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INTERNAL RADIATION FIELD IN THE NONLINEAR TRANSFER PROBLEM FOR A ONE-DIMENSIONAL ANISOTROPIC MEDIUM. I.

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Ambartsumyan's invariance principle is applied to the nonlinear radiative transfer problem of determining the internal radiation field in a one-dimensional, anisotropic, scattering and absorbing medium when both of its boundaries are illuminated by intense radiative fluxes. Formulas are derived for adding and imbedding layers in media with finite geometrical thicknesses. It is shown that to find the internal radiation field in the nonlinear case, as in the linear case, it is not necessary to solve any new equations: it is sufficient to use only the explicit expressions and quantities found by solving the particular problem of the radiation emerging from the medium, i.e., the diffuse reflection and transmission problem. Then a complete set of differential equations is found for invariant imbedding. The standard two-point nonlinear boundary value problem for radiative transfer reduces to an initial value problem— the Cauchy problem. A new Cauchy problem, in which the spatial variables appear only as fixed parameters, is formulated by eliminating derivatives with respect to the layer thickness. In this way we arrive at a semilinear system of the Ambartsumyan's complete invariance.

Keywords: *radiative transfer: nonlinear radiative transfer problem: internal radiation field: invariance principle*

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1. Introduction and purpose of this paper

In a recent paper [1], the author discussed in detail the nonlinear reflection and transmission problem (RTP) for radiation when both sides of a uniform anisotropic layer of finite geometrical thickness are irradiated. Formulas for adding layers were derived by successive application of Ambartsumyan's approach [2,3] and these were shown to be applicable for recurrent or step-by-step construction of various layered (in particular, uniform or periodic) media by increments in the thickness (e.g., by "doubling") or summing the thicknesses of specified initial layers. With these formulas, by limit transition, the following results were obtained: a complete set of differential equations for invariant imbedding, a quasilinear system of the so-called Ambartsumyan's complete invariance, and a satisfactory substantiation of this study together with a rather detailed bibliography. The author found subsequently that two intermediate equations in Ref. 1, Eqs. (57) and (60) had been found first by Bellman, et al. [4], using another method in a study of particle transport (see Eqs. (2.6) and (2.10) of that paper, respectively).

The purpose of the present study, which is in two parts, is to determine the radiation field inside a one-dimensional anisotropic medium whose boundaries are irradiated from outside. First, formulas for the addition of layers are derived for determining the radiation field inside the medium. The prospects for their practical application are discussed. It is shown that after the problem of finding the radiative intensity emerging from the medium is solved, the internal radiation field, as with the linear problem, can be found without solving new equations. Then a classical statement of this problem is given in the form of a "two-point" nonlinear boundary value radiative-transfer problem and the possibility of reducing it to initial value problems, i.e., Cauchy problems. Here the invariance principle is used in a differential form. The layer addition formulas in a limiting approach are used to obtain a complete set of equations for invariant imbedding that describes the intensity of the radiation field in the interior of the medium and, by means of it, the Ambartsumyan's complete invariance system. In it the spatial variables serve as fixed parameters and the differentiation procedure is carried out only with respect to the intensities of two-sided external illumination of the medium. Here, as in the linear case, the solution of the more specific RTP problem is used as additional information, i.e., it is assumed to have been found in advance.

We have given some particular results earlier [5,6] without providing proofs.

2. Layer addition in the nonlinear problem of determining the radiation field inside a medium

Consider two one-dimensional anisotropic media with geometric thicknesses A and B, respectively, in which multiple absorption and scattering of light take place. Here the layer of thickness B, as in Ref. 1, is attached to the layer of thickness A on the right. The resulting composite layer of thickness $A + B$ is illuminated on the left and right by powerful radiation beams with intensities x and y, respectively (Fig. 1). We seek the intensities $I^{\pm}_{A+B}(l; x, y)$ of the radiation directed to the right "+" and left "-" at a depth l in this layer of thickness $A + B$ under the condition that the solutions $I^{\pm}_{A}(l; x, y)$ and $I^{\pm}_{B}(l; x, y)$ of the corresponding problems for the two component sublayers with thicknesses A and B are specified in advance (the geometric thickness and depth increase from left to right).

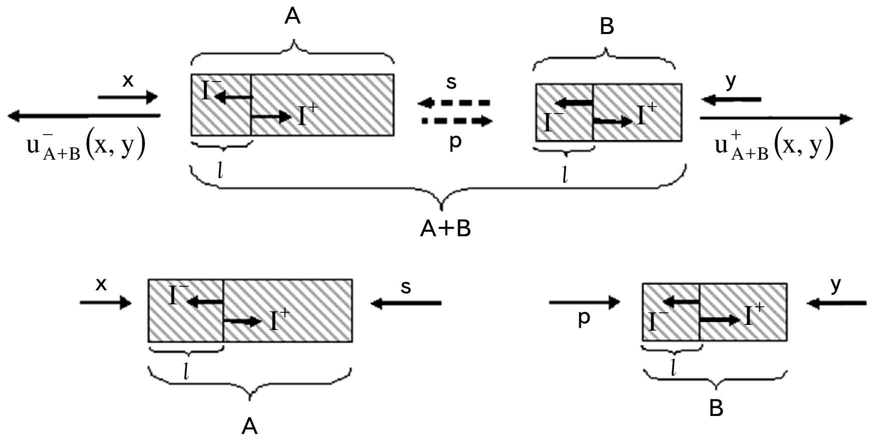

Fig. 1. Adding of layers in the nonlinear problem for determining the radiation field inside a medium.

To start with, we note an obvious from the physical point of view the fact that the field formed in this scattering and absorbing medium depends entirely on the radiation incident on it from outside and on the radiation produced by internal energy sources [1]. This statement ensures that the invariance principle is also applicable to the problem of determining the internal field, which is more general than the reflection and transmission problem of Ref. 1. In fact, the radiant intensity $I^{\pm}_{A+B}(l; x, y)$ at the given depth l of the original composite medium "A + B" will, for $l < A$, be identical to the field intensity $I^{\pm}_{A}(l; x, s)$ in the separate "A" layer if radiation of intensity s, formed in the initial layer of thickness A+B, is only incident on the outer boundary $l = A$. This can be written in the form

$$I^{\pm}_{A+B}(l; x, y) = I^{\pm}_{A}(l; x, s), \tag{1}$$

where

$$s \equiv I^{-}_{A+B}(A; x, y). \tag{2}$$

On the other hand, if we consider the field at depth $A+l$ of the original medium, then analogous arguments yield (here $l < B$)

$$I^{\pm}_{A+B}(A+l; x, y) = I^{\pm}_{B}(l; p, y), \tag{3}$$

where we use the notation

$$p \equiv I^{+}_{A+B}(A; x, y). \tag{4}$$

The quantities s and p in Eqs. (2) and (4) represent the intensities of the radiation directed, respectively, to the left and right inside the medium "$A + B$" at the contact boundary $l = A$. In terms of the invariance principle, Eq. (3) means that the intensity of the radiation field at depth $A+l$ of the original medium "$A + B$" formed by two-sided illumination (x, y) does not vary if the first laser of thickness A is removed from it while retaining the value p as left-sided illumination for the remaining second layer "B." Equation (1) has an analogous significance in the case where the right-hand layer "B" is removed from "$A + B$" without changing the magnitude of the resulting illumination s of part "A" at its right boundary. Assuming, in particular, that $l = A$ in Eq. (1) with Eq. (4) and that $l = 0$ in Eq. (3) with Eq. (2), we arrive at the well known particular equations first obtained by Ambartsumyan [1,2,5,6],

$$p = u^{+}_{A}(x, s), \tag{5}$$

and

$$s = u^{-}_{B}(p, y), \tag{6}$$

where we use the notation of Ref. 1 for the intensities emerging from the medium:

$$I^{+}_{A}(A; x, y) \equiv u^{+}_{A}(x, y), \quad I^{-}_{B}(0; x, y) \equiv u^{-}_{B}(x, y). \tag{7}$$

The addition relations (1) and (3), together with the system of functional equations (5) and (6), can be used to find the internal field in the composite "$A + B$" layer when the corresponding fields of the constituent sublayers "A" and "B" are known. Thus, the transition from the problem of finding the emerging intensities $u^{\pm}_{A+B}$ (from the RTP; see Eqs. (2) and (3) of Ref. 1) to the more general problem of determining the internal fields $I^{\pm}_{A+B}(l; x, y)$, which are specified by Eqs. (1) and (3) as in the linear case [7,8], does not require solving any new equations: it is sufficient to use the already known solution of the system of functional Eqs. (5) and (6) and the explicit expressions of Eqs. (1) and (3). By analogy with the linear case, beginning with concrete computational goals, here also the addition formulas can be used for constructing a set of exact procedures for effective determination of the radiation fields inside different media through step-by-step increments (e.g., doubling) of the thickness ahead of a given layer, for constructing different layered (in particular, periodic) media, and for adding different layers with specified arbitrary characteristics of their inner and outer fields.

Changing to the notation $A+B \equiv L$ and $A \equiv l$, while using the notation of Eqs. (2) and (4), we rewrite Eqs. (5) and (6) as the functional system

$$I^{+}_{L}(l; x, y) = u^{+}_{l}\left(x, I^{-}_{L}(l; x, y)\right), \tag{8}$$

and

$$I_L^-(l; x, y) = u_{L-l}^-\left(I_L^+(l; x, y), y\right). \tag{9}$$

The system of functional Eqs. (8) and (9) can be used for an arbitrary depth l in a medium of thickness L to determine the intensities of radiation entering both sides, if the solutions of the particular problems of emergent radiation for its two component parts with separate thicknesses l and $L - l$ are known.

In some calculations of the internal fields, as in the linear case [7,8], it may be necessary to use algorithms for the more general three-layer addition when one medium is not attached to but is imbedded in the second medium.

Let layer "B" be the middle part of a given composite medium "$A+B+C$". Then for the field inside this layer ($0 < l < B$) we can write

$$I_{A+B+C}^{\pm}(A+l; x, y) = I_B^{\pm}(l; p, \tilde{s}), \tag{10}$$

while the radiation emerging from the medium will be given by the equations

$$I_{A+B+C}^-(0; x, y) = u_A^-(x, s), \quad I_{A+B+C}^+(A+B+C; x, y) = u_C^+(\tilde{p}, y), \tag{11}$$

where the following notation is used:

$$\begin{aligned} p &\equiv I_{A+B+C}^+(A; x, y), \quad \tilde{s} \equiv I_{A+B+C}^-(A+B; x, y), \\ s &\equiv I_{A+B+C}^-(A; x, y), \quad \tilde{p} \equiv I_{A+B+C}^+(A+B; x, y). \end{aligned} \tag{12}$$

To find the unknowns in Eqs. (10) and (11), we have the system

$$p = u_A^+(x, s), \quad s = u_B^-(p, \tilde{s}), \quad \tilde{s} = u_C^-(\tilde{p}, y), \quad \tilde{p} = u_B^+(p, \tilde{s}). \tag{13}$$

It is evident that in the special case of two-layer addition, Eqs. (10)-(13) transform to Eqs. (1)-(7).

3. The boundary value problem for determining the internal radiation field

We now examine the nonlinear stationary radiative transfer problem for the simplest case of a one-dimensional anisotropic medium in its standard differential formulation in the form of a classical two-point boundary value problem. We seek the radiation field inside a medium illuminated on its two sides by powerful radiation sources. It is well known that the equation for the propagation of radiant energy in a scattering-absorbing medium, i.e., the radiative transfer equation, is a classical Boltzmann equation written for the particles of a photon gas and is a boundary value problem:

$$\frac{\partial I_L^{\pm}}{\partial l} = \pm\alpha\left(I_L^{+}, I_L^{-}\right), \tag{14}$$

with

$$I_L^{+}(0; x, y) = x\,, \quad I_L^{-}(L; x, y) = y\,, \tag{15}$$

where $I_L^{\pm} \equiv I_L^{\pm}(l; x, y)$, $0 \le x, y \le \infty$, $L \ge 0$, and $0 \le l \le L$.

Here $I_L^{\pm}$ is the unknown intensity at depth l of a one-dimensional medium of geometric thickness L and depends on the powers (x, y) of radiant fluxes incident on the left ($l = 0$) and right ($l = L$) boundaries of the medium, while $\alpha^{\pm}\left(I_L^{+}, I_L^{-}\right)$ is the collision integral in this transport problem, i.e., the reprocessing law for the radiation at each point of the medium by the matter in an elementary volume exposed to the radiation field in all directions. In particular, for the linear problem of an isotropic medium with isotropic scattering,

$$\alpha^{\pm}\left(I_L^{+}, I_L^{-}\right) = -k \cdot I_L^{\pm} + \frac{\lambda}{2} \cdot k \cdot \left(I_L^{+} + I_L^{-}\right), \tag{16}$$

where k is the absorption coefficient and λ is the probability of reradiation of a photon in an elementary interaction of the radiation with matter. In the linear transport problem, the characteristics of the medium such as k and λ are independent of the intensity of the radiation diffusing through it, so the solution of the boundary value problem for two-sided illumination of the medium from outside (14) becomes much simpler in this case. It is explicitly expressed in terms of a simple linear combination of the solutions $G_L^{\pm}$ and $\tilde{G}_L^{\pm}$ of two more particular problems in which the medium is irradiated separately at each of its boundaries,

$$I_L^{\pm}(l; x, y) = x \cdot G_L^{\pm}(l) + y \cdot \tilde{G}_L^{\pm}(l), \tag{17}$$

i.e., in terms of the surface Green functions

$$G_L^{\pm}(l) = I_L^{\pm}(l; 1, 0) \quad \text{and} \quad \tilde{G}_L^{\pm}(l) = I_L^{\pm}(l; 0, 1). \tag{18}$$

Here "l" denotes unit flux of radiation in the original linear problem incident on the corresponding external boundary of the medium. As can be seen from Eqs. (18), $G_L^{\pm}$ and $\tilde{G}_L^{\pm}$ are the mutually independent intensities of the internal radiation field at geometric depth l in a scattering and absorbing layer of thickness L formed by the action of these unit fluxes separately illuminating the layer on one or the other side. Thus, in the linear case, to find the radiation field inside the medium by solving the boundary value problem (14) with two-sided illumination (x, y), it is sufficient to examine separately the two simpler cases corresponding to the special conditions of one-sided illumination $(x, 0)$ and $(0, y)$ and to construct the combination (17) from these solutions. The situation is more complicated in the nonlinear case, for the direct relation (17) is unsuitable, so that the original boundary value problem with two-sided illumination of the medium must be solved in the more general formulation (14). Now the radiation changes the scattering and absorbing capabilities of the medium and the medium, in turn, changes the characteristics of the radiation diffusing in it. Thus, it becomes necessary to search for effective ways of solving Eq. (14). In the nonlinear case we shall use an implicit form of the collision integral $\alpha^{\pm}\left(I_L^+, I_L^-\right)$, assuming that its form has been specified in advance.

4. Application of the invariance principle

The introduction of the invariance principle by Ambartsumyan [9-14] was a great advance in the theory of linear transfer problems. The invariance principle approach was subsequently used successfully in astrophysics and atmospheric optics by Sobolev [16] and others, as well as in various areas of physics and mathematics [17]. The domain of applicability of this approach has expanded greatly owing to the development and extensive applications by Bellman and others [18] of so-called invariant imbedding in transport theory for particles and waves [19] and in applied mathematics problems [20,21]. Later on, nonlinear transport problems were studied. These have been analyzed using invariant imbedding [4]. By further development of the layer addition method and a limiting transition from functional equations to differential equations, Ambartsumyan [2,3] was able to reduce the nonlinear reflection and transmission problem to a Cauchy problem for quasilinear partial differential equations [4]. As in the linear case, here knowledge of the solution $u_L^{\pm}(x, y)$ of the reflection and transmission problem makes it possible to reduce the two-point boundary value problem (14) for finding the radiation field inside a medium to a simpler initial value problem with the conditions specified at only one of the boundaries of the medium.

For this, it is sufficient [4] to replace the boundary condition (15) by the initial condition

$$I_L^+\Big|_{l=0} = x\,, \quad I_L^-\Big|_{l=0} = u_L^-(x, y) \quad \text{or} \quad I_L^+\Big|_{l=L} = u_L^+(x, y), \quad I_L^-\Big|_{l=L} = y\,. \tag{19}$$

Successive application of the Ambartsumyan approach made it possible to build up a set of differential equations for invariant imbedding in the nonlinear reflection and transmission problem and to obtain a complete set of these equations [1,5,6]. This made it possible to reduce the solution of the reflection and transmission problem to a new quasilinear system of equations— Ambartsumyan's complete invariance system. Here all of the spatial variables serve as fixed parameters and the differentiation procedure is carried out only with respect to the parameters of the external radiation incident on both boundaries of the medium.

Using the invariance principle we now examine new (compared to Eq. (14)) prospects for determining the radiation field inside a medium. In Eqs. (3) and (1), for the geometric thicknesses of the layers we take the values $A \equiv \Delta$ and $B \equiv \Delta$, respectively, let $\Delta \to 0$, and make the necessary expansions on the left sides of these equations:

$$I^{\pm}_{\Delta+L}(\Delta+l;x,y)=I^{\pm}_{L}(l;x,y)+\left(\frac{\partial}{\partial L}+\frac{\partial}{\partial l}\right)I^{\pm}_{L}(l;x,y)\cdot\Delta+O(\Delta^2), \tag{20}$$

and

$$I^{\pm}_{L+\Delta}(l;x,y)=I^{\pm}_{L}(l;x,y)+\frac{\partial}{\partial L}I^{\pm}_{L}(l;x,y)\cdot\Delta+O(\Delta^2). \tag{21}$$

To simplify the right hand sides of Eqs. (3) and (1), we use Eqs. (13, 14) and (20, 21) from Ref. 1:

$$p=x+\alpha^{+}(x,u^{-}_{L})\cdot\Delta+O(\Delta^2),\quad s=u^{-}_{L}+\alpha^{+}(x,u^{-}_{L})\cdot\frac{\partial u^{-}_{L}}{\partial x}\cdot\Delta+O(\Delta^2), \tag{22}$$

and

$$\tilde{s}=y+\alpha^{-}(u^{+}_{L},y)\cdot\Delta+O(\Delta^2),\quad \tilde{p}=u^{+}_{L}+\alpha^{-}(u^{+}_{L},y)\cdot\frac{\partial u^{+}_{L}}{\partial y}\cdot\Delta+O(\Delta^2), \tag{23}$$

where

$$u^{+}_{L}\equiv u^{+}_{L}(x,y)=I^{+}_{L}(L;x,y),\quad u^{-}_{L}\equiv u^{-}_{L}(x,y)=I^{-}_{L}(0;x,y). \tag{24}$$

After substituting Eqs. (22) and (23) in the right hand sides of Eqs. (3) and (1), using Eqs. (20) and (21), and making some simple calculations, we obtain

$$\left(\frac{\partial}{\partial L}+\frac{\partial}{\partial l}\right)I^{\pm}_{L}=\alpha^{+}(x,u^{-}_{L})\cdot\frac{\partial I^{\pm}_{L}}{\partial x}, \tag{25}$$

and

$$\frac{\partial I^{\pm}_{L}}{\partial L}=\alpha^{-}(u^{+}_{L},y)\cdot\frac{\partial I^{\pm}_{L}}{\partial y}. \tag{26}$$

Equations (25) and (26) are new; they differ favorably from the original semilinear systems (14,15) and (14,19) (the equations are classified in accordance with Ref. 22). First, the right hand sides of Eqs. (14), $\alpha^{\pm}(I^{+}_{L},I^{-}_{L})$, depend simultaneously on both of the unknowns I^{+}_{L} and I^{-}_{L} (the coupling properties of the equations for $I^{\pm}_{L}$). Second, here the functional dependence on the intensities I^{+}_{L} and I^{-}_{L} is nonlinear (the semilinearity property of the equations). Equations (25) and (26) also differ favorably from Eq. (14) in that, with respect to I^{+}_{L}, they are both separable and linear. Recall that the Cauchy problem (14,19) was formulated by replacing the two-point boundary condition in Eqs. (14,15) for I_L^- by a new, initial condition which is now specified on the opposite boundary l=0, so that additional information was required on the solution of an auxiliary reflection and transmission problem. As opposed to this, here (Eqs. (25) and (26)), the Cauchy problem arises in a natural way as a result of the separability properties of I_L^+ and I_L^-, since the characteristics of the reflection and transmission problem show up already in the equations themselves. Eliminating the derivative with respect to depth in Eq. (25) using Eq. (14), we obtain a new semilinear and coupled invariant imbedding equation,

$$\frac{\partial I_L^{\pm}}{\partial L} = \alpha^+\left(x, u_L^-\right)\cdot\frac{\partial I_L^{\pm}}{\partial x} \mp \alpha^{\pm}\left(I_L^+, I_L^-\right). \tag{27}$$

In the problem for determining the internal field, the separate linear equations (26), together with the semilinear coupled system (27), form (in the sense of Bellman's terminology) a complete set of invariant imbedding equations. With the aid of initial conditions, e.g.,

$$I_L^{\pm}\Big|_{y=0} \equiv I_L^{\pm}(l; x, 0) = X_L^{\pm}(l; x) \quad \text{и} \quad I_L^{\pm}\Big|_{x=0} \equiv I_L^{\pm}(l; 0, y) = Y_L^{\pm}(l; y), \tag{28}$$

the Cauchy problems are formed for Eqs. (26) and (27), respectively; alternatively, for both of these equations we can adopt

$$I_L^{+}\Big|_{L=l} = u_l^+(x, y), \quad \text{и} \quad I_L^{-}\Big|_{L=l} = y\,. \tag{29}$$

It is noteworthy that combining Eqs. (26) and (27) opens up the possibility of determining the internal radiation field without differentiation with respect to some spatial variable, thereby making it possible to obtain a new semilinear system of the Ambartsumyan's complete invariance,

$$\left[\alpha^+\left(x, u_L^-\right)\cdot\frac{\partial}{\partial x} - \alpha^-\left(u_L^+, y\right)\cdot\frac{\partial}{\partial y}\right] I_L^{\pm} = \pm\alpha^{\pm}\left(I_L^+, I_L^-\right), \tag{30}$$

with

$$I_L^{\pm}\Big|_{x=0} = Y_L^{\pm}(l; y) \quad \text{or} \quad I_L^{\pm}\Big|_{y=0} = X_L^{\pm}(l; x), \tag{31}$$

where, as opposed to Eqs. (26) and (27), the spatial variables appear only as fixed parameters. The problem of Eqs. (30, 31) generalizes the analogous result that we have obtained for the reflection and transmission problem (Eqs. (33, 34) below). Here $X_L^{\pm}$ and $Y_L^{\pm}$ describe the one-sided illumination of the medium and are considered to be known here (see the second article in this study about finding these auxiliary, particular quantities) and, as in the reflection and transmission problem [1,5,6], the differential operator of the Ambartsumyan's complete invariance [1],

$$\hat{A} \equiv \alpha^{+}\left(x, u_L^{-}\right) \cdot \frac{\partial}{\partial x} - \alpha^{-}\left(u_L^{+}, y\right) \cdot \frac{\partial}{\partial y}, \tag{32}$$

describes the change in (reaction of) the established radiation field in the original medium, when the intensities illuminating both of its boundaries from outside are subjected to an infinitesimal variation by the operation of adding an elementary layer of thickness $\Delta \to 0$ on the left and, simultaneously, subtracting another on the right. It is remarkable that the appearance of this operator is a consequence of the nonlinearity of the problem, while it is constructed [1,5,6] by solving the reflection and transmission problem:

$$\begin{cases} \left[\alpha^{+}\left(x, u_L^{-}\right) \cdot \dfrac{\partial}{\partial x} - \alpha^{-}\left(u_L^{+}, y\right) \cdot \dfrac{\partial}{\partial y}\right] u_L^{+} = +\alpha^{+}\left(u_L^{+}, y\right) \\ \left[\alpha^{+}\left(x, u_L^{-}\right) \cdot \dfrac{\partial}{\partial x} - \alpha^{-}\left(u_L^{+}, y\right) \cdot \dfrac{\partial}{\partial y}\right] u_L^{-} = -\alpha^{-}\left(x, u_L^{-}\right), \end{cases} \tag{33}$$

with the initial conditions

$$\begin{aligned} &\text{a)} \quad u_L^{+}\Big|_{y=0} \equiv T_L^{+}(x), \quad u_L^{-}\Big|_{y=0} \equiv R_L^{-}(x) \quad \text{or} \\ &\text{b)} \quad u_L^{+}\Big|_{x=0} \equiv R_L^{+}(y), \quad u_L^{-}\Big|_{x=0} \equiv T_L^{-}(y). \end{aligned} \tag{34}$$

Recall that, in the linear case, instead of the Cauchy problem (33,34), there are the explicit expressions

$$u_L^{+} = x \cdot \tilde{T}_L^{+} + y \cdot \tilde{R}_L^{+}, \quad u_L^{-} = x \cdot \tilde{R}_L^{-} + y \cdot \tilde{T}_L^{-},$$

with

$$\begin{aligned} \tilde{T}_L^{+} &\equiv G_L^{+}(L) = I_L^{+}(L;1,0), \quad \tilde{R}_L^{+} \equiv \tilde{G}_L^{+}(L) = I_L^{+}(L;0,1), \\ \tilde{R}_L^{-} &\equiv G_L^{-}(0) = I_L^{-}(0;1,0), \quad \tilde{T}_L^{-} \equiv \tilde{G}_L^{-}(0) = I_L^{-}(0;0,1). \end{aligned}$$

Here $\tilde{R}_L^{\pm}$ and $\tilde{T}_L^{\pm}$ are the linear reflection and transmission coefficients of the medium with illumination of one of its boundaries by a flux of unit power. Thus, to start with, an independent method is used to find the radiation emerging from the medium, $u_L^{\pm}$, i.e., solve the reflection and transmission problem [1,5,6] by means of the quasilinear system (33) (or any other method [1]), and then this solution is used to determine the desired internal field $I_L^{\pm}$ with the system (30). It should be pointed out that, on going from the special problem of the emerging radiation (33) to the more general problem (30) of determining the field inside the medium, the differential operaror (32), of Ambartsumyan's complete invariance, does the following: (i) it preserves its form (i.e., does not become more complicated); (ii) as in the reflection and transmission problem, it operates only through energy variables while leaving the spatial variables fixed; and, finally, (iii) the unknown functions no longer appear in the differentiation operators, i.e., here the problem with initial conditions is formulated only for the semilinear system (30), as opposed to the quasilinear system (33).

5. Conclusion

We conclude by briefly listing the results of this article:

1. Layer addition and imbedding formulas (Eqs. (1), (3), and (10), (12)) have been obtained in the nonlinear radiative energy transfer problem for determining the internal radiation field during illumination of a one-dimensional anisotropic layer of finite geometrical thickness at both of its boundaries. For this, it has been shown that:

— In the layer addition or imbedding method, on going from the reflection and transmission problem, i.e., determining the radiation emerging from a composite medium, to the problem of finding the radiation field inside the medium there is no need to solve any new equations; explicit expressions and solutions of the systems of Eqs. (5-6) or (13), which have already been found in determining the radiation emerging from the medium, are used.

— The invariance equations (8-9) which relate the internal fields moving to the left and right at each depth in the composite medium can be used for directly determining the intensity of the field at any specified depth by solving the system of functional equations (8-9). There the depth is a fixed parameter and the solution of the reflection and transmission problem for the two corresponding sublayers of this medium are regarded as known.

2. By systematic application of the invariance principle, the original nonlinear boundary problem of determining the radiation field inside a one-dimensional anisotropic medium illuminated by intense radiation fluxes on both of its external boundaries can be reduced to a step-by-step, sequential solution of problems that just have initial conditions; here the earlier solutions are used to find solutions for the subsequent problems. In particular:

— A "complete set" of differential equations, the linear Eq. (26) and the semilinear Eq. (27), for invariant imbedding has been obtained for finding the internal radiation field during two-sided illumination of a medium.

— The boundary value problem for determining the radiation field inside a medium with two-sided illumination reduces to other special problems for this medium: the emerging radiation problem (i.e., the reflection and transmission problem) and one-sided illumination problems.

— A new system of functional equations (30) of the Ambartsumyan's complete invariance has been obtained; it is a Cauchy problem where the values of the spatial variables are fixed parameters and differentiation is done only with respect to energy parameters related to the external radiation that excites the medium.

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