



# Constraints on the equation of state from the stability condition of neutron stars

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**Abstract** The stellar equilibrium and collapse, including mainly white dwarfs, neutron stars and super massive stars, is an interplay between general relativistic effects and the equation of state of nuclear matter. In the present work, we use the Chandrasekhar criterion of stellar instability by employing a large number of realistic equations of state (EoSs) of neutron star matter. We mainly focus on the critical point of transition from stable to unstable configuration. This point corresponds to the maximum neutron star mass configuration. We calculate, in each case, the resulting compactness parameter,  $\beta = GM/c^2R$ , and the corresponding effective adiabatic index,  $\gamma_{cr}$ . We find that there is a model-independent relation between  $\gamma_{cr}$  and  $\beta$ . This statement is strongly supported by the large number of EoSs, and it is also corroborated by using analytical solutions of the Einstein field equations. In addition, we present and discuss the relation between the maximum rotation rate and the adiabatic index close to the instability limit. Accurate observational measurements of the upper bound of the neutron star mass and the corresponding radius, in correlation with present predictions, may help to impose constraints on the high density part of the neutron star equation of state.

**Keywords** Neutron stars · Nuclear equation of state · Stability condition · Adiabatic index

## 1 Introduction

The discovery of the in-spiraling and coalescence of a binary neutron star system (GW170817), by the Laser Interferometer Gravitational-wave Observatory (LIGO, VIRGO) (on 17th August 2017), opened a new window to exploring the neutron star equation of state at high densities (Abbott et al. 2017a,b). In particular, just after the mentioned discovery, a significant effort was made in constraining the upper as well as the lower limit of the maximum neutron star mass and the corresponding radius. In any case, one of the main ingredients is the compactness of the neutron star, which is expected to play an important role in the stability and dynamics processes of neutron stars. It is well known that the maximum mass, which corresponds to the most compact configuration, is the border between the stable–unstable configuration. Very useful and robust information can be gained by studying this extreme case.

The stability of relativistic stars has been studied extensively in the past (Chandrasekhar 1964a,b; Friedman and Stergioulas 2013; Glendenning 2000; Haensel et al. 2007; Harrison et al. 1965; Shapiro and Teukolsky 1983; Weinberg 1972; Zeldovich and Novikov 1978) while various approaches have been used in order to treat this problem (Bardeen et al. 1966). In particular, firstly one can solve the Tolman–Oppenheimer–Volkoff (TOV) (Oppenheimer and Volkoff 1939; Tolman 1939) equations (which provide the equilibrium configuration) for either numerically derived equation of state or trying to find analytical solutions. In any case, both of the solutions lead to an infinite number of configurations. Secondly, one possibility is the use of the criterion of Chandrasekhar (1964a,b) in order to identify, in each case, the stable configuration as well as the interface between stable and unstable configuration.

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Moreover, at a given density, there is an important parameter, called the adiabatic index, which in particular characterizes the stiffness of the equation of state (Harrison et al. 1965; Haensel et al. 2007; Misner et al. 1973; Bludman 1973a,b; Ipson 1970; Glass and Harpaz 1983; Lindblom and Detweiler 1983; Hiscock and Lindblom 1983; Gaertig and Kokkotas 2009). The instability criterion of Chandrasekhar (1964a,b) strongly depends on this parameter (adiabatic index). One of the main motivations of the present work is to examine the possibility to impose constraints on the realistic neutron star equations of state via the instability condition of Chandrasekhar.

In particular, we employ an extended group of realistic equations of state based on various theoretical nuclear models. The abbreviated names of these equations of state are: MDI (Madappa et al. 1997; Moustakidis and Panos 2009), NLD (Gaitanos and Kaskulov 2013, 2015), HHJ (Heiselberg and Hjorth-Jensen 2000), Ska, SkI4 (Chabanat et al. 1997; Farine et al. 1997), HLPS (Hebeler et al. 2013), SCVBB (Sharma et al. 2015), BS (Balberg and Shapiro 2000), BGP (Bowers et al. 1975), W (Walecka 1974), DH (Douchin and Haensel 2001), BL (Bombaci and Logoteta 2018), WFF1, WFF2 (Wiringa et al. 1988), APR (Akmal et al. 1998) and PS (Pandharipande and Smit 1975). All of them satisfy, at least marginally, the observed limit of  $M = 1.97 \pm 0.04 M_{\odot}$  (PRS J1614-2230, Demorest et al. 2010) and  $M = 2.01 \pm 0.04 M_{\odot}$  (PSRJ0348+0432, Antoniadis et al. 2013). Actually, at the moment, the most robust constraints on the neutron star equations of state are based on the measurements of the lower bound of the maximum neutron star mass. Strictly speaking, the suggested equations of state, which do not reproduce the higher measurement of neutron star mass, must be excluded.

It is also well known that the rapidly rotating neutron stars can be used in order to determine the equation of state (see Ref. (Haensel et al. 2007) and the references therein). In particular, the maximum rotating frequency  $f_{\max}$  (Keplerian frequency) depends both on the gravitational mass  $M_{\max}$  and the EoS. Until this moment, the fastest known pulsar, PSR J1748-244ad, is rotating with a frequency of 716 Hz (Hessels et al. 2006). While the theoretical predicted values for  $f_{\max}$  are much more higher than 716 Hz, there is a lack of neutron stars rotating faster than this value. This is an open problem and obviously additional theoretical assumptions must be made in order to solve it.

In the present work we concentrate on the dependence of the effective critical adiabatic index to the compactness of neutron star, for each equation of state. We mainly focus on the interface between stable and unstable configuration which corresponds to the maximum-mass configuration. This region is very important, since it is directly related with the high density part of the neutron star equation of state. Moreover, we propose an additional method

to constrain the equations of state with the help of accurate measurements of the maximum neutron star mass and/or compactness. Finally, we make an effort to relate the maximum rotating frequency  $f_{\max}$  with the critical adiabatic index and the bulk properties corresponding to the maximum-mass configuration of a non-rotating (static) neutron star (including the maximum mass  $M_{\max}^{\text{stat}}$ , the corresponding radius  $R_{\max}^{\text{stat}}$  and the compactness parameter  $\beta_{\max}^{\text{stat}}$ ) so as to indicate how observational measurements of highly rotating neutron stars may impose constraints on the EoS.

The article is organized as follows: In Sect. 2 we present the TOV equations, the Chandrasekhar instability criterion, the definition of the relevant adiabatic indices and we briefly present four relevant analytical solutions of the TOV equations. In Sect. 3 we briefly discuss the maximum rotating frequency in connection with the maximum-mass configuration. The results are presented and discussed in Sect. 4. Section 5 contains the concluding remarks of the study. Appendix contains relevant analytical approximations for the critical adiabatic index.

## 2 The stability criterion and the adiabatic indices

The starting point for determining the mechanical equilibrium of neutron star matter is the well-known Tolman–Oppenheimer–Volkoff (TOV) equations (Glendenning 2000; Oppenheimer and Volkoff 1939; Shapiro and Teukolsky 1983; Tolman 1939). This set of differential equations describes the structure of a neutron star. For a static spherical symmetric system, the metric reads as follows (Glendenning 2000; Shapiro and Teukolsky 1983):

$$ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

and the corresponding TOV equations take the form

$$\frac{dP(r)}{dr} = -\frac{G\mathcal{E}(r)M(r)}{c^2 r^2} \left(1 + \frac{P(r)}{\mathcal{E}(r)}\right) \times \left(1 + \frac{4\pi P(r)r^3}{M(r)c^2}\right) \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1}, \quad (2)$$

$$\frac{dM(r)}{dr} = \frac{4\pi r^2}{c^2} \mathcal{E}(r). \quad (3)$$

By introducing a realistic EoS for the neutron star (e.g. a dependence on the form  $P = P(\mathcal{E})$ ) we solve numerically the TOV equations. Of course, one can try to find analytical solutions of the TOV equations. However, it is worth pointing out that using the analytical solutions, although each one of them describes equilibrium configurations, is not sufficient to tell us if it corresponds to stable ones (Tolman 1939);

this is the case also for any numerical solution. Straightforwardly speaking, any unstable solution is not of physical interest.

Chandrasekhar, in order to solve the instability problem, introduced a criterion for dynamical stability based on the variational method (Chandrasekhar 1964a). To be more specific, the averaged adiabatic index is defined as follows (Merafina and Ruffini 1989; Moustakidis 2017; Negi and Durgapal 2001):

$$\langle \gamma \rangle = \frac{\int_0^R e^{(\lambda+3\nu)/2} \gamma(r) \frac{P}{r^2} \left( \frac{d}{dr} (r^2 e^{-\nu/2} \xi(r)) \right)^2 dr}{\int_0^R e^{(\lambda+3\nu)/2} \frac{P}{r^2} \left( \frac{d}{dr} (r^2 e^{-\nu/2} \xi(r)) \right)^2 dr}, \tag{4}$$

and the effective critical adiabatic index by

$$\begin{aligned} \gamma_{\text{cr}} = & \left[ -4 \int_0^R e^{(\lambda+\nu)/2} r \left( \frac{dP}{dr} \right) \xi^2 dr \right. \\ & + \int_0^R e^{(\lambda+\nu)/2} \left( \frac{dP}{dr} \right)^2 \frac{r^2 \xi^2}{P + \mathcal{E}} dr \\ & \left. - \frac{8\pi G}{c^4} \int_0^R e^{(3\lambda+\nu)/2} P(P + \mathcal{E}) r^2 \xi^2 dr \right] \\ & \times \left( \int_0^R e^{(\lambda+3\nu)/2} \frac{P}{r^2} \left( \frac{d}{dr} (r^2 e^{-\nu/2} \xi) \right)^2 dr \right)^{-1}. \tag{5} \end{aligned}$$

The Chandrasekhar stability condition leads to the inequality

$$\langle \gamma \rangle \geq \gamma_{\text{cr}}, \tag{6}$$

while the case  $\langle \gamma \rangle = \gamma_{\text{cr}}$  corresponds to the onset of the instability (Merafina and Ruffini 1989; Moustakidis 2017). According to Eqs. (4) and (5) the averaged and the critical adiabatic indices are functional of the function  $\xi(r)$  as well as of the compactness parameter  $\beta$ . In particular, the lagrangian displacement away from equilibrium has the form  $\zeta(r) = \xi(r)e^{-i\sigma t}$ , where  $\sigma$  is the pulsation frequency of the oscillations. It is obvious from the lagrangian displacement that  $\sigma^2$  can take both positive and negative values. To be more specific, a positive value of  $\sigma^2$  corresponds to stable configuration while a negative to unstable one (Chandrasekhar 1964a; Kokkotas and Ruoff 2001; Merafina and Ruffini 1989). It is worth pointing out that the stability condition (6) expresses a minimal and not just an external principle (Chandrasekhar 1964a). Moving on to the trial functions that appear in Eqs. (4) and (5), it is widely known that there are infinite numbers of them. However, the most frequently used are the following (where the names mentioned in the paper have also been indicated):

$$\xi(r) = r e^{\nu/2}, \tag{TF-1} \tag{7}$$

$$\xi(r) = r e^{\nu/4}, \tag{TF-2} \tag{8}$$

$$\xi(r) = r(1 + a_1 r^2 + a_2 r^4 + a_3 r^6) e^{\nu/2}, \tag{TF-3} \tag{9}$$

$$\xi(r) = r. \tag{TF-4} \tag{10}$$

Now, considering an adiabatic perturbation, the adiabatic index  $\gamma$  is defined as follows (Chandrasekhar 1964a; Merafina and Ruffini 1989):

$$\gamma \equiv \frac{P + \mathcal{E}}{P} \left( \frac{\partial P}{\partial \mathcal{E}} \right)_S = \left( 1 + \frac{\mathcal{E}}{P} \right) \left( \frac{v_s}{c} \right)^2, \tag{11}$$

where the derivation is performed at constant entropy  $S$ . Moreover,  $(v_s/c)_S = \sqrt{(\partial P/\partial \mathcal{E})_S}$  is the speed of sound in units of speed of light. The speed of sound is an important quantity, related directly with the stiffness of the equation of state, and plays a significant role in the maximum-mass configurations. In general, since the adiabatic index is a function of the baryon density, exhibits radial dependence and consequently, provides local information for each neutron star configuration. Its values vary from 2 to 4 in most of the neutron stars' equations of state (Haensel et al. 2007). In the specific case of a polytropic equation of state, the adiabatic index is a constant. The effective adiabatic indices,  $\langle \gamma \rangle$  and  $\gamma_{\text{cr}}$ , in contrast to  $\gamma$  (Eq. (11)) have a global character. Both of them are directly related with the neutron star equation of state as well as with the strength of the gravitational field (see also Bludman 1973a,b; Herrera et al. 1989; Chan et al. 1994; Ipser 1970; Merafina and Ruffini 1989; Moustakidis 2017; Negi and Durgapal 1999, 2001; Sharif and Yousaf 2015; Yousaf and Bhatti 2016).

Chandrasekhar, using the Schwarzschild constant-density interior solution (see below for more details as regards this analytical solution), found that in the Newtonian limit the stability is ensured when (Chandrasekhar 1964a)

$$\langle \gamma \rangle \geq \gamma_{\text{cr}} = \frac{4}{3} + \frac{19}{42} 2\beta. \tag{12}$$

He employed the approximation that the adiabatic index  $\gamma$  is a constant throughout the star (Chandrasekhar 1964a). In addition, Chandrasekhar (1964a), in the framework of the post-Newtonian approximation using relativistic polytropes, found the relation

$$\gamma_{\text{cr}} = \frac{4}{3} + C \left( \frac{P_c}{\mathcal{E}_c} \right), \tag{13}$$

where  $C = 1.8095, 2.2615, 2.4968, 2.6325$  corresponds to the polytropic index  $n = 0, 1, 2, 3$ , respectively, and  $P_c, \mathcal{E}_c$  are the central values of pressure and energy density. It should be noted that the ratio  $P_c/\mathcal{E}_c$  can also be mentioned as a relativistic index and is closely related with the compactness  $\beta$  (see the extended discussion in Sect. 4). Similar results have also been found by Tooper in a series of papers (Tooper 1964, 1965). Moreover, Bludman (1973a,b) studied

the stability of general relativistic polytropes and provided the formula

$$\gamma_{\text{cr}} \simeq \frac{4}{3} + 1.73 \left( \frac{P_c}{\mathcal{E}_c} \right) - 0.31 \left( \frac{P_c}{\mathcal{E}_c} \right)^2. \quad (14)$$

It is worth extending all these previous studies in order to examine the dependence of  $\gamma_{\text{cr}}$  on the compactness parameter  $\beta_{\text{max}}$  (as well as on the ratio  $P_c/\mathcal{E}_c$ ) close to the instability limit, which corresponds to the maximum-mass configuration. Although the study concerning the Newtonian or post-Newtonian case is universal, meaning that for low values of  $\beta$  ( $\beta \ll 1$ ) the dependence of  $\gamma_{\text{cr}}$  is almost insensitive to the details of the EoS, this is not the case for high values of  $\beta$ . In this case the structure of a neutron star and the corresponding values of  $\gamma_{\text{cr}}$  are very sensitive on the EoS. Since, especially for high values of the densities, the uncertainty on pressure–energy dependence is appreciable, we expect an influence on the values of  $\gamma_{\text{cr}}$ . In view of the above, we conclude that possible constraints on  $\beta_{\text{max}}$  may impose constraints on the high density behavior of the neutron star equations of state.

We can also study the stability of the equilibrium configuration by using the general properties of the central density as well as those of the mass–radius relation (Weinberg 1972). In this case, the configuration is stable when the inequality  $dM/d\mathcal{E}_c > 0$  holds. Actually, this condition, due to its simplicity, has been used extensively in the literature. However, it needs to be noted that this condition is just necessary but not sufficient and consequently, it is weak compared to the criterion (6).

Now we will briefly discuss four analytical solutions of the TOV equations. In the case of the Schwarzschild constant-density interior solution (hereafter called uniform), the density is constant throughout the star (Schutz 1985; Weinberg 1972). The physical realization of the Tolman VII solution has been studied in detail in Ragoonundun and Hobill (2015). In this solution causality is ensured for  $\beta < 0.2698$ . However, useful information and predictions are taken when applied for even higher values of  $\beta$  (see for example Lattimer and Prakash 2001; Moustakidis 2017; Lattimer and Prakash 2005; Sotani and Kokkotas 2018). The Buchdahl solution (Buchdahl 1959, 1967) is applicable only for low values of the compactness ( $\beta \leq 0.2$ ). Actually, it forms a *bridge* which connects the Newtonian and post-Newtonian limits with the relativistic one (Lattimer and Prakash 2001; Moustakidis 2017; Papazoglou and Moustakidis 2016). The Nariai IV solution (Nariai 1950, 1951, 1999), although being very complicated, provides useful insight because it is one of the physically interesting solutions.

In general, the selected solutions exhibit realistic behavior and can be used as a guide to establish some universal approximations. In particular, while the unrealistic uniform solution has been used by Chandrasekhar (1964a) in

order to prove his famous expression (12), its main drawback is the infinite value of the speed of sound. In the case of Tolman VII solution, useful information and predictions are taken when applied for high values of  $\beta$ . Thus, Lattimer and Prakash (2005) have demonstrated, using the Tolman VII solution, that the largest measured mass of a neutron star establishes an upper bound to the energy density of observable cold matter. Moreover, while in the Nariai IV solution the causality was ensured for  $\beta < 0.2277$ , its extension for higher values was applied successfully (Lattimer and Prakash 2005; Moustakidis 2017; Papazoglou and Moustakidis 2016).

### 3 Maximum mass and maximum rotation frequency

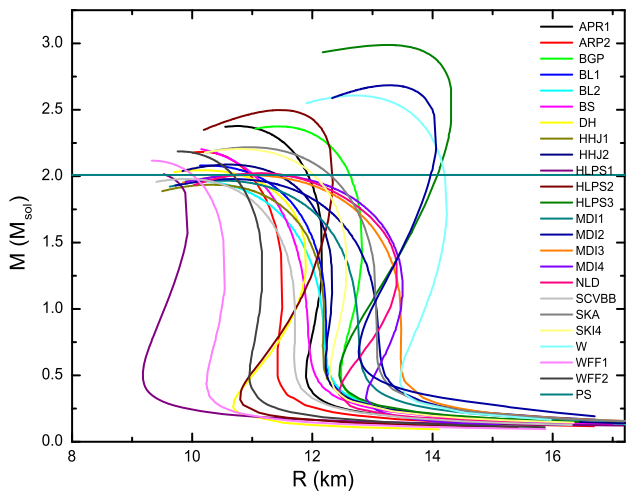
It is well known that rotation increases the maximum mass ( $M_{\text{max}}^{\text{stat}}$ ) of a corresponding stationary neutron star. In this case, we face two extreme configurations: a) maximum mass  $M_{\text{max}}^{\text{rot}}$  and b) maximum rotation frequency  $f_{\text{max}}$  (known as Kepler frequency) (Haensel et al. 2007). These configurations do not coincide but since they are very close to each other (with high accuracy) we do not differentiate them. Moreover, it was found that the maximum frequency can be expressed, with high accuracy, in terms of mass and radius of the non-rotating configuration with the maximum mass (see Haensel et al. 2007 and the references therein). A precise formula which relates  $f_{\text{max}}$  with the maximum mass  $M_{\text{max}}^{\text{stat}}$  and the compactness parameter  $\beta_{\text{max}}^{\text{stat}}$  of the static maximum-mass configuration was found by Haensel et al. (1999, 2016) and Lasota et al. (1996),

$$f_{\text{max}} \simeq 15.125 \beta_{\text{max}}^{3/2} (1 + 1.6164 \beta_{\text{max}}) \left( \frac{M_{\odot}}{M_{\text{max}}^{\text{stat}}} \right) \text{ kHz}. \quad (15)$$

It is worth to point out the strong dependence of  $f_{\text{max}}$  on  $\beta_{\text{max}}^{\text{stat}}$  and consequently, via the adiabatic index, on the high density dependence area of the EoS. The above expression can be used to constrain an absolute lower bound of the maximum frequency of rigid rotation (for example by measuring the upper bound on the surface red-shift of a non-rotating neutron star) and consequently to impose useful constraints on the EoS and vice versa.

## 4 Results and discussion

We employ a large number of published realistic equations of state for neutron star matter based on various theoretical nuclear models. We calculate both the effective averaged and the critical adiabatic indices for each configuration. Mainly, we are interested in the maximum mass, the corresponding radius, the ratio  $P_c/\mathcal{E}_c$  and the corresponding



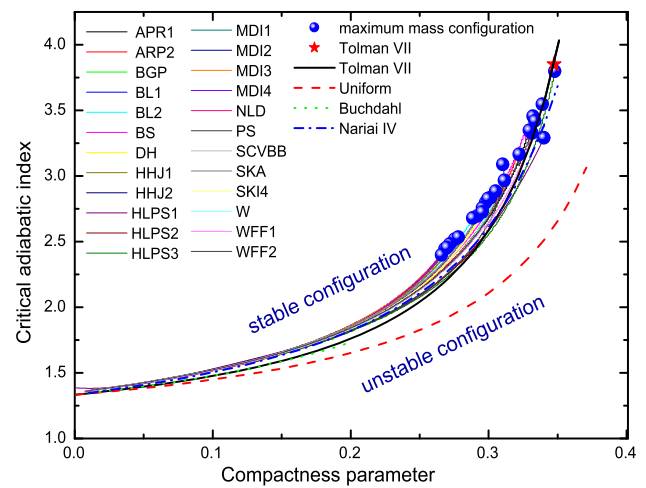
**Fig. 1** Mass–radius trajectories for the selected EoSs

compactness  $\beta$  for each case. For each configuration we determine  $\langle \gamma \rangle$  and  $\gamma_{cr}$ . The onset of instability is found from the equality  $\langle \gamma \rangle = \gamma_{cr}$ . The corresponding compactness parameter, is denoted as  $\beta_{max}$ .

There is also a second criterion, which defines the stability limit according to the equality  $dM/d\mathcal{E}_c = 0$ , providing an additional value of  $\beta$  for the maximum-mass configuration. Now, in general, since  $\langle \gamma \rangle$  and  $\gamma_{cr}$  are functionals of the trial function  $\xi(r)$ , we expect that the calculated values of  $\beta$ , for the two methods, will not coincide. In these cases, we will consider as the most optimal trial function  $\xi(r)$  the one that produces values of  $\beta$  as close as possible to the second method. In particular, we found that the trial function (7) (indicated by TF-1) is the optimal one, leading to an error, in most of the cases, of less than 1%.

In Fig. 1 the radius–mass relation is drawn using the selected EoSs. One can see that the majority of the EoSs reproduce the recent observation of two-solar masses neutron stars. It is obvious that the various predictions cover a wide range of the maximum neutron star masses and the corresponding radii.

In Fig. 2 we display the dependence of  $\gamma_{cr}$  as a function of the compactness parameter  $\beta$  for all the employed EoSs by using the optimal trial function (7). The results of the four analytical solutions have also been included for comparison. The blue dots correspond to all configurations with neutrally stable equilibrium as results of the equality  $\langle \gamma \rangle = \gamma_{cr}$ . These configurations correspond to the ones with the largest possible central density reachable for stable configuration of a given mass. In the case of the Tolman VII solution, the results using the trial function TF-1 (7) have also been included. In this case, the onset of instability is indicated by the red star and corresponds to  $\beta = 0.3475$  and  $\gamma_{cr} = 3.85$ . It is remarkable that the use of the Tolman VII solution, leads to results very close to the predictions by using realistic equations of state. The other two analytical solutions



**Fig. 2** The critical adiabatic index,  $\gamma_{cr}$ , as a function of the compactness parameter  $\beta$ , for the selected EoSs (using the trial function TF-1 (7)). The results of the four analytical solutions, using for consistency the trial function TF-1 (7), have also been included for comparison (for more details see text). The blue dots correspond to the onset of instability as a result of the equality  $\langle \gamma \rangle = \gamma_{cr}$ . The onset of instability for the Tolman VII solution is indicated by a red star (for the TF-1)

(Buchdahl’s and Nariai IV), in each case, lead to a stable configuration (Moustakidis 2017). The uniform solution is always used as a guide for a stable configuration mainly for low values of the compactness  $\beta$  (see Eq. (12)).

The most distinctive feature in Fig. 2 is the remarkable unanimity of all equations of state and consequently the occurrence of a model-independent relation between  $\gamma_{cr}$  and  $\beta_{max}$ , at least for any stable configuration. The above finding is clearly expected for low values of the compactness  $\beta$  (since all equations of state converge for low values of the density). However, at high densities of the equations of state, where there is a considerable uncertainty, this result was not obvious. In any case, as a consequence of the convergence, both for low and high values of the compactness, the majority of the points indicate the onset of the instability located in the mentioned trajectory. In particular, we found that the simple expression

$$\gamma_{cr}(\beta) = \gamma_0 + A_1 e^{\beta/t_1} \tag{16}$$

reproduces very well the numerical results due to the use of realistic equations of state. Equation (16) is the relativistic expression for the critical value of the adiabatic index and can be considered as the relativistic generalization of the post-Newtonian approximation (12). The parametrization is provided in Table 1.

The results of the analytical solutions, in each case, can be parameterized according to the expression (see details in Table 1)

$$\gamma_{cr}(\beta) = \gamma_0 + A_1 e^{\beta/t_1} + A_2 e^{\beta/t_2}. \tag{17}$$

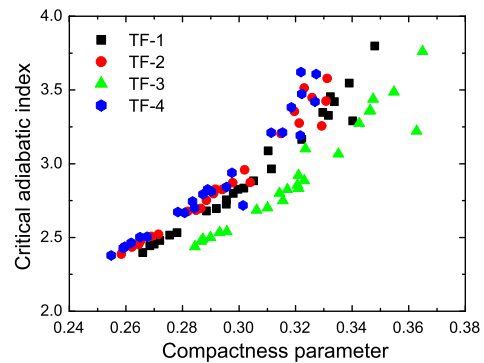
**Table 1** The parametrization of the analytical formulas (16), (17) and (18) is using realistic equations of state as well as four analytical solutions (using the trial function TF-1 (7)). The case called *Realistic EoS* reproduces the averaged results of the realistic equations of state

Solution	$\gamma_0$	$A_1$	$t_1$	$A_2$	$t_2$	$\gamma_0$	$C_1$	$C_2$
Realistic EoS	1.23333	0.10425	0.11007					
Tolman VII	1.18654	0.14938	0.15293	0.00011	0.03731	1.32055	2.45877	-0.36691
Buchdahl	1.04258	0.28285	0.27558	0.00792	0.07695	1.33344	2.25592	-1.28137
Nariai IV	1.13470	0.20200	0.16781	0.00015	0.04016	1.33094	2.68839	-0.45055
Uniform	1.18955	0.14587	0.17682	0.00009	0.04140	1.32743	1.94115	-0.08660

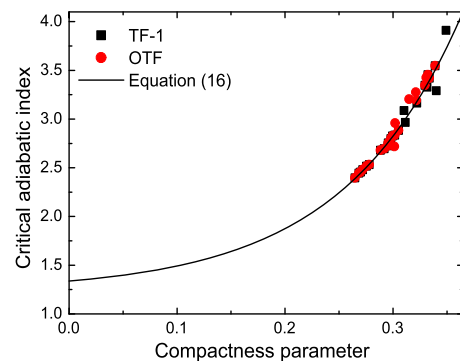
Obviously, there is a small deviation between the results of the realistic equations of state and the analytical solutions of Tolman VII, Nariai IV and Buchdahl. It is worth noticing that the Tolman VII solution reproduces very well the numerical results, especially for high values of the compactness. In general, analytical solutions lead to lower values of the adiabatic index  $\gamma_{cr}$ , compared to the realistic EoS. In particular, the uniform solution provides the lower limit for  $\gamma_{cr}$ , especially for high values of the compactness and close to the instability limit. However, the general trend is similar and useful insight can be gained concerning the reliability of analytical solutions. The stable configurations, independently of the equation of state, correspond to a universal relation between  $\gamma_{cr}$  and  $\beta$ . One can safely conclude that  $\gamma_{cr}$  is an intrinsic property of neutron stars (likewise the parameter  $\beta$ ) which reflects the relativistic effects on their structure. In particular,  $\gamma_{cr}$  exhibits a linear dependence with  $\beta$  in the Newtonian and post-Newtonian regime but a more complicated behavior in the relativistic regime (see also the appendix).

Actually, the above finding may help to impose constraints to the equation of state of neutron star matter. For example, the accurate and simultaneous observation of a possible maximum neutron star mass and the corresponding radius will constrain the maximum value of the compactness and consequently the maximum value of the adiabatic index  $\gamma_{cr}$ . In any case, insight may be gained by the use of Eq. (16) with the parametrization given in Table 1 (realistic EoS).

In order to clarify further the effects of the trial functions  $\xi(r)$  on the results, we present Fig. 3. In particular, in Fig. 3 we display the dependence of the critical adiabatic index,  $\gamma_{cr}$ , which corresponds to the onset of instability ( $\gamma_{cr}=\langle\gamma\rangle$  at this point), as a function of the compactness parameter  $\beta_{max}$  using the selected trial functions (7), (8), (9) and (10). With regard to the trial function (9), we use the parametrization  $a_1 = 1/10R^2$ ,  $a_2 = 1/5R^4$  and  $a_3 = 3/10R^6$ . The most distinctive feature, in this case, is the occurrence of an almost linear dependence (in the region under study, e.g. on the maximum-mass configuration) between the adiabatic index and the compactness  $\beta_{max}$ . Obviously, the use of the trial function  $\xi(r)$  affects mainly the values of  $\gamma_{cr}$  (for the same  $\beta_{max}$ ) but not the linear dependence.



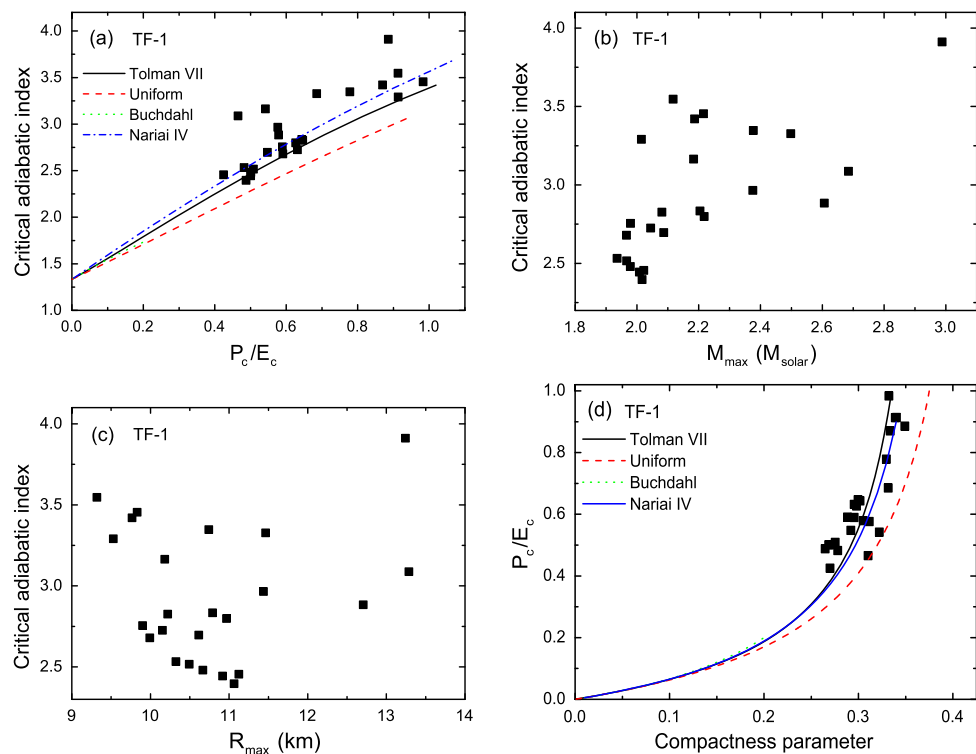
**Fig. 3** The critical adiabatic index  $\gamma_{cr}$  as a function of the compactness parameter  $\beta$  for the selected EoSs. The points correspond to the onset of instability for the four selected trial functions  $\xi(r)$



**Fig. 4** The critical adiabatic index,  $\gamma_{cr}$ , as a function of the compactness parameter  $\beta$ , for the selected EoSs, using the trial function TF-1 (7) (squares) and the optimal trial function (OTF) in each EoS (dots). Equation (16) (the parametrization is provided in Table 1) which reproduces the numerical results corresponding to the trial function (7) is also included

Moreover, in Fig. 4 we display the  $\gamma_{cr}$ , as a function of the compactness parameter  $\beta$ , for the selected EoSs, using the trial function TF-1 (7) and the optimal trial function (OTF) in each EoS, which corresponds to the one with the smallest error. Equation (16), which reproduces the numerical results corresponding to the trial function (7), is also included. Obviously, using the optimal trial function (OTF) in each EoS the rearrangement of the results becomes more ordered. However, the deviation of using the trial function

**Fig. 5** (a) The critical adiabatic index,  $\gamma_{\text{cr}}$ , as a function of the ratio  $P_c/\mathcal{E}_c$  for the selected EoSs (the dots correspond to the onset of instability in each case) and for the trial function TF-1 (7). The results of the four analytical solutions have also been included for comparison, (b) the  $\gamma_{\text{cr}}$  as a function of the maximum mass,  $M_{\text{max}}$ , for the EoSs, (c) the  $\gamma_{\text{cr}}$  as a function of the radius,  $R_{\text{max}}$ , corresponding to  $M_{\text{max}}$  for the selected EoSs, and (d) the ratio  $P_c/\mathcal{E}_c$  as a function of the compactness parameter,  $\beta$ , for the selected EoSs, while the results of the four analytical solutions have also been included for comparison



TF-1 (which is the optimal one in the most of the cases) is negligible.

In Fig. 5(a) we display the dependence of  $\gamma_{\text{cr}}$  on the ratio  $P_c/\mathcal{E}_c$  (which corresponds to the maximum-mass configuration). The symbols correspond to the results originating from the use of realistic equations of state while the results of the four analytical solutions have also been included for comparison. In general, in the case of realistic equations of state,  $\gamma_{\text{cr}}$  is an increasing function of the ratio  $P_c/\mathcal{E}_c$  without obeying specific formulas. However, we found that the expression

$$\gamma_{\text{cr}} = \gamma_0 + C_1 \left( \frac{P_c}{\mathcal{E}_c} \right) + C_2 \left( \frac{P_c}{\mathcal{E}_c} \right)^2 \quad (18)$$

reproduces very well the numerical results of the analytical solutions. The parameters  $\gamma_0$ ,  $C_1$ ,  $C_2$  are displayed in Table 1. In Fig. 5(b) the dependence of  $\gamma_{\text{cr}}$  is displayed with respect to  $M_{\text{max}}$ . In Fig. 5(c) we plot  $\gamma_{\text{cr}}$  as a function of the radius corresponding to the maximum-mass configuration,  $R_{\text{max}}$ . Obviously, in these cases, the dependence is almost random and consequently is unlikely to impose constraints from these kind of correlations.

It is known that for low values of  $\beta$  (in the framework of Newtonian and post-Newtonian approximation) there is a very simple and universal linear correlation between  $\beta$  and the ratio  $P_c/\mathcal{E}_c$ . In particular, in the case of the analytical solutions of the TOV equations (uniform, Tolman VII, Buchdahl's and Nariai IV) we get in each case, by employing a

Taylor expansion, the approximated simple relation

$$\frac{P_c}{\mathcal{E}_c} \simeq \frac{\beta}{2}. \quad (19)$$

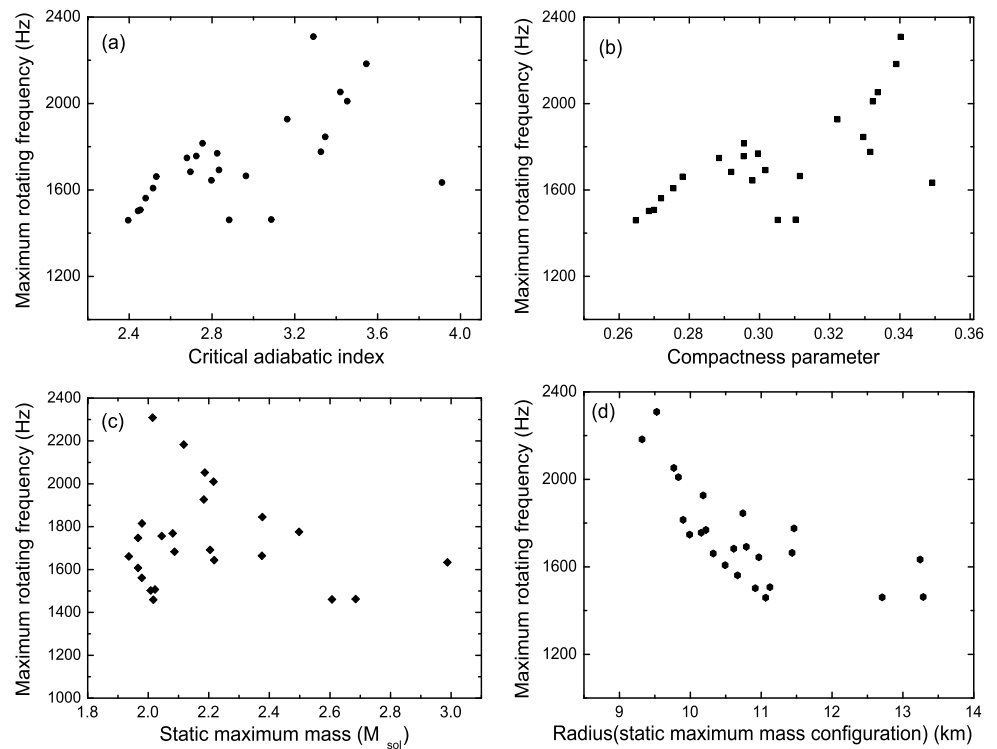
Moreover, in the case of the Newtonian limit e.g. using the Lane–Emden equation with the polytropic equation of state  $P = K(\mathcal{E}/c^2)^\Gamma = K(\mathcal{E}/c^2)^{1+\frac{1}{n}}$  (Shapiro and Teukolsky 1983) we found

$$\frac{P_c}{\mathcal{E}_c} = \frac{\beta}{2} \mathcal{F}(\xi_0, n), \quad (20)$$

where  $\mathcal{F}(\xi_0, n)$  is a function of  $\xi_0$  (where  $\theta(\xi_0) = 0$ ) and the polytropic index  $n$ . More precisely, we found that for  $n = 0, 0.5, 1, 1.5, 2, 3, 4$  (correspondingly  $\Gamma = \infty, 3, 2, 5/3, 3/2, 4/3, 5/4$ ) the function is  $\mathcal{F}(\xi_0, n) = 1, 0.97, 1, 1.077, 1.204, 1.709, 3.332$ , respectively. In conclusion, for  $0 < n < 2$  we get  $\mathcal{F}(\xi_0, n) \simeq 1$ .

Since it is worth examining this dependence in the relativistic limit, we display in Fig. 5(d) the dependence of  $P_c/\mathcal{E}_c$  with respect to the compactness parameter  $\beta_{\text{max}}$ . Firstly, we can see that quantities originating from the use of realistic EoSs obey a general trend. A similar trend is obtained by employing the analytical solutions (Moustakidis 2017). In particular, the Tolman VII and Nariai IV solutions reproduce very well the results of realistic calculations. Consequently, the Tolman VII solution may be used as a guide for an almost universal dependence between  $P_c/\mathcal{E}_c$  and  $\beta_{\text{max}}$ . That is, in the critical point between stable and

**Fig. 6** The maximum rotating frequency  $f_{\max}$  for the selected EoSs, as a function: (a) of the critical adiabatic index,  $\gamma_{\text{cr}}$ , (b) of the compactness parameter  $\beta_{\max}^{\text{stat}}$ , which corresponds to the static maximum-mass configuration, (c) of the static maximum mass  $M_{\max}^{\text{stat}}$  and (d) of the radius  $R_{\max}^{\text{stat}}$ , which corresponds to the static maximum-mass configuration



unstable configuration. Moreover, this correlation may help to constrain the maximum value of the ratio  $P_c/\mathcal{E}_c$  and, consequently, the maximum density in the universe with the help of accurate measurements of the maximum value of the compactness.

To be more specific, from recent observations of the GW170917 binary system merger, a method to constrain some neutron stars properties was proposed (Bauswein et al. 2017). In particular, they found that the radius  $R_{\max}$  of the non-rotating maximum-mass configuration must be larger than  $9.6^{+0.15}_{-0.04}$  km. Almost simultaneously, Margalit and Metzger (Margalit and Metzger 2017), combining electromagnetic and gravitational-wave information on the binary neutron star merger GW170817, constrain the upper limit of  $M_{\max}$  according to  $M_{\max} \leq 2.17M_{\odot}$ . The combination of the two suggestions leads to an absolute maximum value of compactness equal to  $\beta_{\max} = 0.333^{+0.001}_{-0.005}$ . The use of this value with the help of Figs. 2 and 5(d) will impose constraints both on the maximum values of the index  $\gamma_{\text{cr}}$  and the ratio  $P_c/\mathcal{E}_c$ . According to Eq. (16), a constraint on the  $\gamma_{\text{cr}}$  can be imposed,  $\gamma_{\text{cr,max}} = 3.381^{+0.020}_{-0.095}$ , correspondingly to  $\beta_{\max}$ . Moreover, a large number of realistic equations of state must be excluded. Some previous and recent efforts to constrain the compactness of neutron stars have been provided in Alsing et al. (2018), Chen and Piekarewicz (2015), Hambaryan et al. (2017), Miller and Lamb (1998), Ravenhall and Pethick (1994), Rosso et al. (2017).

In Fig. 6(a) we display the dependence of the maximum rotating frequency on the critical adiabatic index (it

is worth indicating, in order to avoid any confusion, that in the present study  $M_{\max}^{\text{stat}}$ ,  $R_{\max}^{\text{stat}}$  and  $\beta_{\max}^{\text{stat}}$  correspond to  $M_{\max}$ ,  $R_{\max}$  and  $\beta_{\max}$ , respectively). Obviously, while  $f_{\max}$  is an increasing function of  $\gamma_{\text{crit}}$ , the correlation is not so restrictive. However, the most important finding (see also Fig. 6(b)) is the derivation of an absolute lower upper bound of the maximum rotation rate close to the value of 1460 Hz. The observation of neutron stars rotating with a spin  $f > 1460$  Hz will exclude a number of the selected EoSs. In Fig. 6(b) we display also the dependence of  $f_{\max}$  on  $\beta_{\max}^{\text{stat}}$ , while in Fig. 6(c) the dependence of  $f_{\max}$  on the mass which corresponds to the static maximum-mass configuration is provided. In this case, the dependence is random. However, the dependence of  $f_{\max}$  on the radius, which corresponds to the static maximum-mass configuration, exhibits a more restrictive dependence. In particular,  $f_{\max}$  is a decreasing function of  $R_{\max}^{\text{stat}}$ , i.e. the maximum rotation rate is expected to be observed in small sized neutron stars.

We expect that in the near future, the detectability of radio pulsars will be appreciably increased due to the Square Kilometer Array (SKA) (Bourke et al. 2015). In particular, the SKA is expected to increase the number of measured neutron stars masses by a factor of, at least, 10 and, in this way, to help constraining the existing EoSs (Watts et al. 2016). Information about the radius of neutron stars can also be gained by radio observations through measurements of the moment of inertia. Moreover, the two telescopes Advanced LIGO and VIRGO will be able to detect gravitational waves from the late in-spiraling of binary neutron stars sys-



tems, which are also sensitive to the EoS. These two telescopes are expected to achieve uncertainties of 10%, or less, in radius for the closest detected binaries (Watts et al. 2016). The discovery of a neutron star rotating very fast would impose strong constraints on the EoS. According to the most reliable scenario, the formation of very rapid rotating neutron stars is via spin-up due to accretion. Consequently, the study of X-ray and radio waves will help to confirm the theory that accretion could spin-up stars, close to the instability limit. In this case, we will be able to gain rich information about the neutron stars structure and about the bulk properties, and to impose robust constraints on the EoSs (Watts et al. 2016).

## 5 Concluding remarks

We suggested a new method to constrain the neutron star equation of state by means of the stability condition introduced by Chandrasekhar (1964a). We found that the predicted critical adiabatic index, as a function of the compactness, for most of the equations of state considered here (although they differ considerably at their maximum masses and in how their masses are related to the radii) satisfies a universal relation. In particular, the exploitation of these results leads to a model-independent expression for the critical adiabatic index as a function of the compactness. Equation (16) (with the specific parametrization given in Table 1) reproduces very well this relation. The above finding may be added to the rest of approximately EoS-independent relations (Breu and Rezzolla 2016; Maselli et al. 2017; Ravenhall and Pethick 1994; Silva et al. 2016; Silva and Yunes 2018; Yagi and Yunes 2013a,b, 2017; Yagi et al. 2014). These universal relations break the degeneracies among astrophysical observations and lead to a variety of applications. We also found that observations of high rotating neutron stars may help to impose useful constraints on the EoSs, by using the dependence of the maximum frequency on the compactness parameter corresponding to the maximum-mass configuration of a non-rotating neutron star and consequently on the adiabatic index (instability limit). We state that additional theoretical and observational measurements of the bulk neutron star properties close to the maximum-mass configuration will help to impose robust constraints on the neutron star equation of state or, at least, to minimize the numbers of proposed EoSs.

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## Appendix

The numerical integration of the integrals related with the definition of  $\langle \gamma \rangle$  and  $\gamma_{\text{cr}}$  can easily be performed. However, following this procedure it is difficult to perceive the final results. Actually, this is easy only in some approximated cases, e.g. in the Newtonian and post-Newtonian limit. In the following, we try to generalize the finding of Chandrasekhar (1964a) to even higher values of the compactness where the relativistic effects become important. The expression of the critical adiabatic index, with the help of the TOV equations (2), (3), using the trial function  $\xi(r) = re^{v/2}$  (in order to be consistent with the pioneering work of Chandrasekhar (1964a)), and, performing a Taylor expansion inside the integrals in each case, we found for the uniform and the Tolman VII solution that (see also Merafina and Ruffini 1989)

$$\gamma_{\text{cr}}(\beta) = \frac{4}{3} + \frac{38}{42}\beta\mathcal{P}(\beta) \quad (21)$$

where, for the uniform solution,  $\mathcal{P}(\beta)$  takes the form

$$\mathcal{P}_{\text{uniform}}(\beta) = 1 + 2.13\beta + 4.65\beta^2 + 10.22\beta^3 + \mathcal{O}(\beta^4) \quad (22)$$

and for the Tolman VII solution:

$$\mathcal{P}_{\text{Tolman}}(\beta) = 1.19 + 2.93\beta + 7.34\beta^2 + 19.36\beta^3 + \mathcal{O}(\beta^4). \quad (23)$$

Obviously, the approximation (21), using Eq. (22), to a linear term, confirms the Chandrasekhar expression (12). The above expressions are good approximations for  $\beta < 0.2$ . However, they fail for higher values of  $\beta$  and consequently additional terms must be included. In particular,  $\gamma_{\text{cr}}$  increases very fast for  $\beta > 0.25$  due to the strong effects of general relativity.

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