



How deterministic is the Earth ionosphere's response to solar activity?

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Abstract

This contribution is aimed at an analysis of the dynamics of free-electron density fluctuations in the ionospheric critical plasma frequency f_0F_2 by using some tools from the theory of nonlinear dynamical systems. The results suggest the existence of low-dimensional attractors that point to a characterization of the free electron density fluctuations in the f_0F_2 as a deterministic chaotic system. The study carried out focused on the response of the ionosphere to solar activity as a function of the ascending and descending phases of the solar cycle.

Keywords Earth's ionosphere · Dynamical systems · Determinism · Chaos · Strange attractor

1 Introduction

The ionospheric critical plasma frequency f_0F_2 is not steady, it varies with latitude, season, time of the day, and other factors. Also, the f_0F_2 critically depends on the solar activity stage. Related to the latter, different indices of solar activity, as well as geomagnetic indices, have been widely used as input variables for numerous ionospheric models based on the statistical correlations between them (e.g., Buresova et al. 2014; Perna and Pezzopane 2016; Blagoveshchensky 2018; Tshisaphungo et al. 2018; Ippolito et al. 2020; and references therein). The time series profile of the ionospheric critical plasma frequency f_0F_2 variation is highly structured in time representing a real-world system that strongly depends on several variables and therefore full determinism is not expected. In consequence, some fundamental tools of the Nonlinear Dynamics and Chaos Theory are used and applied to two different profiles of the ionospheric critical plasma frequency f_0F_2 fluctuations corresponding to different phases of the solar cycle (ascending and descending). State-space, Lyapunov exponent, and strange attractor dimension are calculated to deduce if the fluctuations are generated by a deterministic but chaotic dynamic corresponding to different solar cycle stages.

2 Data description

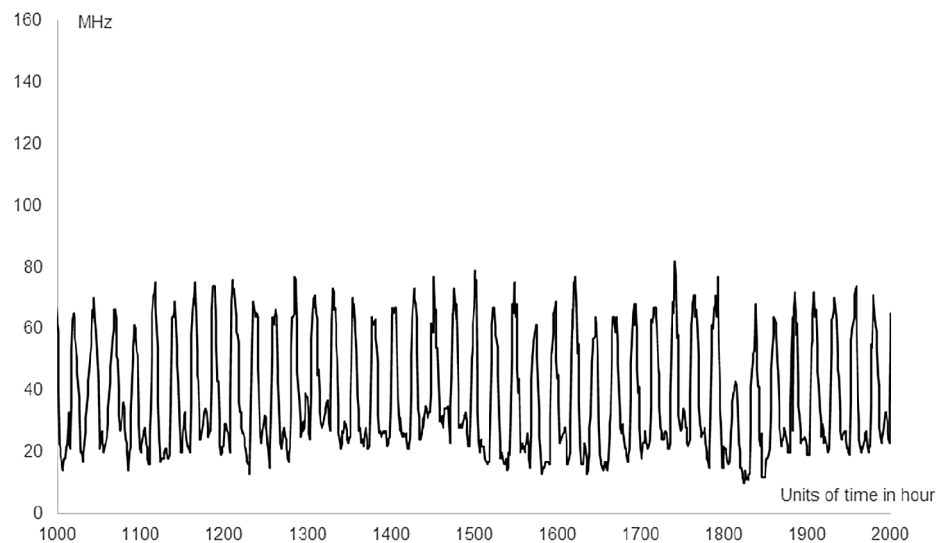
The dynamic response of the ionospheric critical frequency f_0F_2 to the solar activity depending on the ascending and descending phase of the solar cycle is analyzed. The ionospheric data (f_0F_2 profiles, hourly data) is from the Ionosonde Station Juliusruh (Germany, 54.6 N 13.4 E) and was obtained from Space Weather Service-Australian Government-Bureau of Meteorology (https://www.sws.bom.gov.au/World_Data_Centre/1/3). We selected two same seasonal periods (winter) corresponding to two different phases of the solar cycle: ascending and descending phases. The analyzed periods were November 01, 1987 to February 10, 1988 and November 01, 2002 to February 10, 2003. These two periods correspond to solar cycle 22 ascending phase, and solar cycle 23 descending phase respectively. Table 1 displays the main characteristics of these two solar cycles. The data for the solar cycles was obtained from the Space Weather Prediction Center, National Oceanic and Atmospheric Administration, NOAA (<https://www.swpc.noaa.gov/products/solar-cycle-progression>). Figure 1 shows segments of the corresponding profiles of the f_0F_2 fluctuations under study.

Deterministic chaotic behavior can be detected from the inspection of observational time series representing real-world systems, revealing the nature of the temporal evolution of the underlying process. Calculations of parameters related to quantifying how chaotic (or deterministic) a time series is, are sensitive to the length of the time series and the reliable limit for the number of structures for dimension in the state-space. These conditions are satisfied if

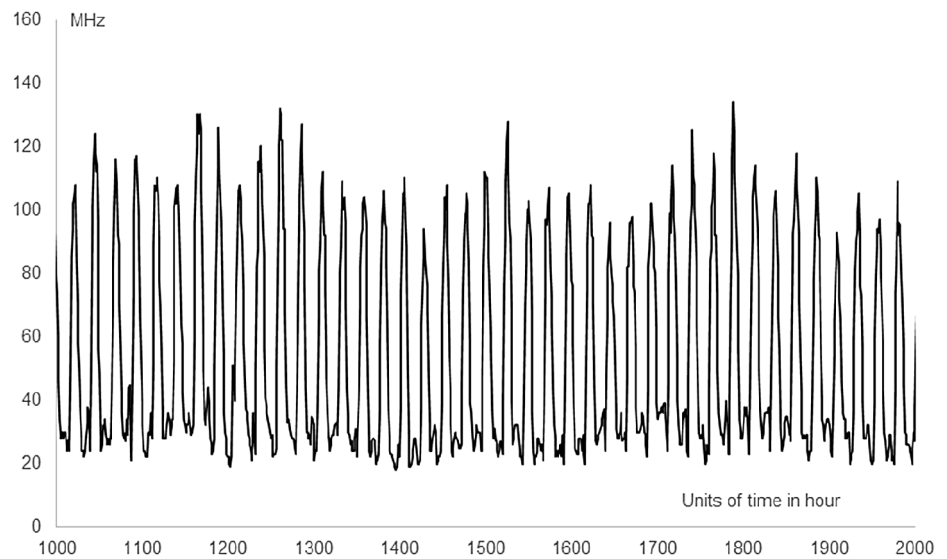
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Fig. 1 f0F2 fluctuations. Segments of the profile for Nov. 01, 1987-Feb. 10, 1988 (a) and for Nov. 01, 2002-Feb. 10, 2003 (b)



(a)



(b)

Table 1 Main characteristics of the solar cycles 23 and 23

Solar cycle	Duration	Average spots per day	Slope of the corresponding phase
22	9.9 years	106	(ascending) 8.5 units
23	12.3 years	82	(descending) 3.4 units

(Islaker 1992):

$$N_s := N\Delta/t_{\text{autocorr}} \geq 100 \tag{1}$$

where N_s is the number of structures representing the number of full orbits in state-space, N is the length of the time series in number of data-points, Δ is the temporal resolution of the time series, and t_{autocorr} is the autocorrelation time, the time at which autocorrelation decays to $1/e$. Table 2 resume

Table 2 Condition (1) for the two analyzed time series, showing that both satisfied it

Data	N	Δ	t_{autocorr}	N_s
Nov. 1987-Feb. 1988	2424	1 hour	4	606
Nov. 2002-Feb. 2003	2448	1 hour	5	490

condition (1) for the two analyzed time series, showing that both satisfied it.

3 Methods and results

3.1 State-space reconstruction

State-space is a multidimensional abstract space of all possible system states. A point in the state-space specifies the

state of the system, and vice versa. Dynamical systems are usually defined by a set of first-order ordinary differential equations acting on a state-space. The coordinates of the state-space are defined by the independent variables of the dynamic system under study. The state-space can be reconstructed from a one-dimension scalar time series according to an m -dimensional vector $x(t)$ by the embedding method of time delay (Takens' theorem, Takens 1981):

$$x(t) = [x(t), x(t + \tau), \dots, x(t + (m - 1)\tau)] \quad (2)$$

where t represents the time and $x(t)$ is the measured value at time t , τ represents a time delay (any multiple of time resolution), and m is the embedding dimension of the system. Takens' theorem asserts that the state-space of a dynamical system reconstructed according to (2) will have the same mathematical properties as the original system. The fundamental assumption underlying Takens' idea is that an observable time series is the realization of some dynamical process since all degrees of freedom are intertwined: every single coordinate of the time series contains the information on all others.

From a mathematical point of view, the time delay between successive elements is, in general, arbitrary and therefore almost any choice of time delay would be acceptable (Takens 1981). In consequence, the optimum time delay for the state-space reconstruction is a sort of conjecture, and there is no unique guess for an optimum time delay, therefore, for choosing it any tool for detecting correlations in a time series can be used. The main argument for choosing the optimum time delay for the reconstruction of the state-space is: it should be as large as possible because a small value means successive elements of the delay vectors are strongly correlated, resulting in all elements of the state-space are clustered around the diagonal line $f(x) = x$. On the other hand, it should not be too large because successive elements are already almost independent, and the points fill a large cloud in the state-space. The seminal paper of Fraser and Swinney (1986) states the first minimum of the mutual information as the better criterion for choosing time delay for state-space reconstruction than the autocorrelation function. This was the used method to find the optimum time delay τ for the reconstruction of state-space associated with the two analyzed time series.

A set of nontransient trajectories in a state-space is the typical portrait of a deterministic chaotic system (Ruelle and Takens 1971). This set of nontransient trajectories is a bounded region in the state-space to which all trajectories are asymptotically attracted: the attractor. Once a system reaches its attractor, it does not leave such state unless strong external perturbation is applied. Figure 2 shows the reconstructed state-space for the two dynamical systems under study corresponding to analyzed periods: November 01, 1987 to February 10, 1988, and November 01, 2002, to

February 10, 2003. The plotted state-space in two dimensions is for visualizing, in a first approximation, the underlying dynamics of the time series after selecting the optimum time delay for the state-space reconstruction. A two-dimension plot with coordinates $x(t_i)$ and $x(t_i + \tau)$, where τ is the selected optimum time delay, is intuitively reasonable and then enough to be used as a visual inspection and for inferring the existence of an attractor (Kantz and Schreiber 1999). The presence of an attractor region in both can be inferred, suggesting a chaotic but deterministic behavior of such dynamical systems.

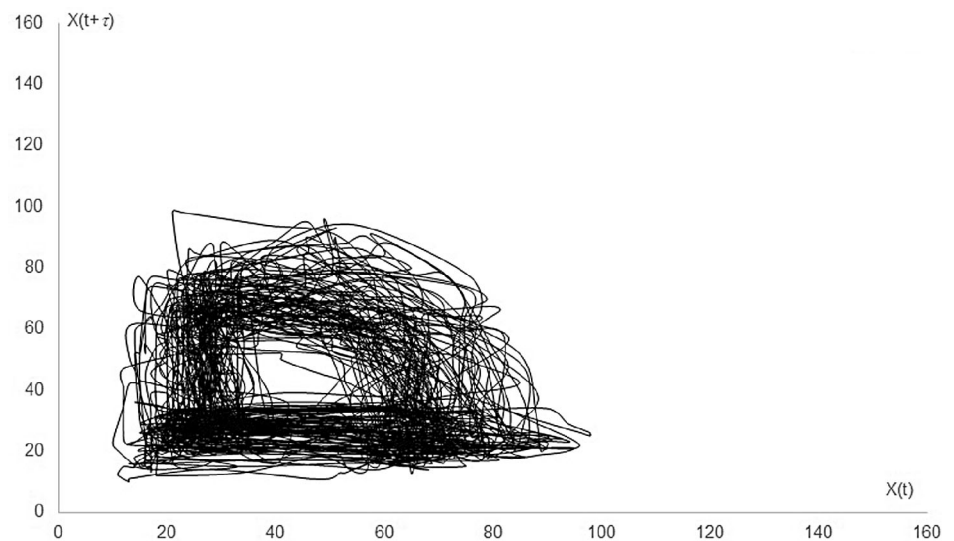
3.2 Lyapunov exponents

State-space reconstruction is a 2-dimensional map of an m -dimensional space, therefore, for most real physical systems, it is too difficult to visualize the actual trajectories from one state to the next in the dynamics of the system. In a completely deterministic system, for example, in a periodic regime, all the trajectories from one state to the next, in its state-space remain close to each other: in this case, each state of the system is determined by the previous states. In this way, if nearby trajectories separate very quickly, or say, if the trajectories diverge in the course of time during which the system evolves, the system is going to be less deterministic. In completely deterministic systems, the trajectories in their state space do not diverge. But, on the contrary, if the trajectories diverge with an exponentially fast rate of separation, the system is chaotic.

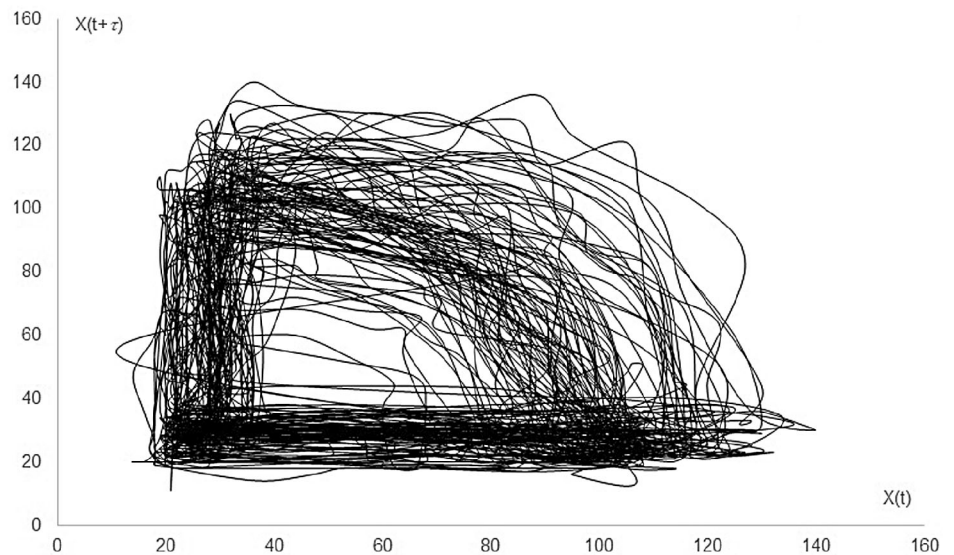
Lyapunov exponents determine the rate of divergence or convergence of initially nearby trajectories in the reconstructed state-space. According to Eckmann and Ruelle (1985) if at least one of these exponents is larger than zero, the system is chaotic. Lyapunov exponents uniquely determine whether the system is chaotic or not; it is not a quantitative measure of how deterministic a system is. The Lyapunov exponents, Λ , represents a characteristic quantity of any dynamical system and it is expressed as an inverse of time and gives a typical time scale for the divergence or convergence of nearby trajectories in the state-space. Actually, the Lyapunov exponent value is not a quantification about the dimension of the attractor in the state-space; it is just to point out if the system is fully deterministic ($\Lambda \leq 0$), chaotic ($0 < \Lambda < \infty$), or full stochastic ($\Lambda \rightarrow \infty$). In order to quantify how chaotic a system is (how close or far to be fully deterministic), the dimension of the attractor must be calculated by using other tools.

The maximal Lyapunov exponents were calculated (Wolf et al. 1985) for the reconstructed state-space of the examined data sets to confirm the presence of attractor and the inquired time series are originated from a chaotic system. Table 3 shows the obtained values, pointing to the presence of an attractor as a signature of a deterministic but chaotic behavior in both state-space as was suggested from Fig. 1.

Fig. 2 Reconstructed 2D state-space for Nov. 01, 1987-Feb. 10, 1988. Delay, $\tau = 6$ (a) and for Nov. 01, 2002-Feb. 10, 2003. Delay, $\tau = 7$ (b)



(a)



(b)

Table 3 Calculated quantities characterizing the nonlinear dynamics of the systems generating the f0F2 fluctuations considered in this paper: maximal Lyapunov exponent (Λ), deterministic factor (λ), and pointwise correlation dimension, ($PD^{(2)}$)

Time series	Λ	λ	$PD^{(2)}$
Nov. 1987-Feb. 1988	0.9156	0.8079	3.30 ± 0.52
Nov. 2002-Feb. 2003	0.5543	0.8597	2.99 ± 0.40

3.3 Determinism

One of the goals of the non-linear dynamical systems theory is to find out how deterministic the system is. Determinism is an important qualitative key to decide if the dynamic of the system is generated by a deterministic, rather than a stochastic, process. In Kaplan and Glass (1992) a re-

liable determinism test is introduced based on a proper reconstruction of the state-space and it is used to distinguish between deterministic and stochastic dynamics. In case the presence of an attractor in state-space can be inferred, the Kaplan-Glass determinism test is a measure of the degree to which the attractor results from the evolution of a deterministic system. The degree of determinism is given by the factor λ , being close to 1 for deterministic systems and close to zero for stochastic systems. The applied determinism test shows considerably high values for the deterministic factors (see Table 3) and this can be considered as pointing out an indication for deterministic processes. However, this result should be tested with other calculations.

Once the degree of determinism of the system is estimated, the correlation dimension is used to quantify how deterministic a dynamical system is when it is known that

Fig. 3 Average pointwise correlation dimension, $PD^{(2)}$ for Nov. 01, 1988–Feb. 10, 1989 (a) and for Nov. 01, 2002–Feb. 10, 2003 (b)



those systems it is. The correlation dimension $D^{(2)}$ is related to the scaling properties of spatial correlations between attractor points and how many degrees of freedom play a role, that is, how much sensitivity to the initial conditions is present (Eckmann and Ruelle 1985) giving a lower limit for the number of independent variables or degree of freedom involved in the system and then the minimum number of ordinary differential equations needed to fully describe the system. In order to describe quantitatively the complexity of the dynamics of time series associated with the fluctuations of f0F2, we calculated the pointwise correlation dimensions, $PD^{(2)}$, (Farmer and Ott 1983) as a locally alternative variant of the correlation dimension $D^{(2)}$ to characterize the chaos regime based on the fact that local dimension estimations have the property that they can be used with non-stationary data. The way to calculate the pointwise dimension is calculating the probability p_i to find points in a neighborhood of a point χ_i with size r in the state-space (r is the radius

of a state-space neighborhood around χ_i). The pointwise dimension is calculated over an embedding dimension range centered on the optimum value of the embedding dimension. The used embedding dimension range was between 2 and 10, which is considered large enough. The scanning resolution or the radius of a state-space neighborhood around every point was 10 ($r = 10$). The calculated pointwise correlation dimension is the average of all $PD^{(2)}(\chi_i)$ over all points of the state-space. In Table 3 the calculated average of pointwise correlation dimensions, $PD^{(2)}$ is shown.

The optimum embedding dimension value was estimated from the false nearest neighbors' method (Kennel et al. 1992). In the state-space reconstruction, the optimum embedding dimension, according to Taken's theorem, is an upper bound of the correlation dimension, that is, the embedding dimension must be larger or at least, equal than the dimension of the attractor. The calculated embedding dimension for the two analyzed time series was 5, which agreed

according to the obtained values for the dimensions (3.30 and 2.99) with the mentioned condition.

4 Discussion and conclusions

The reconstructed state-space using Takens's delay-embedded theorem suggests the presence of an attractor region, as well as the obtained values for the maximal Lyapunov exponent, Λ , and deterministic factor, λ , seems to confirm the existence of a chaotic-deterministic process underlying the generation of the f0F2 fluctuations. The dimension corresponds to the number of degrees of freedom that are activated in a process. It is therefore a lower limit to the number of involved system of variables. By having the dimension (in our case calculated from pointwise dimension) it is possible to have an idea about the lower limit of the number of degrees of freedom of the system and therefore, how many variables are activated. The calculated pointwise correlation dimension, $PD^{(2)}$, indicates a low dimensional chaotic deterministic process that can be characterized by a relatively low number of exited degrees of freedom and consequently a relatively low minimum number of independent non-linear equations needed to specify the evolution of the f0F2 ionospheric layer.

On the other hand, the calculated pointwise correlation dimension, $PD^{(2)}$, shows a little difference between the two analyzed time series, that is, different strangeness of the corresponding attractors in its state-space. In this sense, Fig. 3 shows the two profiles of the average pointwise correlation dimension. Is noticeable the greater variability for Nov. 01, 1988-Feb. 10, 1989 f0F2 fluctuations time series (Fig. 3a). Sudden leaps of the average pointwise correlation dimension, indicate changes in the system's dynamic complexity, showing more complexity in the Nov. 01, 1988-Feb. 10, 1989 f0F2 fluctuations time series, which is associated with the ascending phase of the solar cycle 22.

It is attractive for us, to find such a difference for the two f0F2 profiles corresponding to different phases of the solar cycles also with different ascending and descending slopes. This fact can be an indicator of a natural phenomenon (fluctuations of the free electron density in the f0F2 ionospheric layer) with sensitive dependence on the solar cycle stage, in this case, different ascending and descending phases of the solar cycles 22 and 23. This can be in consonance with many cited and controversial papers regarding the strong connection between the solar cycles' factors and its influence on the Earth's environment (example Friis-Christensen and Lassen 1991).

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Declarations

Declaration of Competing Interest The author declares that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. The data used for this contribution is public data and it is available at the addresses cited in the manuscript.

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