Astrophysics, Vol. 53, No. 2, 2010

NONLINEAR DIFFUSE REFLECTION AND TRANSMISSION OF RADIATIVE ENERGY BY A LAYER OF FINITE THICKNESS

H. V. Pikichian

The major results for the linear problem of diffuse reflection and transmission of radiation by a layer of finite thickness are carried over to the nonlinear case by successive application of Ambartsumyan's approach for a one dimensional anisotropic medium. Formulas are given for nonlinear addition of layers which can be used to construct recurrence calculation procedures for uniform, periodic, and arbitrary stratified media. A complete set of differential equations for invariant imbedding is derived with the aid of these formulas. These equations are used to obtain a system of total invariance *equations, which, in turn, offer the possibility of reducing the nonlinear problem of diffuse reflection and transmission during irradiation of a layer from both sides to the simpler problem of illuminating this medium from only one side, with the thickness of the layer remaining only as a fixed parameter. Finally, it is shown that the results obtained for the single frequency case (two-level atom) remain valid in the polychromatic case (multilevel atom), which is important for interpreting astrophysical data.*

Keywords: *radiative transfer: nonlinear diffuse reflection and transmission*

1. Introduction

Many papers on linear radiative transfer problems have been published in the last sixty years, while the nonlinear problems remain little studied because of their complexity. However, the significance of nonlinear problems continues to increase inexorably in connection with the discovery and interpretation of new higher energy physical phenomena taking place in astrophysical objects. As opposed to the linear case, for high radiation field densities the

V. A. Ambartsumyan Byurakan Astrophysical Observatory, Armenia; e-mail: hovpik@bao.sci.am

Original article submitted September 14, 2009; accepted for publication March 3, 2010. Translated from Astrofizika, Vol. 53, No. 2, pp. 285-299 (May 2010).

0571-7256/10/5302-0251

characteristics of a scattering and absorbing medium (such as the radiative absorption and scattering coefficients for an elementary volume, optical thickness, the radiative frequency and directional redistribution function, etc.) are not quantities that can be specified in advance. Here the resulting characteristics of the radiation field and the medium must be mutually consistent.

It is well known that the invariance principle [1-7] introduced by Ambartsumyan in 1942 greatly simplified the study of linear problems in the theory of radiative transfer. In particular, as opposed to the traditional approach, it made it possible to solve the classical problem of diffuse reflection and transmission directly, without first determining the characteristics of the radiation field inside the medium. Later, this served as the basis for the development of an extensive arsenal of analytic numerical techniques for analyzing various transport problems involving radiation, particles, and waves [8].

Ambartsumyan extended this approach to the area of nonlinear problems [9,10]. With the nonlinear problem of reflection and transmission for a one dimensional, isotropic medium in the monochromatic case (two level atom) as an example, he formulated the "nonlinear addition" relations for layers. Later they were used to develop a quasilinear differential equation for "invariant imbedding." Ultimately, the two point boundary radiative transfer problem was reduced to a problem with initial conditions (i.e., a Cauchy problem). In the case of a semi-infinite medium, Ambartsumyan found an explicit solution for this equation. An analytic solution to this problem for the more general case of polychromatic scattering (three level atom) was obtained by Nikogosyan [11,12]. Later on, analytic solutions were also obtained for the internal radiative field in a semi-infinite medium in the case of a two level atom [13]; here several analogs were found between the invariance principle approach used in radiative transfer theory and the group renormalization method of quantum field theory. An algorithm based on the method of linear layer addition was also developed for approximate calculation of nonlinear diffuse reflection and transmission; this was carried out for the case of a four level atom with a continuum [14-16]. Nonlinear problems have been studied analytically using this and other methods [17-21]. In a recent paper [22] a recurrence procedure for the nonlinear addition of layers has been developed axiomatically and the problem studied mathematically for solvability and for convergence conditions leading to a solution.

2. Purpose of this paper and statement of the problem

Ambartsumyan's study of linear diffuse reflection and transmission included both a derivation of the differential equations for invariant imbedding [5], when each of the boundaries of the initial layer is separately subjected to an infinitely small variation [this can be arbitrarily referred to as a procedure for separate variation of the boundaries], and an application of he combined procedure, when an infinitely small layer is added at one boundary and the same boundary is removed from the other side [procedure for joint variation]. The latter, of course, is already an example of the invariant variation of boundaries, since it does not change the initial geometry of the medium. By equating all the calculated changes in the radiation field to zero, Ambartsumyan obtained a functional relationship of a new type which no longer involved differentiating with respect to the layer thickness [2]. We shall refer to this type of relation as *total invariance* relations in order to distinguish them from the now widely used class of *invariant imbedding* equations [23-25]. In the nonlinear diffuse reflection and transmission problem, on the other hand, Ambartsumyan carried out only one of the procedures for separate variation of boundaries, so that only one total invariance differential equation was obtained [9]. Our task is to derive a complete set of total invariance differential equations and use them to find the total invariance relations. It appears that successive application of Ambartsumyan's approach makes it possible, in a way analogous to the linear case, to simplify the solution of the nonlinear reflection-transmission problem. To begin, formulas are given for the nonlinear addition of layers which are suitable for constructing different algorithms for the recurrent accumulation of layers to a previously specified structure and thickness. Then they are used to derive new differential equations for invariant imbedding which, together with the already known equations, make it possible, in particular, by eliminating derivatives with respect to the thickness, to obtain a new system of functional equations where the thickness of the layer now shows up only as fixed parameter. This makes it possible, in particular, to reduce the original nonlinear reflection-transmission problem with two sided illumination of the layer from outside to a simpler problem with one sided illumination. These results are generalized further to polychromatic scattering, which is of special importance in astrophysical applications. Some of the results given here have been reported previously [26,27].

Consider a one dimensional, anisotropic, scattering and absorbing medium of geometrical thickness L with its left and right boundaries illuminated by powerful beams of radiation with intensities j^+ and j^-, respectively. We seek to determine the radiant intensities $u_L^+\left(j^+, j^-\right)$ and $u_L^-\left(j^+, j^-\right)$ emerging through the right and left boundaries of this medium. Following Ambartsumyan, we shall use the "reflection-transmission" properties for a layer of "small" geometric thickness $\Delta \to 0$ as initial information in the representation

$$u_\Delta^\pm(x, y) = \begin{Bmatrix} x \\ y \end{Bmatrix} + \alpha^\pm(x, y) \cdot \Delta + O\left(\Delta^2\right), \tag{1}$$

where, to begin with, we take the coefficient $\alpha^\pm(x, y)$ as given.

3. Equations for the "nonlinear addition" of layers

Let a second layer of thickness B be applied to the right side a layer of geometric thickness A. The resulting layer of thickness $A+B$ is illuminated from the left and right by powerful beams of radiation with intensities x and y, respectively. We seek the intensities $u_{A+B}^\pm(x, y)$ of the radiation emerging from the combined layer $A+B$ under the condition that the corresponding quantities $u_A^\pm(x, y)$ and $u_B^\pm(x, y)$ for the separate layer A and B are given beforehand. (See Fig. 1.)

We shall proceed from the obvious physical fact that, both the radiation emerging from the medium and the radiation field within it are uniquely determined by the intensities of the radiation entering it from outside. Then, each of the intensities $u^\pm$ emerging through the given (right or left) boundary can be represented in two ways. For example, on one hand, u^+ can be written as the intensity of the radiation emerging from the combined medium $A+B$ and formed

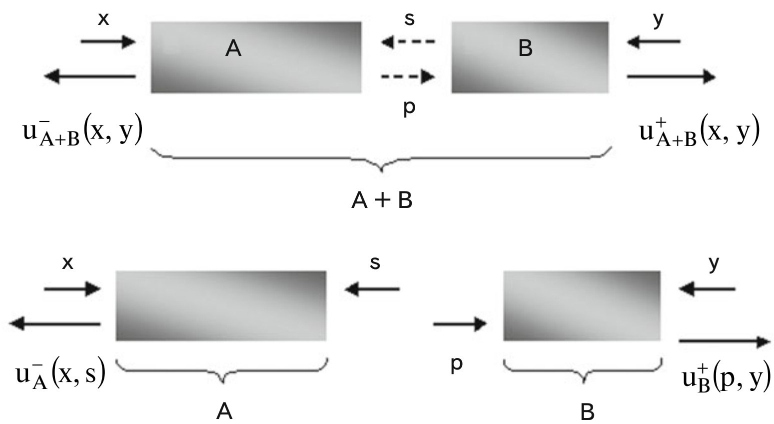

Fig. 1. The addition of layers in the nonlinear diffuse reflection and transmission problem.

by the interaction of the external fluxes (*x, y*), and, on the other, as the intensity of the radiation emerging from the medium *B* alone and formed by the action of the external fluxes (*p, y*). Similarly, u^- can be treated as the intensity of the radiation emerging from the medium *A+B* and formed by the action of the external fluxes (*x, y*) and, on the other, as the intensity of the radiation emerging from medium *A* as a result of the action of the external fluxes (*x, s*); i.e.,

$$u^+_{A+B}(x, y) = u^+_B(p, y), \tag{2}$$

$$u^-_{A+B}(x, y) = u^-_A(x, s). \tag{3}$$

In determining u^+ from the initial medium *A+B*, its part *A* was cut away with retention of the resultant field p formed at the contact boundary between its parts, meaning that now there is only the medium *B* irradiated from the left by a field *p* and, as before, from the right by *y*. Similarly, in determining u^-, the initial medium was replaced by its part *A* irradiated from the left by the field *s* and from the right by *x*. *p* and *s*, which represent the intensities of the resulting radiation moving to the right and left at the contact surface between *A* and *B*, are still unknown. It is easy to obtain the following system of equations for the latter:

$$p = u^+_A(x, s), \quad s = u^-_B(p, y). \tag{4}$$

The equations for nonlinear layer addition, (2)-(4), can, as in the linear case [28], be used for recurrence calculations in the reflection-transmission problem. In each successive step of the addition:

1. The unknowns u_{A+B} are calculated from the specified values of u_A and u_B using the explicit expressions (2)-(3), while the auxiliary functions are determined by solving Eq. (4) (e.g., iteratively).

2. If *A* and *B*, in particular, are chosen to be the same, with $A = B \equiv L$, then we obtain the formulas for

nonlinear doubling of the layers: $L \to 2L \to 4L \to \ldots$.

3. If, on the other hand, the calculation is begun with a small value $L = \Delta + O(\Delta^2)$, for example using the representation (1), then we obtain a procedure for buildup of the layer according to the scheme $(L = 2^{i-1} \cdot \Delta) \to (2L = 2^i \cdot \Delta) \to \ldots$, with successive values of $i = 1, 2, 3, \ldots$, etc.

The scheme in item 3 can obviously be used to calculate the nonlinear problem for uniform media, while that in item 2, on the other hand, can be used to proceed to the calculations for "periodic media" once the solution of the problem for a single period L has already been found. Finally, item 1 describes a method for calculating arbitrary layered media, when the corresponding problems for each of the initial layers A and B have already been calculated separately.

4. Equations for invariant imbedding

Using a slightly modified version of Ambartsumyan's method [9], with the aid of Eqs. (2)-(4) we now derive a complete set of differential equations for invariant imbedding for determining the emerging intensities $u_L^{\pm}(j^+, j^-)$ in the case of a layer with arbitrary finite geometrical thickness L whose left and right boundaries are acted on by fluxes j^+ and j^-, respectively. Taking $A \equiv \Delta \to 0$ and $B \equiv L$, as well as $x \equiv j^+$ and $y \equiv j^-$ in Eqs. (2)-(3), we obtain

$$u_{\Delta+L}^{+}(j^+, j^-) = u_L^{+}(p, j^-), \tag{5}$$

$$u_{\Delta+L}^{-}(j^+, j^-) = u_{\Delta}^{-}(j^+, s). \tag{6}$$

For the unknowns s and p from Eq. (4), we will obviously have

$$p = u_{\Delta}^{+}(j^+, s), \quad s = u_L^{-}(p, j^-). \tag{7}$$

Using the representation (1) and the expansion

$$u_{\Delta+L}^{\pm} = u_L^{\pm} + \frac{\partial u_L^{\pm}}{\partial L} \cdot \Delta + O(\Delta^2), \tag{8}$$

in Eqs. (5) and (6) and in the first of Eqs. (7) yields the following relationships:

$$u_L^+(j^+, j^-) + \frac{\partial u_L^+(j^+, j^-)}{\partial L} \cdot \Delta + O(\Delta^2) = u_L^+(p, j^-), \tag{9}$$

$$u_L^-(j^+, j^-) + \frac{\partial u_L^-(j^+, j^-)}{\partial L} \cdot \Delta = s + \alpha^-(j^+, s) \cdot \Delta + O(\Delta^2), \tag{10}$$

and

$$p = j^+ + \alpha^+(j^+, s) \cdot \Delta + O(\Delta^2), \tag{11}$$

respectively. Substituting Eq. (11) in the second of Eqs. (7), and noting the possibility of expanding the function $u_L^-(p, j^-)$ in terms of the small quantity Δ in its right argument, we obtain

$$s = u_L^-(j^+, j^-) + \alpha^+(j^+, s) \cdot \frac{\partial u_L^-(j^+, j^-)}{\partial j^+} \cdot \Delta + O(\Delta^2). \tag{12}$$

Equations (11) and (12) are an expanded form of the system of Eqs. (7), and, to within $O(\Delta^2)$, its explicit solution is given by

$$p \cong j^+ + \alpha^+(j^+, u_L^-) \cdot \Delta, \tag{13}$$

$$s \cong u_L^-(j^+, j^-) + \alpha^+(j^+, u_L^-) \cdot \frac{\partial u_L^-(j^+, j^-)}{\partial j^+} \cdot \Delta. \tag{14}$$

Substituting Eqs. (13) and (14) in Eqs. (9) and (10) and carrying out the corresponding expansions and cancellations, we finally obtain

$$\frac{\partial u_L^+(j^+, j^-)}{\partial L} = \alpha^+(j^+, u_L^-) \cdot \frac{\partial u_L^+(j^+, j^-)}{\partial j^+}, \tag{15}$$

$$\frac{\partial u_L^-(j^+, j^-)}{\partial L} = \alpha^+(j^+, u_L^-) \cdot \frac{\partial u_L^-(j^+, j^-)}{\partial j^+} + \alpha^-(j^+, u_L^-). \tag{16}$$

If we take $A \equiv L$ and $B \equiv \Delta \to 0$, as well as $x \equiv j^+$ and $y \equiv j^-$, in Eqs. (2) and (3), then, in place of Eqs. (5) and

(6), we find

$$u^{+}_{L+\Delta}\left(j^{+}, j^{-}\right)=u^{+}_{\Delta}\left(\tilde{p}, j^{-}\right), \tag{17}$$

$$u^{-}_{L+\Delta}\left(j^{+}, j^{-}\right)=u^{-}_{L}\left(j^{+}, \tilde{s}\right), \tag{18}$$

and the system analogous to Eq. (7) takes the form

$$\tilde{p}=u^{+}_{L}\left(j^{+}, \tilde{s}\right), \quad \tilde{s}=u^{-}_{\Delta}\left(\tilde{p}, j^{-}\right). \tag{19}$$

Repeating the above procedure using Eqs. (1) and (17)-(19), to within terms of order $O\left(\Delta^{2}\right)$, for the solution of the system of Eqs. (19) we obtain

$$\tilde{s} \cong j^{-}+\alpha^{-}\left(u^{+}_{L}, j^{-}\right)\cdot \Delta , \tag{20}$$

$$\tilde{p} \cong u^{+}_{L}\left(j^{+}, j^{-}\right)+\alpha^{-}\left(u^{+}_{L}, j^{-}\right)\cdot \frac{\partial u^{+}_{L}\left(j^{+}, j^{-}\right)}{\partial j^{-}}\cdot \Delta , \tag{21}$$

so that Eqs. (15) and (16) can be supplemented by a second pair of differential equations for invariant imbedding for the same unknowns $u^{\pm}_{L}\left(j^{+}, j^{-}\right)$:

$$\frac{\partial u^{+}_{L}\left(j^{+}, j^{-}\right)}{\partial L}=\alpha^{-}\left(u^{+}_{L}, j^{-}\right)\cdot \frac{\partial u^{+}_{L}\left(j^{+}, j^{-}\right)}{\partial j^{-}}+\alpha^{+}\left(u^{+}_{L}, j^{-}\right), \tag{22}$$

$$\frac{\partial u^{-}_{L}\left(j^{+}, j^{-}\right)}{\partial L}=\alpha^{-}\left(u^{+}_{L}, j^{-}\right)\cdot \frac{\partial u^{-}_{L}\left(j^{+}, j^{-}\right)}{\partial j^{-}}. \tag{23}$$

Thus, for finding the two functions $u^{+}_{L}\left(j^{+}, j^{-}\right)$ and $u^{-}_{L}\left(j^{+}, j^{-}\right)$ we have obtained four equations (15), (16), (22), and (23). Two of these, Eqs. (22) and (16), are individual quasilinear differential equations which, with the initial conditions

$$u^{+}_{L}\left(j^{+}, j^{-}\right)\Big|_{L=0}=j^{+}, \quad u^{+}_{L}(0,0)\Big|_{L\geq 0}=0, \tag{24}$$

$$u_L^-\left(j^+, j^-\right)\Big|_{L=0} = j^-, \quad u_L^-(0,0)\Big|_{L\geq 0} = 0 \tag{25}$$

are sufficient for calculating the unknowns $u_L^+\left(j^+, j^-\right)$ and $u_L^-\left(j^+, j^-\right)$, respectively. In addition, if at least one of the unknowns is first determined from the corresponding Eq. (16) or (22), then the other can be found from Eq. (15) or (23). Taken together, Eqs. (15) and (23),

$$\frac{\partial u_L^+\left(j^+, j^-\right)}{\partial L} = \alpha^+\left(j^+, u_L^-\right)\cdot\frac{\partial u_L^+\left(j^+, j^-\right)}{\partial j^+}, \quad \frac{\partial u_L^-\left(j^+, j^-\right)}{\partial L} = \alpha^-\left(u_L^+, j^-\right)\cdot\frac{\partial u_L^-\left(j^+, j^-\right)}{\partial j^-}, \tag{26}$$

offer a new, third possibility for the joint determination of the unknowns. Here, as opposed to the separate equations (22) and (16), the quasilinear differential equations are homogeneous. Equations (15), (16) and (22), (23) can also be solved with the alternative initial conditions

$$u_L^+\left(j^+, 0\right) = T_L^+\left(j^+\right), \quad u_L^+\left(0, j^-\right) = R_L^+\left(j^-\right), \tag{27}$$

$$u_L^-\left(j^+, 0\right) = R_L^-\left(j^+\right), \quad u_L^-\left(0, j^-\right) = T_L^-\left(j^-\right). \tag{28}$$

$R_L^\pm\left(j^\mp\right)$ and $T_L^\pm\left(j^\pm\right)$ are the intensities of the radiation reflected and transmitted by the layer when one of its boundaries is irradiated by beams of intensity j^+ or j^-. To find these particular quantities using Eqs. (15) and (16), (22) and (23), and (27) and (28), it is easy to write down both the separate equations

$$\frac{\partial R_L^+\left(j^-\right)}{\partial L} = \alpha^-\left(R_L^+\left(j^-\right), j^-\right)\cdot\frac{\partial R_L^+\left(j^-\right)}{\partial j^-} + \alpha^+\left(R_L^+\left(j^-\right), j^-\right), \tag{29}$$

$$\frac{\partial R_L^-\left(j^+\right)}{\partial L} = \alpha^+\left(j^+, R_L^-\left(j^+\right)\right)\cdot\frac{\partial R_L^-\left(j^+\right)}{\partial j^+} + \alpha^-\left(j^+, R_L^-\left(j^+\right)\right), \tag{30}$$

and the “mixed” equations

$$\frac{\partial T_L^+\left(j^+\right)}{\partial L} = \alpha^+\left(j^+, R_L^-\left(j^+\right)\right)\cdot\frac{\partial T_L^+\left(j^+\right)}{\partial j^+}, \quad \frac{\partial T_L^-\left(j^-\right)}{\partial L} = \alpha^-\left(R_L^+\left(j^-\right), j^-\right)\cdot\frac{\partial T_L^-\left(j^-\right)}{\partial j^-}. \tag{31}$$

The initial conditions for Eqs. (29)-(31) can obviously be chosen easily from the set

$$R_L^\pm\left(j^\mp\right)\Big|_{L=0} = 0, \quad R_L^\pm(0)\Big|_{L\geq 0} = 0, \quad T_L^\pm\left(j^\pm\right)\Big|_{L=0} = j^\pm, \quad T_L^\pm(0)\Big|_{L\geq 0} = 0. \tag{32}$$

In the particular case of an isotropic medium, because of the symmetry

$$\alpha^{+}(x, y) = \alpha^{-}(y, x) \equiv \alpha(y, x), \tag{33}$$

$$u_L^{-}(x, y) = u_L^{+}(y, x) \equiv u_L(y, x), \tag{34}$$

Eqs. (22) and (16) coincide, transforming into the differential equation for invariant imbedding first obtained by Ambartsumyan [9],

$$\frac{\partial u_L(x, y)}{\partial L} = \alpha(u_L, y) \cdot \frac{\partial u_L(x, y)}{\partial y} + \alpha(y, u_L), \tag{35}$$

and the system of Eqs. (26) transforms into

$$\frac{\partial u_L(x, y)}{\partial L} = \alpha(u_L(y, x), x) \cdot \frac{\partial u_L(x, y)}{\partial x}. \tag{36}$$

Using the notation

$$\upsilon_L(x, y) \equiv u_L(y, x), \tag{37}$$

in Eq. (36), it is possible to preserve the form of Eqs. (26),

$$\frac{\partial u_L(x, y)}{\partial L} = \alpha(\upsilon_L, x) \cdot \frac{\partial u_L(x, y)}{\partial x}, \quad \frac{\partial \upsilon_L(x, y)}{\partial L} = \alpha(u_L, y) \cdot \frac{\partial \upsilon_L(x, y)}{\partial y}. \tag{38}$$

Equation (36) (like Eq. (38)) was obtained first and, together with Eq. (35), represents a complete set of differential equations for invariant imbedding in the case of an isotropic medium. In the case of an anisotropic medium this set is evidently specified by the four equations (15)-(16) and (22)-(23), which are actually obtained by variation of the thickness L of the layer from its right, Eqs. (15)-(16), or left, Eqs. (22)-(23), boundary. The differential operators

$$\hat{E}_{+} = \alpha^{+}(j^{+}, u_L^{-}) \cdot \frac{\partial}{\partial j^{+}}, \quad \hat{E}_{-} = \alpha^{-}(u_L^{+}, j^{-}) \cdot \frac{\partial}{\partial j^{-}}, \tag{39}$$

here indicate the changes in the unknown emerging fields as a consequence of variations in the incident fluxes j^{+} and j^{-} caused by this "infinitely small" variation in the thickness of the layer at its left, $\hat{E}_{+}$, or right, $\hat{E}_{-}$, boundary, respectively. The operator form of the equations of invariant imbedding,

$$\left(\frac{\partial}{\partial L}-\hat{E}_{+}\right)u_L^{+}=0\,,\qquad (15a)$$

$$\left(\frac{\partial}{\partial L}-\hat{E}_{+}\right)u_L^{-}=\alpha^{-}\left(j^{+},u_L^{-}\right), \qquad (16a)$$

$$\left(\frac{\partial}{\partial L}-\hat{E}_{-}\right)u_L^{+}=\alpha^{+}\left(u_L^{+},j^{-}\right), \qquad (22a)$$

$$\left(\frac{\partial}{\partial L}-\hat{E}_{-}\right)u_L^{-}=0\,, \qquad (23a)$$

shows intuitively that the combined action $\left(\frac{\partial}{\partial L}-\hat{E}_{\pm}\right)$ of the thickness variation operator $\partial/\partial L$ and the operator $\hat{E}_{\pm}$ for the "radiation response" created by it, leaves unchanged the radiant intensity emerging through the opposite (with respect to the varied one) boundary of the medium (Eqs. (15a) and (23a)), while at the varied boundary the radiation emerging from the medium is produced by emission only from its infinitely thin boundary layer (Eqs. (16a) and (22a)).

5. Total invariance equations

Although the invariant imbedding equations (15a), (16a), (22a), (23a) are sufficient for solving the reflection-transmission problem (see above), the presence of the differential operator $\partial/\partial L$ implies calculating a family of functions $u_L^{\pm}$ over all the instantaneous values of the imbedding parameter over the interval [0, L]. It is also possible to eliminate the differentiation procedure entirely from this set of equations, thereby reducing the problem to solving a new system,

$$\begin{cases}\alpha^{+}\left(j^{+},u_L^{-}\right)\cdot\dfrac{\partial u_L^{+}\left(j^{+},j^{-}\right)}{\partial j^{+}}-\alpha^{-}\left(u_L^{+},j^{-}\right)\cdot\dfrac{\partial u_L^{+}\left(j^{+},j^{-}\right)}{\partial j^{-}}=\alpha^{+}\left(u_L^{+},j^{-}\right)\\ \alpha^{+}\left(j^{+},u_L^{-}\right)\cdot\dfrac{\partial u_L^{-}\left(j^{+},j^{-}\right)}{\partial j^{+}}-\alpha^{-}\left(u_L^{+},j^{-}\right)\cdot\dfrac{\partial u_L^{-}\left(j^{+},j^{-}\right)}{\partial j^{-}}=-\alpha^{-}\left(j^{+},u_L^{-}\right),\end{cases} \qquad (40)$$

where the layer thickness enters only as a fixed parameter, while the differentiation operation includes only the parameters of the two sided external radiative interaction, j^{+} and j^{-}, with the medium. It is noteworthy that both equations in this system contain one and the same operator

$$\hat{A} \equiv \alpha^{+}\left(j^{+}, u_L^{-}\right) \cdot \frac{\partial}{\partial j^{+}} - \alpha^{-}\left(u_L^{+}, j^{-}\right) \cdot \frac{\partial}{\partial j^{-}}, \tag{41}$$

i.e., the difference between the "one sided radiative response" operators of the medium in response to the procedure of an infinitesimally small variation in one of its boundaries,

$$\hat{A} = \hat{E}_{+} - \hat{E}_{-} \tag{42}$$

and depends on both of the unknowns $u_L^{\pm}$. It can be used to rewrite the system of Eqs. (4) in the more compact form

$$\hat{A}u_L^{+} = +\alpha^{+}\left(u_L^{+}, j^{-}\right), \quad \hat{A}u_L^{-} = -\alpha^{-}\left(j^{+}, u_L^{-}\right). \tag{40a}$$

It is easy to see (see Eq. (42)) that the effect of the operator $\hat{A}$ corresponds to a procedure of adding an infinitely thin layer to one boundary of the original medium with simultaneous removal of the same sort of layer from the opposite boundary, which obviously leaves the physical picture of the problem *totally invariant*. This is the sort of procedure first used by Ambartsumyan [2] for the diffuse reflection and transmission of radiation by a layer of finite optical thickness in the linear case, but the same "invariant" procedure in the nonlinear case has led us to the operator (41), i.e., to a new procedure, "differentiation with respect to the parameters of a two sided radiative interaction with the layer from outside," resulting precisely from the *nonlinearity* of the problem. It is natural to refer to $\hat{A}$ as the *Ambartsumyan total invariance operator*. For an isotropic medium, this operator takes (see Eqs. (33) and (41)) the simpler form

$$\hat{A} \equiv \alpha\left(\upsilon_L, x\right) \cdot \frac{\partial}{\partial x} - \alpha\left(u_L, y\right) \cdot \frac{\partial}{\partial y}, \tag{43}$$

and on going to the linear case, it "vanishes" completely. In fact, after completing this transition in Eq. (40a) with Eq. (43), using the obvious relations

$$\alpha\left(x, y\right) = -k \cdot y + \frac{\lambda}{2} \cdot k \cdot \left(x + y\right), \tag{44}$$

$$u_L\left(x, y\right) = x \cdot T + y \cdot R, \tag{45}$$

$$\upsilon_L\left(x, y\right) = x \cdot R + y \cdot T, \tag{46}$$

and some simple calculations, we arrive at the well known Ambartsumyan formula [5]

$$T^2 = R^2 + \frac{2}{\lambda} \cdot (2-\lambda) \cdot R + 1, \tag{47}$$

which, together with Eqs. (45) and (46) has the property of *total invariance* in the linear, one dimensional, isotropic case of diffuse reflection-transmission. Here k and λ are the absorption coefficient and the probability of a photon's surviving an elementary scattering event, and R and T are the diffuse reflection and transmission coefficients of the medium. Characteristically, the layer thickness enters in both (nonlinear and linear) cases as a fixed parameter (Eqs. (40a) and (45)-(47), respectively). In solving the total invariance system (40) (likewise (40a)), the dependence on the layer thickness must evidently be specified through quantities included in the initial conditions (27) and (28). Thus, in the nonlinear case (see Eqs. (40), (40a) and (27), (28)), similarly to the linear case (Eqs. (45)-(47),

1. it is possible to reduce the problem of two sided irradiation of the medium to the simpler problem of irradiating the medium only on one of its boundaries; and
2. the layer thickness is left over only as a fixed parameter.

6. Polychromatic case

The monochromatic case which we have examined here serves in astrophysics only as a "zeroth" order model approximation, since it corresponds to the case of a two level atom. Interpreting the observed spectra of astrophysical objects requires dealing with multilevel atoms, so the problem of polychromatic scattering gains in practical importance. It is obvious that every polychromatic scattering problem is, in fact, essentially nonlinear. Hence, it is appropriate to ask how the formulas derived above change in the multifrequency case. Let a one dimensional anisotropic medium of geometric thickness L be acted on simultaneously by external radiation fluxes at n different frequencies, specified in the form of the row vectors $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$, respectively on the left and on the right, with

$$\boldsymbol{J}^\pm \equiv \left\{ j_1^\pm, \ldots, j_n^\pm \right\}. \tag{48}$$

We denote the intensities of the radiation emerging through the right and left boundaries of the medium at each individual frequency k ($k = 1,\ 2,\ \ldots,\ n$) by

$$u_k^\pm\left(j_1^+, \ldots, j_n^+; j_1^-, \ldots, j_n^-; L\right) \equiv u_k^\pm\left(\boldsymbol{J}^+, \boldsymbol{J}^-; L\right), \tag{49}$$

and construct the row vector

$$\boldsymbol{U}_{\boldsymbol{L}}^\pm \equiv \left\{ u_1^\pm\left(\boldsymbol{J}^+, \boldsymbol{J}^-; L\right), \ldots, u_n^\pm\left(\boldsymbol{J}^+, \boldsymbol{J}^-; L\right) \right\}, \tag{50}$$

so that the "nonlinear layer addition" relations (2)-(4) in the polychromatic case take the form

$$\boldsymbol{U}_{A+B}^+(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{U}_B^+(\boldsymbol{P}, \boldsymbol{y}), \tag{51}$$

$$\boldsymbol{U}^-_{\boldsymbol{A+B}}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{U}^-_{\boldsymbol{A}}(\boldsymbol{x},\boldsymbol{S}), \tag{52}$$

$$\boldsymbol{P}=\boldsymbol{U}^+_A(\boldsymbol{x},\boldsymbol{S}),\quad \boldsymbol{S}=\boldsymbol{U}^-_B(\boldsymbol{P},\boldsymbol{y}) \tag{53}$$

where the following row vectors have been introduced:

$$\boldsymbol{x}\equiv\{x_1,\dots,x_n\},\quad \boldsymbol{y}\equiv\{y_1,\dots,y_n\}, \tag{54}$$

$$\boldsymbol{P}\equiv\{p_1,\dots,p_n\},\quad \boldsymbol{S}\equiv\{s_1,\dots,s_n\}. \tag{55}$$

If the reflection and transmission properties of an elementary layer are specified in the form

$$u_k^\pm(\boldsymbol{J}^+,\boldsymbol{J}^-;\Delta)=\begin{Bmatrix} j_k^+ \\ j_k^- \end{Bmatrix}+\alpha_k^\pm(\boldsymbol{J}^+,\boldsymbol{J}^-)\cdot\Delta+O(\Delta^2), \tag{56}$$

then, using Eqs. (51)-(56) and proceeding as in the monochromatic case, we obtain the complete set of invariant imbedding equations,

$$\frac{\partial u_k^+}{\partial L}=\sum_{l=1}^n\alpha_l^-(\boldsymbol{U}^+,\boldsymbol{J}^-)\cdot\frac{\partial u_k^+}{\partial j_l^-}+\alpha_k^+(\boldsymbol{U}^+,\boldsymbol{J}^-), \tag{57}$$

$$\frac{\partial u_k^-}{\partial L}=\sum_{l=1}^n\alpha_l^+(\boldsymbol{J}^+,\boldsymbol{U}^-)\cdot\frac{\partial u_k^-}{\partial j_l^+}+\alpha_k^-(\boldsymbol{J}^+,\boldsymbol{U}^-), \tag{58}$$

$$\frac{\partial u_k^+}{\partial L}=\sum_{l=1}^n\alpha_l^+(\boldsymbol{J}^+,\boldsymbol{U}^-)\cdot\frac{\partial u_k^+}{\partial j_l^+}, \tag{59}$$

$$\frac{\partial u_k^-}{\partial L}=\sum_{l=1}^n\alpha_l^-(\boldsymbol{U}^+,\boldsymbol{J}^-)\cdot\frac{\partial u_k^-}{\partial j_l^-}. \tag{60}$$

Eliminating the derivatives with respect to the layer thickness from Eqs. (57)-(58), we finally obtain the total invariance system for the vector case being examined here:

$$\begin{cases} \sum_{l=1}^{n} \alpha_l^+\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right)\cdot\frac{\partial u_k^+}{\partial j_l^+} - \sum_{l=1}^{n} \alpha_l^-\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right)\cdot\frac{\partial u_k^+}{\partial j_l^-} = +\alpha_k^+\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right) \\ \sum_{l=1}^{n} \alpha_l^+\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right)\cdot\frac{\partial u_k^-}{\partial j_l^+} - \sum_{l=1}^{n} \alpha_l^-\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right)\cdot\frac{\partial u_k^-}{\partial j_l^-} = -\alpha_k^-\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right). \end{cases} \tag{61}$$

Introducing the operators

$$\hat{\boldsymbol{E}}_+ \equiv \sum_{l=1}^{n} \alpha_l^+\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right)\cdot\frac{\partial}{\partial j_l^+}, \quad \hat{\boldsymbol{E}}_- \equiv \sum_{l=1}^{n} \alpha_l^-\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right)\cdot\frac{\partial}{\partial j_l^-}, \tag{62}$$

$$\hat{\boldsymbol{A}} \equiv \hat{\boldsymbol{E}}_+ - \hat{\boldsymbol{E}}_- = \sum_{l=1}^{n} \alpha_l^+\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right)\cdot\frac{\partial}{\partial j_l^+} - \sum_{l=1}^{n} \alpha_l^-\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right)\cdot\frac{\partial}{\partial j_l^-}, \tag{63}$$

we write the invariant imbedding equations in the compact form

$$\left(\frac{\partial}{\partial L} - \hat{\boldsymbol{E}}_-\right)u_k^+ = \alpha_k^+\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right), \quad \left(\frac{\partial}{\partial L} - \hat{\boldsymbol{E}}_+\right)u_k^- = \alpha_k^-\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right), \tag{64}$$

$$\left(\frac{\partial}{\partial L} - \hat{\boldsymbol{E}}_+\right)u_k^+ = 0, \quad \left(\frac{\partial}{\partial L} - \hat{\boldsymbol{E}}_-\right)u_k^- = 0, \tag{65}$$

and the total invariance system then takes the form

$$\hat{\boldsymbol{A}}u_k^+ = +\alpha_k^+\left(\boldsymbol{U}^+,\boldsymbol{J}^-\right), \quad \hat{\boldsymbol{A}}u_k^- = -\alpha_k^-\left(\boldsymbol{J}^+,\boldsymbol{U}^-\right). \tag{66}$$

The initial conditions are written by analogy with Eqs. (24), (25), (27), and (28):

$$u_k^\pm\left(\boldsymbol{J}^+,\boldsymbol{J}^-;L\right)\Big|_{L=0} = j_k^\pm, \quad u_k^\pm\left(\boldsymbol{0},\boldsymbol{0};L\right)\Big|_{L\geq 0} = 0, \tag{67}$$

$$u_k^+\left(\boldsymbol{J}^+,\boldsymbol{0};L\right) = T_k^+\left(\boldsymbol{J}^+;L\right), \quad u_k^+\left(\boldsymbol{0},\boldsymbol{J}^-;L\right) = R_k^+\left(\boldsymbol{J}^-;L\right), \tag{68}$$

$$u_k^-\left(\boldsymbol{J}^+,\boldsymbol{0};L\right) = R_k^-\left(\boldsymbol{J}^+;L\right), \quad u_k^-\left(\boldsymbol{0},\boldsymbol{J}^-;L\right) = T_k^-\left(\boldsymbol{J}^-;L\right), \tag{69}$$

and for the components of the "auxiliary" vector functions $\boldsymbol{T}^\pm \equiv \left\{T_1^\pm\left(\boldsymbol{J}^\pm;L\right),\ldots,T_n^\pm\left(\boldsymbol{J}^\pm;L\right)\right\}$ and

$\boldsymbol{R}^{\pm} \equiv \left\{R_1^{\pm}\left(\boldsymbol{J}^{\mp}; L\right), ..., R_n^{\pm}\left(\boldsymbol{J}^{\mp}; L\right)\right\}$ which appear here, we easily find the equations

$$\frac{\partial R_k^{+}\left(\boldsymbol{J}^{-}; L\right)}{\partial L} = \sum_{l=1}^{n} \alpha_l^{-}\left(\boldsymbol{R}^{+}, \boldsymbol{J}^{-}\right) \cdot \frac{\partial R_k^{+}\left(\boldsymbol{J}^{-}; L\right)}{\partial j_l^{-}} + \alpha_k^{+}\left(\boldsymbol{R}^{+}, \boldsymbol{J}^{-}\right), \tag{70}$$

$$\frac{\partial R_k^{-}\left(\boldsymbol{J}^{+}; L\right)}{\partial L} = \sum_{l=1}^{n} \alpha_l^{+}\left(\boldsymbol{J}^{+}, \boldsymbol{R}^{-}\right) \cdot \frac{\partial R_k^{-}\left(\boldsymbol{J}^{+}; L\right)}{\partial j_l^{+}} + \alpha_k^{-}\left(\boldsymbol{J}^{+}, \boldsymbol{R}^{-}\right), \tag{71}$$

$$\frac{\partial T_k^{+}\left(\boldsymbol{J}^{+}; L\right)}{\partial L} = \sum_{l=1}^{n} \alpha_l^{+}\left(\boldsymbol{J}^{+}, \boldsymbol{R}^{-}\right) \cdot \frac{\partial T_k^{+}\left(\boldsymbol{J}^{+}; L\right)}{\partial j_l^{+}}, \tag{72}$$

$$\frac{\partial T_k^{-}\left(\boldsymbol{J}^{-}; L\right)}{\partial L} = \sum_{l=1}^{n} \alpha_l^{-}\left(\boldsymbol{R}^{+}, \boldsymbol{J}^{-}\right) \cdot \frac{\partial T_k^{-}\left(\boldsymbol{J}^{-}; L\right)}{\partial j_l^{-}}, \tag{73}$$

with the initial conditions $R_k^{\pm}\left(\boldsymbol{J}^{\mp}; L\right)\Big|_{L=0} \equiv 0$, $R_k^{\pm}\left(\mathbf{0}; L\right)\Big|_{L\geq 0} = 0$, $T_k^{\pm}\left(\boldsymbol{J}^{\pm}; L\right)\Big|_{L=0} = j_k^{\pm}$, and $T_k^{\pm}\left(0; L\right)\Big|_{L\geq 0} = 0$. In the special case of a semi-infinite anisotropic medium, the conditions $\left.\frac{\partial R_k^{\pm}\left(\boldsymbol{J}^{\mp}; L\right)}{\partial L}\right|_{L\to\infty} \equiv 0$ evidently hold and then Eqs. (70) and (71) simplify to

$$\begin{aligned} &\sum_{l=1}^{n} \alpha_l^{-}\left(\boldsymbol{R}^{+}, \boldsymbol{J}^{-}\right) \cdot \frac{\partial R_k^{+}\left(\boldsymbol{J}^{-}\right)}{\partial j_l^{-}} + \alpha_k^{+}\left(\boldsymbol{R}^{+}, \boldsymbol{J}^{-}\right) = 0, \\ &\sum_{l=1}^{n} \alpha_l^{+}\left(\boldsymbol{J}^{+}, \boldsymbol{R}^{-}\right) \cdot \frac{\partial R_k^{-}\left(\boldsymbol{J}^{+}\right)}{\partial j_l^{+}} + \alpha_k^{-}\left(\boldsymbol{J}^{+}, \boldsymbol{R}^{-}\right) = 0, \end{aligned} \tag{74}$$

where the notation $R_k^{\pm}\left(\boldsymbol{J}^{\mp}; \infty\right) \equiv R_k^{\pm}\left(\boldsymbol{J}^{\mp}\right)$ and $R^{\pm} \equiv \left\{R_1^{\pm}\left(\boldsymbol{J}^{\mp}\right), ..., R_n^{\pm}\left(\boldsymbol{J}^{\mp}\right)\right\}$ is used. The first of Eqs. (74) corresponds to removal of the left boundary of the medium to infinity, and the second, of the right boundary. For a semi-infinite medium the latter are the equations of total invariance and in the case of an isotropic medium, as noted above, have been derived and solved analytically for two- [9] and three-level [11,12] atoms.

Thus, on going from a single frequency to the polychromatic case, the form of the main equations is retained with replacement of scalar quantities by vectors. This means that all the advantages of solving the linear reflection-transmission problem by Ambartsumyan's method are adequately retained in the general case of polychromatic scattering in a one dimensional anisotropic medium.

7. Conclusion

We now briefly enumerate the major results of this paper. With successive application of Ambartsumyan's approach it is possible to transfer its advantages for solving the linear diffuse reflection-transmission problem for a finite layer to the nonlinear case in an adequate manner. For one dimensional anisotropic media, formulas are first derived for the nonlinear addition of layers, Eqs. (2)-(4), which are suitable for constructing "recurrence" computation procedures—for uniform, periodic, and arbitrary layered media— and then, these relationships are used to derive the *complete* set of differential equations for invariant imbedding, Eqs. (15)-(16) and (22)-(23) (likewise for Eqs. (15a)-(16a) and (22a)-(23a)). The latter yield the *total invariance* equations (40) (likewise, Eq. (40a)), which offer the possibility of reducing the nonlinear diffuse reflection-transmission problem with irradiation of a layer on *both* sides by "intense" radiative fluxes, to the simpler problem of irradiation at just *one* boundary. The layer thickness then remains as a fixed quantity, and the new *Ambartsumyan total invariance* differential operator (41) that shows up here represents differentiation only with respect to the parameters of the external radiative interaction with the medium and are the same for the intensities emerging through the two boundaries of the medium. At the end of this paper it was shown that on going from the single frequency case (two level atom) to the multifrequency case (polychromatic scattering, multilevel atom), which is important for the interpretation of astrophysical data, all the results retain their previous form, with a transition from scalar quantities to vectors with a number of components equal to the number of discrete frequencies being considered.

I thank Dr. A. G. Nikogosyan for comprehensive support and help in all the stages of preparation of this paper.

REFERENCES

1. V. A. Ambartsumyan, *Astron. zh.* **19**, 30 (1942).
2. V. A. Ambartsumyan, *DAN SSSR* **38**, 257 (1943).
3. V. A. Ambartsumyan, *ZhÉTF* **13**, 323 (1943).
4. V. A. Ambartsumyan, *DAN SSSR* **43**, 106 (1944).
5. V. A. Ambartsumyan, *Izv. AN Arm. SSR*, Estestv. nauki, Nos. 1-2, 31 (1944).
6. V. A. Ambartsumyan, *DAN Arm. SSR* **7**, 199 (1947).
7. V. A. Ambartsumyan, *DAN Arm. SSR* **8**, 149 (1948).
8. M. A. Mnatsakanyan and O. V. Pikichyan, eds., *The Invariance Principle and its Applications* (Proc. Symp. at Burakan, Oct. 26-30, 1981) [in Russian], *Izd. AN Arm. SSR*, Erevan (1989), 522 pp.
9. V. A. Ambartsumyan, *DAN Arm. SSR* **38**, 225 (1964).
10. V. A. Ambartsumyan, in: V. V. Sobolev, ed., *Theory of Stellar Spectra* [in Russian] (1966), pp. 91-104.
11. A. G. Nikogosyan, *DAN Arm. SSR* **39**, 227 (1964).
12. A. G. Nikogosyan, *Astrofizika* **1**, 285 (1965).
13. M. A. Mnatsakanyan, *DAN SSSR* **262**, 856 (1982).
14. M. Gros and C. Magnan, *Astron. Astrophys.* **93**, 150 (1981).
15. C. Magnan, *Astron. Astrophys.* **271**, 543 (1993).

16. C. Magnan and P. De Laverny, *Astrofizika* **37**, 313 (1994).
17. V. A. Ambartsumyan, *DAN Arm. SSR* **39**, 159 (1964).
18. N. B. Engibaryan, *Astrofizika* **1**, 297 (1965).
19. V. Yu. Terebizh, *Astrofizika* **3**, 281 (1967).
20. N. B. Engibaryan, *Astrofizika* **3**, 325 (1967).
21. R. S. Vardanyan and N. B. Engibaryan, *Uch. zap. EGU, estestv. n.*, No. 3, 28 (1969).
22. N. B. Yengibarian, On non-linear discrete boundary-value problems and semigroups, Preprint 2009-02, Inst of Math. NAS of Armenia, July 08 (2009), 10p.
23. R. Bellman, G. M. Wing, *An introduction to Invariant Imbedding, Classics in Appl. Math.*, vol. 8, SIAM, Philadelphia (1992).
24. V. I. Klyatskin, *Dynamics of Stochasitc Systems: A Course of Lectures* [in Russian], Fizmatlit, Moscow (2003), 240 pp.
25. J. Casti and R. Kalaba, *Imbedding Methods in Applied Mathematics* [Russian translation], Mir, Moscow (1976), 244 pp.
26. H. Pikichian, in: J. J. Carrol and T. A. Goldman, eds., *Proc. International Conf. on "Lasers'97"*, New Orleans, LA, December 15-19 (1997), STS press., McLean, VA (1998), pp. 226-232.
27. H. V. Pikichian, *In materials of Byurakan Conf.* (Ambartsumian's 100-th anniversary, 18 Sept. 2008) (in press).
28. H. V. Pikichian, *Soobshch. Byurak. obs.* 55, 5 (1984).