the theory of the hot universe, in the initial stages of its evolution all matter was in a state of thermodynamic equilibrium with a large set of variegated particles contained in it (in the broad sense). During cosmological expansion, when all the linear scales of the universe (distances) increase in proportion to a scale factor a and matter cools, some of its components vanished, were annihilated (e.g., antiprotons and protons, mesons of opposite signs, positrons with electrons) or combined with others to form new components (e.g., nucleons and pi-mesons formed from quarks, neutrons with protons during primary nucleosynthesis to form the nuclei of the simplest elements). Other components ceased to interact among themselves or with the others, became separated, and began free expansion. The first to separate were gravitons, i.e., gravitational field quanta. Next, as the temperature fell to on the order of 10^{10} K, the neutrinos separated.

Although the process of their separation takes some amount of time, we assume that this took place instantaneously at a temperature T_ν^* and a value of the scale factor of a_ν^*. We denote the dimensionless reciprocal temperature at the time of separation by $y_\nu^* = m_\nu c^2 / k_B T_\nu^*$. The different types of neutrinos generally separate at different times, but each can be traced separately from the others, so that we examine a single type of neutrino or antineutrino.

The temperature T_ν^* as a neutrino separated was rather high, so that all the component of the matter were ultrarelativistic and their entropy was proportional to the third power of the temperature. We are interested in electrons, positrons, neutrinos, and photons. The entropy of the three Fermion gases at the time of separation were the same (9) and the entropy of the photon gas differed by a factor. These entropies in volume V_* were then

$$S_{e^\pm}^* = S_\nu^* = \frac{7}{6} \grave{a}_{SB} V_* \left(T_\nu^*\right)^3, \quad S_r^* = \frac{4}{3} \grave{a}_{SB} V_* \left(T_\nu^*\right)^3. \tag{16}$$

Cosmological expansion proceeds as an adiabatic process, in which the entropy of a given volume does not change. Thus, the temperature of all the ultrarelativistic gases varies in inverse proportion to the scale factor.

After separation, the neutrinos escaped freely, while the electrons and positrons continued to interact. When the temperature fell significantly below $6 \cdot 10^9$ K, all the positrons were annihilated with electrons, and the annihilated pairs transferred their energy, including the rest energy, to the photon gas. As a result of this, the entropy of the electron-positron pairs also was transferred to the photons. The photons continued to interact with the remaining electrons and the photon gas reached an equilibrium state at a higher temperature which subsequently decreased in accordance with the overall expansion of space. After this, the entropy of the radiation was conserved:

$$S_r = \left(\frac{4}{3} + 2\frac{7}{6}\right) \grave{a}_{SB} V_* \left(T_\nu^*\right)^3 = \frac{11}{3} \grave{a}_{SB} V_* \left(T_\nu^*\right)^3 = \frac{4}{3} \grave{a}_{SB} V T^3. \tag{17}$$

The entropy of the neutrino gas did not change, i.e., was given by Eq. .(9):

$$S_\nu = \frac{7}{6} \grave{a}_{SB} V_* \left(\frac{a}{a_\nu^*}\right)^3 T_\nu^3 = \frac{7}{6} \grave{a}_{SB} V T_\nu^3 = \frac{7}{6} \grave{a}_{SB} V_* \left(T_\nu^*\right)^3. \tag{18}$$

Because of the cosmological expansion, the volume at times corresponding to a value of the scale factor of a_ν^* and a are related by $V_* \left(a_\nu^*\right)^3 = V a^3$. For temperature T_ν we define the dimensionless reciprocal temperature $y_\nu = m_\nu c^2 / k_B T_\nu$.

Substituting the product $V_*\left(T_\nu^*\right)^3$ from Eq. (18) in Eq. (17), we obtain a relationship between the temperatures of the photon and neutrino gases after annihilation of the electron-positron pairs:

$$\frac{11}{3}\grave{a}_{SB}VT_\nu^3 = \frac{4}{3}\grave{a}_{SB}VT^3\,, \quad T^3 = \frac{11}{4}T_\nu^3\,, \quad T_\nu = \sqrt[3]{\frac{4}{11}}\,T = 0.713765856T\,. \tag{19}$$

In particular, the current temperature of massless neutrinos is $T_\nu^0 = \sqrt[3]{4/11}\,T_0 = 1.95\ \mathrm{K}$ so that their mass density (all types together) and the fraction in the critical density are

$$6\rho_\nu^0 = 6\,a_\rho\left(T_\nu^0\right)^4 = 6.35\cdot 10^{-34}\ \mathrm{g/cm^3}\,, \quad \Omega_\nu^0 = 6\frac{\rho_\nu^0}{\rho_c^0} = 6.90\cdot 10^{-5}\,. \tag{20}$$

Because the densities of radiation and massless neutrinos have the same temperature dependence, they can be combined and the joint contribution of radiation and these kinds of neutrinos to the critical density can be determined:

$$\rho_{r\nu}^0 = \rho_r^0 + 6\rho_\nu^0 = 1.10\cdot 10^{-33}\ \mathrm{g/cm^3}\,, \quad \Omega_{r\nu}^0 = \Omega_r^0 + \Omega_\nu^0 = 1.196\cdot 10^{-4}\,. \tag{21}$$

The density and fraction of dusty matter under the same conditions are given by the differences

$$\rho_d^0 = \rho_c^0 - \rho_\Lambda^0 - \rho_{r\nu}^0 = 2.58\cdot 10^{-30}\ \mathrm{g/cm^3}\,, \quad \Omega_d^0 = 1 - \Omega_\Lambda^0 + \Omega_{r\nu}^0 = 0.27988 \approx 0.28\,. \tag{22}$$

3.3. Basic equations of the model. The two Friedman equations for the scale factor in a flat space and massless neutrinos are

$$\ddot{a} = -\frac{4\pi G}{3}\rho_g\,a\,, \quad \dot{a}^2 = \frac{8\pi G}{3}\rho_t\,a^2\,. \tag{23}$$

The dependence of the densities on a is understood.

These equations include the total and gravitating densities of the masses of the components:

$$\rho_t = \rho_d + \rho_\Lambda + \rho_{r\nu}\,, \quad \rho_g = \rho_t + 3\frac{P_t}{c^2}\,, \quad P_t = P_d + P_\Lambda + P_{r\nu}\,, \tag{24}$$

where P_t is the total pressure of all components (including the six types of neutrinos). Given the fulfillment of the equations of state of the components,

$$P_d = 0\,, \quad P_r = \frac{c^2}{3}\rho_r\,, \quad P_\nu = \frac{c^2}{3}\rho_\nu\,, \quad P_{r\nu} = \frac{c^2}{3}\rho_{r\nu}\,, \quad P_\Lambda = -c^2\,\rho_\Lambda\,, \tag{25}$$

the gravitating density is rewritten in terms of the densities of the components as

$$\rho_g = \rho_d + 2\rho_{r\nu} - 2\rho_\Lambda\,. \tag{26}$$

The consistency condition for Eqs. (23) is the equation

$$\dot{\rho}_t = -3\left(\rho_t + \frac{P_t}{c^2}\right)H\,, \quad H = \frac{\dot{a}}{a}\,, \tag{27}$$

which reflects the adiabaticity of the cosmological expansion and is satisfied for each component separately. Integrating these equations, we obtain relations describing the evolution of the densities of the components:

$$\rho_d = \frac{\rho_d^0}{a^3} = \rho_c^0\,\frac{\Omega_d^0}{a^3}\,, \quad \rho_r = \frac{\rho_r^0}{a^4} = \rho_c^0\,\frac{\Omega_r^0}{a^4}\,, \quad \rho_{r\nu} = \frac{\rho_{r\nu}^0}{a^4} = \rho_c^0\,\frac{\Omega_{r\nu}^0}{a^4}\,, \quad \rho_\Lambda = \rho_\Lambda^0\,. \tag{28}$$

The dependences on a for the mass density and pressure for each type of massless neutrino can be rewritten in the form

$$\rho_\nu(a) = a_\rho T_\nu^4 = a_\rho\left(\frac{a_\nu^*}{a}T_\nu^*\right)^4\,, \quad P_\nu(a) = \frac{c^2}{3}\rho_\nu(a). \tag{29}$$

3.4. Solution of the equations. Equations (28) can be used to rewrite the second of Eqs. (23) in the form

$$H = \frac{\dot{a}}{a} = \frac{H_0}{a^2}\sqrt{\Omega(a)}\,, \quad \Omega(a) = \Omega_{r\nu}^0 + \Omega_d^0\,a + \Omega_\Lambda^0\,a^4\,. \tag{30}$$

Separating the variables and integrating give a relationship of the scale factor to time:

$$H_0 t = \int_0^a \frac{a\,\mathrm{d}a}{\sqrt{\Omega(a)}}\,. \tag{31}$$

For $a = 1$ an age of the universe of $t_0 = 13.7\cdot 10^9$ years is obtained. As $a \to \infty$, the age of the universe $\tilde{t} \sim (\ln a)/H_\Lambda$ where $H_\Lambda = \sqrt{\Omega_\Lambda^0}\,H_0 = 59.397\,\mathrm{km/s/Mpc} = 1.9249\cdot 10^{-18}\,\mathrm{s}^{-1}$, which reflects the exponential increase in the scale factor at times $t > 1/H_\Lambda$, i.e., a secondary inflation.

Now we drop the idea of a massless neutrino and see what consequences this leads to. For simplicity we assume that all neutrinos are the same, i.e., they have equal masses and separated from the other matter at the same time.

4. Distributions of cosmological neutrinos with nonzero mass

4.1. The concentration of cosmological neutrinos. The mass density of cosmological neutrinos with mass (often called massives) depends on the scale factor a in a more complicated way than the density of massless neutrinos. To obtain this dependence we examine the distributions of three quantities, as for thermodynamic equilibrium. To distinguish them we mark these quantities with a tilde. We begin with the concentration.

As mentioned above, at some time t_ν^* during the expansion of the universe, neutrinos cease to interact with other particles and among themselves, separate from the remaining material, and propagate freely. At separation all types of neutrinos were in equilibrium, after which the equilibrium is destroyed.

Since the neutrinos propagate freely after separation, their concentration falls off only because of the increasing volume during the cosmological expansion:

$$\tilde{n}_\nu(a) = \left(\frac{a_\nu^*}{a}\right)^3 \tilde{n}_\nu\left(a_\nu^*\right) = \left(\frac{a_\nu^*}{a}\right)^3 \frac{8\pi}{h^3} \int_0^\infty \frac{p_*^2 \,\mathrm{d}\, p_*}{e^{\varepsilon(p_*)/\left(k_B T_\nu^*\right)} + 1} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 \,\mathrm{d}\, p}{e^{\varepsilon\left(ap/a_\nu^*\right)/\left(k_B T_\nu^*\right)} + 1} \,. \tag{32}$$

The preceding formulas imply that the mean occupation numbers for the neutrino states, expressed as a fraction $1/[e^{\varepsilon\left(ap/a_\nu^*\right)/\left(k_B T_\nu^*\right)}+1]$ remain unchanged and only the momenta of the neutrinos vary by the same law as photons: $p = \dfrac{a_\nu^*}{a} p_*$. This fact is derived in [24] by calculating the variation in the momentum with a local transformation from the system of reference comoving the particle to one infinitely close along the expansion. The product $\tilde{n}_\nu(a)V$ also does not vary during the cosmological expansion.

4.2. Mass density and pressure. As in Eq. (1) for the concentration, when transforming to the distributions of the cosmological neutrinos in the equations for the mass density and pressure (2)-(3), in the denominators in the exponent (and only there) the energy $\varepsilon(p)$ must be replaced by $\varepsilon\left(ap/a_\nu^*\right)$. After this substitution we replace the variable of integration $z = \dfrac{cp}{k_B T_\nu^*} \dfrac{a}{a_\nu^*}$:

$$\tilde{n}_\nu(a) = n_\nu(a) G_n\left(y_\nu^*\right), \quad n_\nu(a) = a_n\left(\frac{a_\nu^*}{a} T_\nu^*\right)^3, \tag{33}$$

$$\tilde{\rho}_\nu(a) = \frac{8\pi}{h^3 c^2} \int_0^\infty \frac{p^2\, \varepsilon(p)\, \mathrm{d}\, p}{e^{\varepsilon\left(ap/a_\nu^*\right)/\left(k_B T_\nu^*\right)} + 1} = \rho_\nu(a) G_\rho\left(y_\nu^*, \frac{a}{a_\nu^*}\right), \tag{34}$$

$$\tilde{P}_\nu(a) = \frac{8\pi c^2}{3h^3} \int_0^\infty \frac{p^4}{\varepsilon(p)} \frac{\mathrm{d}\, p}{e^{\varepsilon\left(ap/a_\nu^*\right)/\left(k_B T_\nu^*\right)} + 1} = P_\nu(a) G_P\left(y_\nu^*, \frac{a}{a_\nu^*}\right). \tag{35}$$

Here functions analogous to those introduced before are introduced, but they depend on two arguments:

$$G_\rho(y, A) = \frac{120}{7\pi^4} \int_0^\infty \frac{z^2 \sqrt{z^2 + (yA)^2}\, \mathrm{d}\, z}{e^{\sqrt{z^2+y^2}} + 1}\ , \ G_P(y, A) = \frac{120}{7\pi^4} \int_0^\infty \frac{z^4}{\sqrt{z^2 + (yA)^2}} \frac{\mathrm{d}\, z}{e^{\sqrt{z^2+y^2}} + 1}\ . \tag{36}$$

The neutrino mass appears in these functions only through the argument y and when $y = 0$ they go to unity, so that the distributions return to those valid for massless particles. The functions in Eqs. (6)-(7) are special cases of Eq. (36): $G_\rho(y) = G_\rho(y, 1)$, $G_P(y) = G_P(y, 1)$.

It is possible to introduce a dimensionless neutrino distribution function

$$f_\nu(z) = \frac{2}{3\zeta(3)} a_n \left(\frac{a_\nu^*}{a} T_\nu^*\right)^3 V \frac{z^2}{e^{\sqrt{z^2 + \left(y_\nu^*\right)^2}} + 1} = \frac{2}{3\zeta(3)} a_n \left(T_\nu^*\right)^3 V_* \frac{z^2}{e^{\sqrt{z^2 + \left(y_\nu^*\right)^2}} + 1}, \tag{37}$$

that is normalized with respect to z by the number of particles $\tilde{n}_\nu(a) V$ in the volume V. This function also does not vary during the expansion, like the Boltzmann H-function determined by it,

$$H = \int_0^\infty f_\nu(z) \ln f_\nu(z) \mathrm{d}\, z, \tag{38}$$

in accord with the adiabaticity of the cosmological expansion.

4.3. Properties of the distribution. The functions G_ρ and G_P obey the directly verifiable equation

$$a\frac{\partial G_\rho\left(y, a/a_\nu^*\right)}{\partial a} = A\frac{\partial G_\rho(y, A)}{\partial A} = G_\rho(y, A) - G_P(y, A), \tag{39}$$

which can be used to monitor calculations of these functions. With its help it is easy to verify that the adiabaticity condition for the expansion (27) is met with respect to the neutrino gas, i.e., that

$$\dot{\tilde{\rho}}_\nu(a) = -3\left(\tilde{\rho}_\nu(a) + \frac{\tilde{P}_\nu(a)}{c^2}\right)H. \tag{40}$$

Given that $\dot{\tilde{\rho}}_\nu(a) = \dot{a}\,\mathrm{d}\tilde{\rho}_\nu(a)/\mathrm{d}a$, while $H = \dot{a}/a$, the derivative $\dot{a}$ can be eliminated in this equation and it can be rewritten as

$$\frac{\mathrm{d}\tilde{\rho}_\nu(a)}{\mathrm{d}a} = -\frac{3}{a}\left(\tilde{\rho}_\nu(a) + \frac{\tilde{P}_\nu(a)}{c^2}\right), \tag{41}$$

which also easily verified.

In the sense of a definition, the parameter $A = a/a_\nu^* \geq 1$. The integral $G_\rho(y, A)$ increases with increasing A but $G_P(y, A)$ decreases. The values of y_ν^* are very small, since the neutrino mass is small, while their temperature as they separate is very high. On the other hand, the parameter A can be very large, since the scale factor is very small during the separation period, while $a \geq a_\nu^*$ varies to 1 within the current epoch and to infinity in the future. The product $y_\nu^* A = y_\nu^* a/a_\nu^*$ can be arbitrary.

For small y and large yA, the following expansions hold:

$$G_\rho(y, A) \sim \frac{yA}{a_{\rho n}}\left[1 + \frac{15}{2}\frac{\zeta(5)}{\zeta(3)}\frac{1}{y^2A^2} - \frac{945}{16}\frac{\zeta(7)}{\zeta(3)}\frac{1}{y^4A^4}\right], \tag{42}$$

$$G_P(y, A) \sim \frac{120}{7\pi^4}\frac{1}{yA}\left[\frac{45}{2}\zeta(5) - \frac{2835}{8}\frac{\zeta(7)}{y^2A^2} + \frac{240975}{16}\frac{\zeta(9)}{A^4y^4}\right]. \tag{43}$$

The first formula ensures accuracy better than 10^{-4} for $Ay > 15$, 10^{-5} for $Ay > 23$, 10^{-6} for $Ay > 34$, 10^{-7} for $Ay > 49$ and 10^{-8} for $Ay > 72$. The accuracy of the second formula is lower: its error is less than 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, and 10^{-7}, respectively, for $Ay > 13$, 19, 29, 42, 61, and 90.

Graphs of the functions $G_\rho\left(y_\nu^*, a/a_\nu^*\right)$ and $G_P\left(y_\nu^*, a/a_\nu^*\right)$ are shown in Fig. 3.

If the neutrino mass were zero, then they would be ultrarelativistic throughout the evolution of the universe. For a transition to this case, it is sufficient to set $y_\nu^* = 0$. Then the integrals in the correction functions become

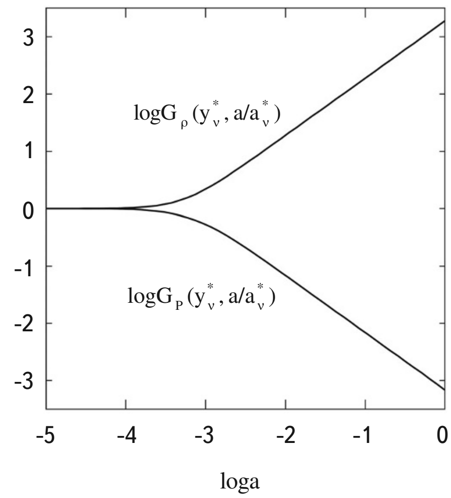

Fig. 2. The degree of relativity as a function of $\log G_\rho(y^*_\nu, a/a^*_\nu)$ and $\log G_P(y^*_\nu, a/a^*_\nu)$ (for three values of the particle mass in eV).

constants, while the functions, themselves, go to unity, so we return to the formulas of the preceding section.

A quantity analogous to Eq. (15) that also characterizes the degree of relativity of the distribution can be introduced, but now as a function of the scale factor a:

$$r(a) = y^*_\nu \frac{a}{a^*_\nu}\left[\frac{\tilde{\rho}_\nu(a)}{m_\nu \tilde{n}_\nu(a)} - 1\right] = y^*_\nu \frac{a}{a^*_\nu}\left[\frac{a_\rho}{m_\nu a_n}\frac{a^*_\nu}{a} T^*_\nu \frac{G_\rho\left(y^*_\nu, a/a^*_\nu\right)}{G_n\left(y^*_\nu\right)} - 1\right] =$$
$$= a_{\rho n}\frac{G_\rho\left(y^*_\nu, a/a^*_\nu\right)}{G_n\left(y^*_\nu\right)} - y^*_\nu \frac{a}{a^*_\nu}. \tag{44}$$

For $a = a^*_\nu$ this difference is very close to $a_{\rho n}$ and for $a \geq 1$ it has the asymptote

$$r(a) \sim \frac{15}{2}\frac{\zeta(5)}{\zeta(3)}\frac{a^*_\nu}{a}\frac{1}{y^*_\nu}\left[1 - \frac{63}{8}\frac{\zeta(7)}{\zeta(5)}\left(\frac{a^*_\nu}{a y^*_\nu}\right)^2\right]. \tag{45}$$

Figure 4 shows plots of $r(a)$ for four values of the neutrino mass in eV.

The difference between the distributions of the neutrinos and the distributions of the radiation is explained

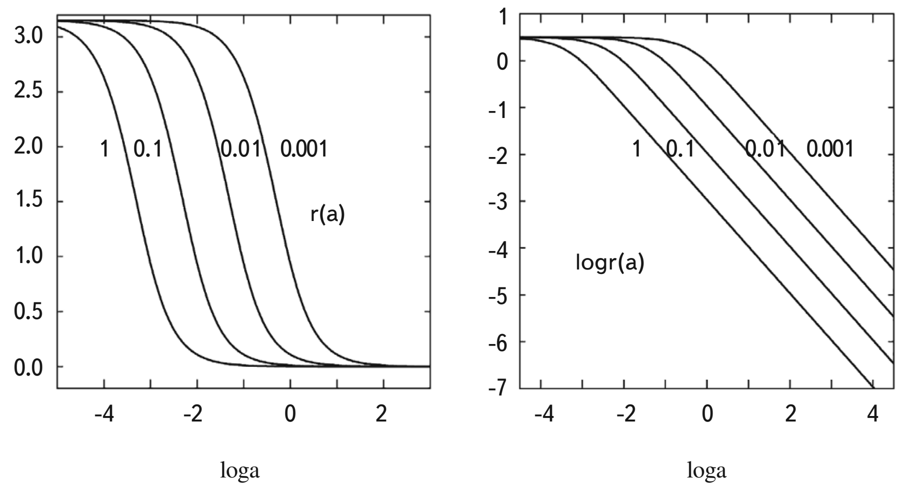

Fig. 4. The degree of relativity of cosmological neutrinos for four values of their masses.

formally by the fact that, because of their nonzero mass the energy of the neutrinos is not proportional to their momentum, as in the case of photons, which are described by the Planck law throughout the evolution of the universe, where frequency and temperature change in the same way with a.

4.4. Temperature of neutrinos with finite mass. Having specified the neutrino mass m_ν, a temperature value T_ν^*, and scale factor a_ν^* at the time they separate, we find the parameter y_ν^* for the distributions (33)-(35) and the integral $G_n\left(y_\nu^*\right)$, and then the dependences of the functions $G_\rho\left(y_\nu^*, a/a_\nu^*\right)$ and $G_P\left(y_\nu^*, a/a_\nu^*\right)$ on a. For quantitative estimates we assume that $k_B T_\nu^* = 10^6$ eV $= 1.6022\cdot 10^{-6}$ erg, so that $T_\nu^* = 1.1605\cdot 10^{10}$ K and $y_\nu^* = m_\nu \cdot 10^{-6}$, where the mass is also in eV. Equations (18)-(19) yield $a_\nu^* = \sqrt[3]{4/11}\, T_0 / T_\nu^* = 1.6777\cdot 10^{-10}$.

Since the neutrino distributions cease to be in equilibrium, the concept of temperature no longer applies to them. Various auxiliary temperatures can be introduced, as is done in astrophysics as applied to the radiation from stars. We use the concept of an "effective" temperature.

We define the effective temperature T_{eff} of the cosmological neutrino gas by equating the cosmological mass density (34) to its equilibrium value at that temperature:

$$T_\nu^4 G_\rho\left(y_\nu^*, \frac{a}{a_\nu^*}\right) = T_{eff}^4 G_\rho\left(y_{eff}, 1\right), \quad T_\nu = \frac{a_\nu^*}{a} T_\nu^*, \quad y_{eff} = \frac{m_\nu c^2}{k_B T_{eff}}. \tag{46}$$

If the neutrino mass is zero, i.e., $m_\nu = 0$, then $y_\nu^* = y_{eff} = 0$ and $T_{eff} = \left(a_\nu^*/a\right) T_\nu^* = T_\nu = \sqrt[3]{4/11}\, T$.

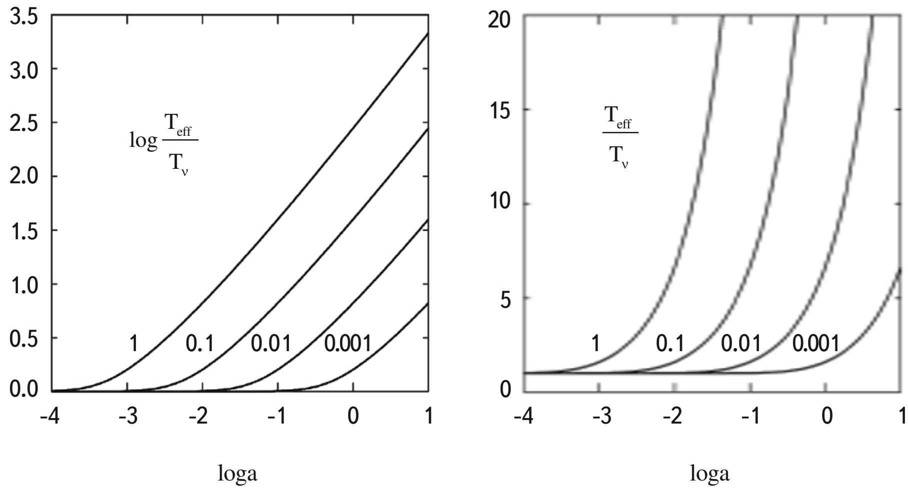

Fig. 5. The ratio of the temperatures T_{eff} and T_ν as a function of a for four values of the mass.

TABLE 1. Temperature of Massless Neutrinos and Effective Temperature for Some Values of a

m_ν			0.001		0.01		0.1		1.0	
log$a$	a	T_ν	T_{eff}	T_{eff}/T_ν	T_{eff}	T_{eff}/T_ν	T_{eff}	T_{eff}/T_ν	T_{eff}	T_{eff}/T_ν
-4.0	0.0001	19469	19469	1.0	19469	1.0	19472	1.0001	19717	1.0127
-3.0	0.001	1946.9	1946.9	1.0	194.72	1.0001	1971.7	1.0127	3117.3	1.6011
-2.0	0.01	194.69	194.72	1.0001	197.17	1.0127	311.73	1.6011	1292.9	6.6405
-1.0	0.1	19.469	19.717	1.0127	31.173	1.6011	129.29	6.6405	775.10	39.811
0.0	1.0	1.9469	3.1173	1.6011	12.929	6.6405	77.510	39.811	545.18	280.02
1.0	10	0.19469	1.2929	6.6405	7.7510	39.811	54.518	280.02	418.05	2147.2

When $m_\nu \neq 0$ Eq. (46) can be written in terms of the dimensionless reciprocal temperatures:

$$\frac{y_\nu}{\sqrt[4]{G_\rho\left(y_\nu^*, a/a_\nu^*\right)}} = \frac{y_{eff}}{\sqrt[4]{G_\rho\left(y_{eff}\right)}}. \tag{47}$$

For small values of the ratio a/a_ν^*, the functions G_ρ are close to unity so that y_{eff} and y_ν coincide, which also means that the temperatures T_{eff} and T_ν are the same. If, on the other hand, the ratio a/a_ν^* is large, the asymptotic of the function G_ρ (42) can be used. Substituting it in Eq. (47), we arrive at an asymptotic relationship between y_{eff} and a:

$$a_{\rho n}^{1/4} y_\nu^{3/4} = \frac{y_{eff}}{\sqrt[4]{G_\rho\left(y_{eff}\right)}} . \tag{48}$$

Thus, for large ratios a/a_ν^* the effective temperature T_{eff} does not depend separately on T_ν^* and the scale factor a, but only on their combination, specifically, the temperature of the massless neutrinos T_ν (and y_{eff}, correspondingly, on $y_\nu = \left(a_\nu^*/a\right) y_\nu^*$). Including the later terms in the expansion (2) does not change this conclusion.

This statement is illustrated in Fig. 5, where it can be seen that the curves proceed exactly the same way and can be replaced by parallel displacement. This is also indicated by the data of Table 1 where the effective temperatures for equal products am_ν are the same to five significant figures.

Naturally, for larger neutrino masses the difference between T_{eff} and T_ν is greater.

5. Effect of the neutrino mass on the evolution of the universe

5.1. Cosmological equation for a finite neutrino mass. The two basic equations of cosmology retain the form of Eq. (23) when the neutrino mass is taken into account (we do not change the notation for the scale factor; at this point we omit the dependences of the densities on it):

$$\ddot{a} = -\frac{4\pi G}{3} \tilde{\rho}_g \, a , \quad \dot{a}^2 = \frac{8\pi G}{3} \tilde{\rho}_t \, a^2 , \tag{49}$$

where now the total and gravitational mass densities of the components are

$$\tilde{\rho}_t = \tilde{\rho}_d + \rho_\Lambda + \rho_r + 6\tilde{\rho}_\nu , \quad \tilde{\rho}_g = \tilde{\rho}_t + 3\frac{\tilde{P}_t}{c^2} , \quad \tilde{P}_t = P_\Lambda + P_r + 6\tilde{P}_\nu . \tag{50}$$

The compatibility condition for Eqs. (27) is also retained:

$$\dot{\tilde{\rho}}_t = -3\left(\tilde{\rho}_t + \frac{\tilde{P}_t}{c^2}\right) H . \tag{51}$$

As before, it can be applied to each component separately. It is also satisfied for neutrinos, as shown in Eq. (40).

The equations of state of dusty matter, radiation, and dark energy are the same, as in Eq. (25), but now the neutrinos cannot be combined with the radiation. The condition (51) shows that the second and fourth formulas in Eq. (28), which describe the evolution of radiation and dark energy, are retained. Equation (34) must be taken as the neutrino mass density, thereby correcting the density of dusty matter.

The determination of the evolution of the densities of the components means that the first of Eqs. (49) has already been used and it remains to solve the second.

5.2. Solution of the equations. We express the fraction of the neutrino mass density in the critical density in terms of the fraction of massless neutrinos according to the representation of the density itself in Eq. (34):

$$\tilde{\Omega}_\nu^0 = 6\frac{\tilde{\rho}_\nu^0}{\rho_c^0} = 6\frac{\tilde{\rho}_\nu(1)}{\rho_c^0} = \frac{\rho_\nu^0}{\rho_c^0} G_\rho\left(y_\nu^*, a/a_\nu^*\right) = \Omega_\nu^0\, G_\rho\left(y_\nu^*, a/a_\nu^*\right). \tag{52}$$

The fraction of dusty matter also has to be changed:

$$\begin{aligned}\tilde{\Omega}_d^0 &= 1-\Omega_r^0-\Omega_\Lambda^0-\tilde{\Omega}_\nu^0 = 1-\Omega_r^0-\Omega_\Lambda^0-\Omega_\nu^0-\Omega_\nu^0\left[G_\rho\left(y_\nu^*, a/a_\nu^*\right)-1\right]= \\ &= \Omega_d^0-\Omega_\nu^0\left[G_\rho\left(y_\nu^*, 1/a_\nu^*\right)-1\right].\end{aligned} \tag{53}$$

Here the law for the change in the density of this component does not depend on the value of the neutrino mass; it falls off as $1/a^3$:

$$\tilde{\rho}_d(a) = \rho_c^0\frac{\tilde{\Omega}_d^0}{a^3}. \tag{54}$$

On substituting the laws of evolution for the components, the second of Eqs. (49) becomes

$$\frac{\dot{a}}{a} = \frac{H_0}{a^2}\sqrt{\Omega(a)+\Delta(a)}, \quad \Delta(a) = \Omega_\nu^0\left\{G_\rho\left(y_\nu^*, a/a_\nu^*\right)-1-\left[G_\rho\left(y_\nu^*, 1/a_\nu^*\right)-1\right]a\right\}. \tag{55}$$

The function $\Delta(a)$ is a correction because of the finiteness of the neutrino mass. It is clear that $\Delta(1)=0$ and the expression under the root sign goes to unity, because it is attached to the contemporary epoch.

After separation of variables and integration, we obtain a refined relationship between a and t:

$$\int_0^a \frac{a\,\mathrm{d}a}{\sqrt{\Omega(a)+\Delta(a)}} = H_0 t. \tag{56}$$

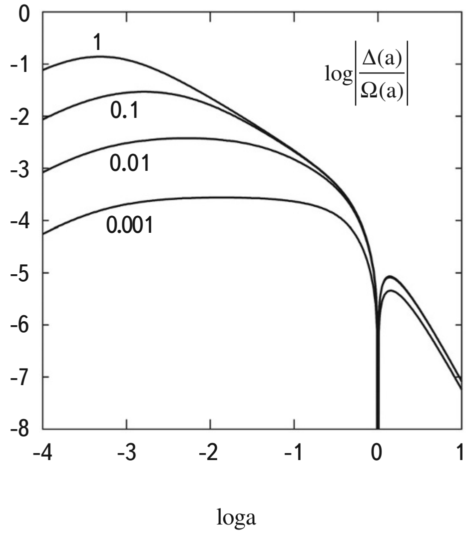

Fig. 6. The relative contribution of the correction for finite mass of the neutrino as a function of the scale factor.

We now estimate the relative contribution of the terms under the root that depend and do not depend on the neutrino mass. Figure 6 shows the logarithms of the moduli of the ratio $\Delta(a)/\Omega(a)$ as a function of log$a$ for four values of the mass m_ν indicated in eV near the curves. For $a<1$ the ratios are negative and take minima, while for $a>1$ they are positive, merge for $m_\nu > 0.001$ and have maxima which are smaller than the moduli of the negative minima. When $a=1$, the function $\Delta(a)$ and, with it, the ratio, go to zero (the logarithm to $- -\infty$). As $a \to \infty$ the ratio approaches zero. It can be seen from the figure that even for $m_\nu = 1\,\mathrm{eV}$ this minimum of the corresponding ratio does not exceed –0.14; it is reached for a small value of a = 0.00048, i.e., for a red shift greater than 2000. Thus, if they are not greater than 1 eV, nonzero neutrino masses have no effect on the expansion of the universe.

6. Conclusion

With the simplest assumptions about the existence of three types of neutrinos and antineutrinos as separate particles and equality of the chemical potential to zero, formulas for their distributions have been introduced to describe the state of the neutrino gases under thermodynamic equilibrium conditions. A parameter, referred to as the degree of relativity, has been introduced which can be used to evaluate the closeness of the distributions to ultrarelativistic and nonrelativistic.

The equations and formulas for the expansion of the universe in terms of the Standard ΛCDM model are reproduced here with radiation and massless neutrinos taken into account. Then it is shown how the cosmological model is generalized to include neutrinos with finite mass. The degree of relativity is generalized for cosmological neutrinos. It has been shown that the existence of neutrino mass leads to significant differences in the (effective) temperature of the cosmological neutrino gas from its value if they are assumed massless. At the same time, the effect of neutrino mass on the course of cosmological expansion is insignificant.

Indirect effects of neutrinos and their mass will presumably show up in other cosmological processes. Theoretical estimates of these kinds of effects have already been made in some of the papers cited in the *Introduction*.

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