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NONSTATIONARY RADIATIVE TRANSFER: EVOLUTION OF A SPECTRUM BY MULTIPLE COMPTON SCATTERING

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A new method is used for numerical solution of the nonlinear integro-differential radiative transfer equation for the evolution of a homogeneous emission spectrum owing to Compton scattering on equilibrium free electrons in an infinite uniform space. The temperature of the electron gas is assumed constant with no limits placed on it: the electrons can be both nonrelativistic and relativistic. The evolution of the spectrum is found to depend substantially on the initial dimensionless photon density. There is a bounday value for this density such that at lower values, there is a limiting equilibrium Bose-Einstein photon distribution, but not at higher values. In the latter case a quasi-line develops which shifts to shorter wavelengths with time while its width decreases and its maximum intensity increases. Calculations are carried out using two frequency redistribution functions of the photons, exact and simplified (assuming an isotropic distribution in the laboratory coordinate system). The results are compared with solutions of the Kompaneets equation.
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1. Introduction

Multiple Compton scattering plays an important role in the physics of various astrophysical objects. But it has been a subject of physical research in itself [1], especially in the theory of radiative transfer. In astrophysical applications, an approximate description of radiative transfer with Compton scattering is used that is, nevertheless, adequate for explaining the spectra of specific objects. As for the exact theory, because of the nonlinearity of the basic equation, further progress is possible only by solving relatively simple model problems by numerical methods. One such problem is calculating the evolution of the spectrum of isotropic homogeneous emission in an infinite space owing to multiple Compton scattering on electrons. It is assumed that an equilibrium electron gas fills the space and that its temperature is independent of position and time. Zeldovich and Syunyaev [2] showed that in this case, quasilines can appear in the spectrum of the scattered light which shift to longer wavelengths with time. In this paper we apply a new numerical method to solve the basic equation for this problem. Both the exact and approximate (from the papers of Nagirner, et al. [3]) frequency redistribution functions are used.

2. Basic equations and methods for solving them numerically

The basic integro-differential radiative transfer equation has the form

$$\frac{\partial f(x,t)}{\partial t} = -\int_0^\infty \left[r(x,x_1) f(x,t)\left(x_1^2 + f(x_1,t)\right) - r(x_1,x) f(x_1,t)\left(x^2 + f(x,t)\right)\right] dx_1 , \qquad (1)$$

where $f(x,t) \equiv x^2 n(x,t)$ is the dimensionless photon density and $n(x,t)$ is the average occupation number of the photon states with energy x at time t, with x and t measured in units of mc^2 and $1/c\,\sigma_0\,n_e$, respectively; σ_0 is the Thomson cross section; and n_e is the electron density in the accompanying system. The frequency redistribution function is $r(x,x_1) \equiv \overline{R}(x,x_1)/(xx_1)$ with $r(x_1,x) = e^{y(x_1-x)} r(x,x_1)$, where $y = mc^2/kT$. Simplified (assuming isotropic scattering in the laboratory coordinate system) and exact expressions (see Nagirner, et al. [3,4]) exist for the averaged redistribution function $\overline{R}(x,x_1)$.

For calculating the integrals on the right hand side of the basic equation (1), the unknown function was represented by a quadratic spline

$$f(x) = f_j + b_j(x - x_j) + c_j(x - x_j)^2, \quad x_j \le x \le x_{j+1}, \; j = 0,1,...,N, \; c_j = \frac{b_{j+1} - b_j}{2(x_{j+1} - x_j)}, \qquad (2)$$

with the coefficients b_j determined by the recurrence relation

$$b_j = -b_{j+1} + 2\frac{f_{j+1} - f_j}{x_{j+1} - x_j}. \tag{3}$$

We assume that $b_N = 0$ at the last point. Ultimately, we have

$$\int_0^\infty r(x_k, x) f(x) dx = \sum_{m=0}^{N} q_{km} f_m, \tag{4}$$

where

$$q_{k0} = \Delta x_0 (A_{k0} - 2B_{k0} + C_{k0}), \tag{5}$$

$$\begin{aligned} q_{km} &= \Delta x_m (A_{km} - 2B_{km} + C_{km}) + \Delta x_{m-1} C_{km-1} - \\ &- 2(-1)^m \frac{\Delta x_m + \Delta x_{m-1}}{\Delta x_m \Delta x_{m-1}} \sum_{j=0}^{m-1} (-1)^j (\Delta x_j)^2 (B_{kj} - C_{kj}), \quad 1 \le m \le N-1, \end{aligned} \tag{6}$$

$$q_{kN} = \Delta x_{n-1} C_{kN-1} - 2\frac{(-1)^N}{\Delta x_{N-1}} \sum_{j=0}^{N-1} (\Delta x_j)^2 (B_{kj} - C_{kj}). \tag{7}$$

Here $\Delta x_j = x_{j+1} - x_j$ for $j = 0, 1, ..., N-1$ and the coefficients

$$\{A_{kj}, B_{kj}, C_{kj}\} \equiv \int_0^1 r(x_k, x_j + t\Delta x_j)\{1, t, t^2\} dt. \tag{8}$$

The integrals that show up here were calculated by gaussian quadrature with 12 points.

Similarly, we find for the integral

$$\int_0^\infty r(x, x_k) f(x) dx = \sum_{m=0}^{N} \tilde{q}_{km} f_m, \tag{9}$$

where the coefficients $\tilde{q}_{km}$ are calculated using the same formulas as for the q_{km}, but with replacement of $r(x_k, x_j + t\Delta x_j)$ by $r(x_j + t\Delta x_j, x_k)$ for calculating the coefficients according to Eq. (8).

Finally, the main equation (1) can be rewritten in the following discrete form:

$$\frac{\partial f_k(t)}{\partial t} = -f_k(t)\alpha_k + x_k^2 \sum_{m=0}^{N} \tilde{q}_{km} f_m(t) + f_k(t) \sum_{m=0}^{N} (\tilde{q}_{km} - q_{km}) f_m(t), \quad k = 0, 1, ..., N, \tag{10}$$

where $f_k(t) = f(x_k, t)$ and

$$\alpha_k = \int_0^\infty r(x_k, x_1) x_1^2 \, dx_1, \quad k = 0, 1, ..., N, \tag{11}$$

We have solved the system of Eqs. (10) by two different methods. Let $\{t_i\}$, $i = 0, 1, ..., I$ be a time grid with $t_0 = 0$. Let us integrate Eq. (10) with respect to time from time t_i to t_{i+1}, taking t_{i+1} to be the instantaneous time. We have

$$\int_{t_i}^{t_{i+1}} f_k(t) dt = f_k(t_{i+1/2}) + O\left[(\Delta t_i)^3\right], \tag{12}$$

$$\int_{t_i}^{t_{i+1}} f_k(t) f_m(t) dt = f_k(t_{i+1/2}) f_m(t_{i+1/2}) + O\left[(\Delta t_i)^3\right], \tag{13}$$

where $t_{i+1/2} = (t_i + t_{i+1})/2$ and $\Delta t_i = t_{i+1} - t_i$. For the value of the function at a half integer point, we obviously have

$$f_k(t_{i+1/2}) = \frac{1}{2}(\hat{f}_k + f_k) + O\left[(\Delta t_i)^2\right], \tag{14}$$

where we use the notation $\hat{f}_k = f_k(t_{i+1})$ and $f_k = f_k(t_i)$. Finally, for determining the instantaneous vector $\hat{f}_k$, $k = 0, 1, ..., N$ we obtain the following nonlinear system of equations:

$$\begin{aligned} &\hat{f}_k \left[1 + \frac{\Delta t_i}{2} - \frac{\Delta t_i}{4} \sum\nolimits_{m=0}^{N} (\tilde{q}_{km} - q_{km}) f_m \right] - \\ &- \frac{\Delta t_i}{2} \sum\nolimits_{m=0}^{N} \hat{f}_m \left[x_k^2 \tilde{q}_{km} + \frac{1}{2} (\tilde{q}_{km} - q_{km}) (\hat{f}_k + f_k) \right] = \\ &= f_k \left[1 - \frac{\Delta t_i}{2} \alpha_k + \frac{\Delta t_i}{4} \sum\nolimits_{m=0}^{N} (\tilde{q}_{km} - q_{km}) f_m \right] + \frac{\Delta t_i}{2} x_k^2 \sum\nolimits_{m=0}^{N} \tilde{q}_{km} f_m, \quad k = 0, 1, ..., N. \end{aligned} \tag{15}$$

This system was solved by Newton's iteration method: the instantaneous value of the vector was written as a sum of the previous value and a correction ($\hat{f}_k = \hat{f}_k^{(s)} + \delta \hat{f}_k^{(s)}$), and the system was linearized with respect to the corrections. The resulting linear system for the corrections was solved by the Gauss elimination method. An initial approximation $\hat{f}_m^{(0)}$ was obtained by replacing $(\hat{f}_k + f_k)/2$ with f_k in the last term on the left hand side of the system of Eqs. (15).

The iterations were stopped when the condition $\max_k \left|\delta \hat{f}_k^{(s)}\right| < \varepsilon$ was satisfied (usually it was assumed that $\varepsilon = 10^{-5}$).

The second method, which we have proposed previously [5] for solving other nonstationary problems, involves using the Taylor expansion

$$f(x, t_{i+1}) = f(x, t_i) + \sum_{n=1}^{\infty} \frac{(\Delta t_i)^n}{n!} a_n(x), \tag{16}$$

which converges on a sufficiently dense time grid. Substituting this expansion in the basic equation (10) yields the following recurrence relation (with respect to *n*) for determining the expansion coefficients $\alpha_{nk} \equiv \alpha_n(x_k)$:

$$a_{n+1k} = -\alpha_k a_{nk} + \sum_{m=0}^{N} \left[x_k^2 \tilde{q}_{km} a_{nm} + (\tilde{q}_{km} - q_{km}) \sum_{l=0}^{n} C_n^l a_{lk} a_{n-lm} \right], \quad k = 0, 1, ..., N, \tag{17}$$

where C_n^l are the binomial coefficients, with the initial value obviously determined from the distribution at the previous time: $a_{0k} = f(x_k, t_i)$, where $k = 0, 1, ..., N$. In this (noniterative) method there is no need to solve any additional equations.

An adaptive frequency-time grid was used. The initial distributions were chosen to be in the form of a very narrow line,

$$f(x, 0) = \frac{C}{\varepsilon\sqrt{\pi}} e^{-(xy - x_1 y)^2/\varepsilon^2}, \tag{18}$$

where $\varepsilon = 10^{-2}$ and $x_1 y = 1$, and in the form of a exponential-power function

$$f(x, 0) = \frac{Cs}{\Gamma(p+1)} (sxy)^p e^{-2sxy} \tag{19}$$

where $p = 3$ and $s = 2$. Both a simplified and the exact frequency redistribution function were used. The results were compared with previously obtained (Nagirner, et al. [6]) solutions of the Kompaneets equation.

It should be noted that, besides the new noniterative method described above, an ordinary iterative method from the theory of heat conduction was also used, but it did not always yield results within an acceptable time; sometimes the iterations converged too slowly. In the cases where the iterations converged sufficiently rapidly, both methods yielded the same results. The computational time using the new method was always considerably shorter.

3. Computational results

Figures 1-3 show the results of the calculations using the simplified redistribution function. Figure 3 also shows some solutions of the Kompaneets equation for comparison. Solutions for the simplified and exact redistribution functions are compared in Fig. 4.

It should be noted that the equilibrium solutions of Eq. (1) have the form of a Bose-Einstein distribution $f_{eq} = y^3 x^2 / \left(e^{\mu + yx} - 1\right)$, so that the integral of the equilibrium solution over all x cannot exceed $C_{max} = 2\zeta(3)$, where $\zeta(z)$ is the Riemann function. Thus, for an initial distribution with $C > C_{max}$ no limiting (as $t \to \infty$) equilibrium solution exists.

The solution depends strongly on the value of the integral *C*. So the initial distribution for *C* = 1 (see Fig. 1) (its "residues" can be seen in the form of a very narrow line at xy=1) evolves to an equilibrium state (indicated by the asterisks in Fig. 2). For *C* = 50, when there is no limiting equilibrium distribution, a narrow frequency distribution (quasiline) develops and moves with time toward the ordinate axis while its width decreases. Note that here the integral of the frequency distribution is conserved with an accuracy no worse than 10^{-4}. The formation of a quasiline was predicted earlier on the basis of qualitative arguments by Zeldovich and Syunyaev [2].

In Fig. 3 solutions of the Kompaneets equation (smooth curves) are compared with solutions of the exact integral-differential equation (asterisks), from which the Kompaneets equation can be derived in the diffusion approximation. Before time *t* = 0.02 these solutions are in good agreement with one another at all frequencies, while at *t* = 0.025 large differences show up on the left edge of the distribution. The exact equation yields a quasiline, while the Kompaneets equation does not. In fact, for the large frequency gradients that appear at times $t \geq 0.025$, the Kompaneets equations is not applicable, since the condition under which it was derived from the exact equation is violated.

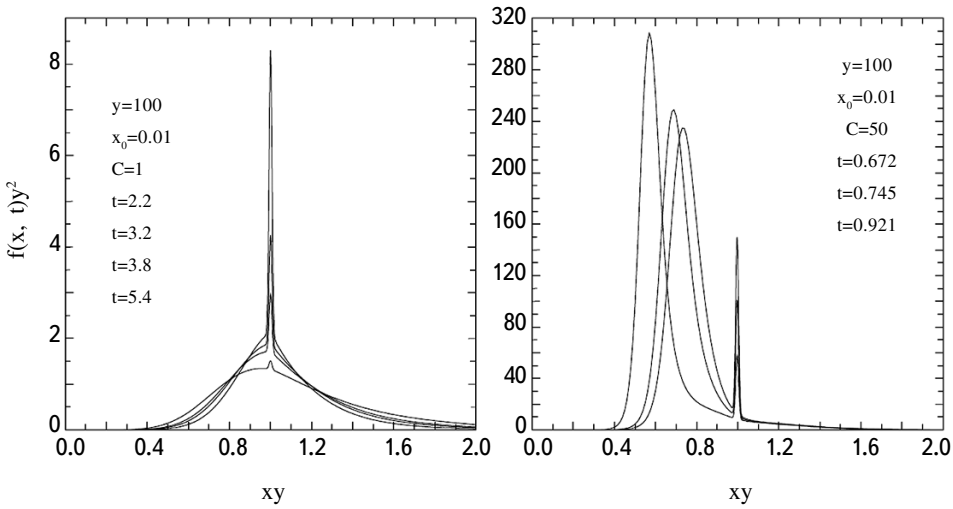

Fig. 1. Evolution of an initial distribution of the form (18) for $y = 100$ and $x_0 = 0.01$. (The height of the initial peak at $xy = 1$ decreases with time.)

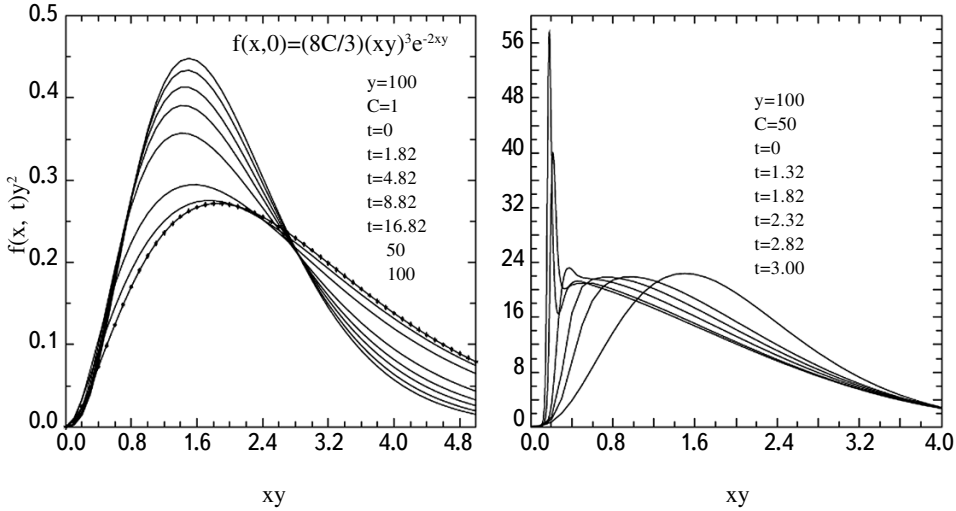

Fig. 2. Evolution of an initial distribution of the form (19) (exponential-power distribution) for $y = 100$, $p = 3$, and $s = 3$. The asterisks denote the limiting equilibrium solution.

It should be noted that the solutions of the Kompaneets equation shown in Fig. 3 were obtained previously by Nagirner, et al. [6], who showed that in those cases where a limiting equilibrium solution does not exist, a very steep leading edge develops in the photon frequency distribution and moves with time toward the ordinate axis. The formation of these fronts (and even their "breaking," i.e., the appearance of nonuniqueness) was predicted by

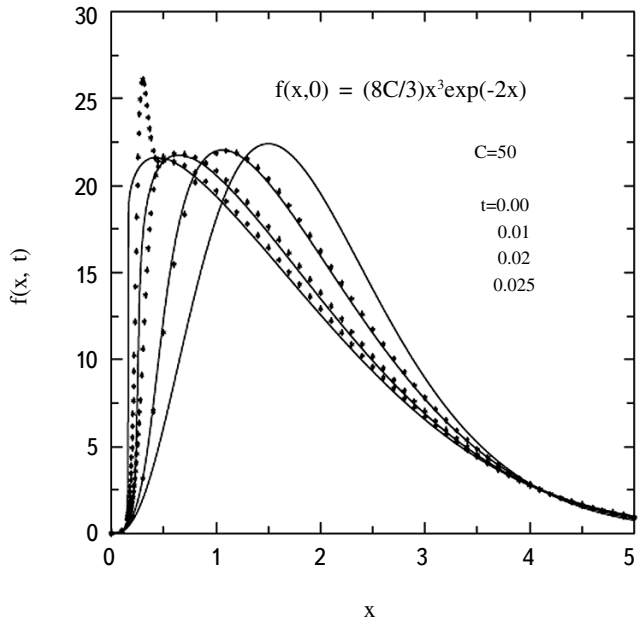

Fig. 3. A comparison of the solutions to the nonlinear Kompaneets equation (the curves) and the nonlinear integral equations (asterisks) for an initial distribution of the form (19) (exponential-power distribution) with $y = 100$, $p = 3$, $s = 3$, and $C = 50$.

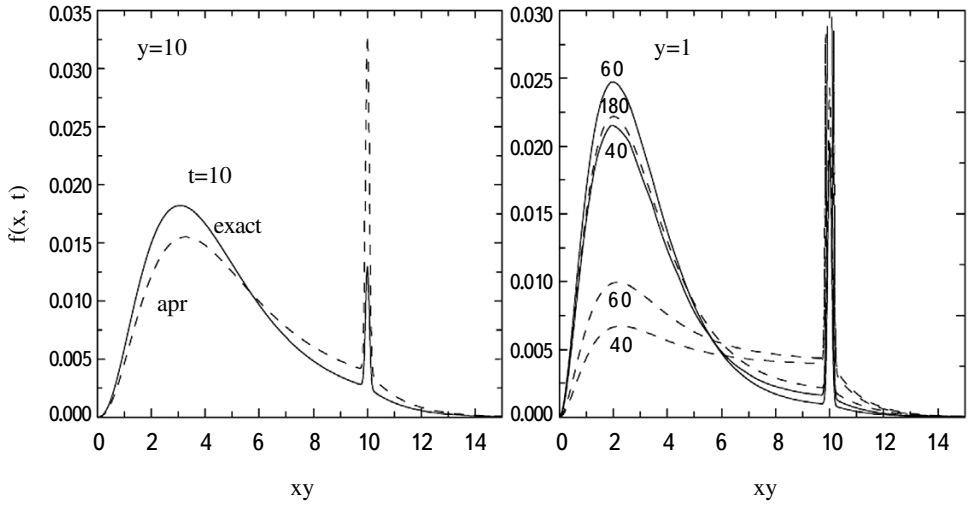

Fig. 4. A comparison of the exact (smooth curve) and simplified (dashed curve) solutions to the nonlinear integral equation for an initial distribution of the form (18) with $C = 0.1$. The numbers on the curve denote the time t.

Zeldovich and Levich [7]. Our numerical simulations show that these fronts do not break at a finite distance from the ordinate, and their evolution terminates by merging at a certain time $t_*(C)$ with the ordinate axis, so that the value of f at zero changes discontinuously from 0 to $t_*(C)$. The solution should then evolve to a stationary, but nonequilibrium distribution, $f(x,\infty)$ that differs from 0 at the axis and falls off at infinity $\propto x^{-2}$. From a physical standpoint, the development of a stable, steep leading front is explained by a competition between diffusion, which tends to smooth the gradients, and nonlinearity, which tends to magnify them. This phenomenon is essentially analogous to the development of solitary waves (solitons) in hydrodynamics.

As for comparing the solutions obtained using the simplified and exact frequency redistribution functions with scattering, the agreement between these solutions improves as the parameter $y = mc^2/kT$ becomes larger. As an illustration, we have done some calculations with an initial distribution in the form of a narrow line (18) with $C = 0.1$, so that there is a limiting (in time) equilibrium distribution. It turned out that for $y = 100$ (nonrelativistic electrons) the difference between these solutions becomes negligible for $t > 5$. As the parameter y increases, the emergence of the simplified solution to the stationary regime takes place more slowly than for the exact solution (see Fig. 4).

It should be noted that the new numerical method we have used can also be used to solve other kinetic equations and not just the equations of radiative transfer. For example, in the multiple Compton scattering problem it is possible to include the mutual influence of the radiation field and the electron velocity distribution, and solve jointly the kinetic equations for the photons and electrons.

4. Conclusion

The new noniterative method for numerical solution of the nonlinear integro-differential equation of radiative transfer with multiple Compton scattering turns out to be much more effective than the standard numerical techniques of the theory of heat conduction. These calculations show that the evolution of the scattered radiation spectrum depends substantially on the initial dimensionless photon density. When this density is high, no limiting equilibrium Bose-Einstein energy distribution of the photons exists, and in that case a quasiline develops which shifts toward lower frequencies with time as its width decreases and its peak intensity increases.

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