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ORIGINAL ARTICLE

On the radiative and thermodynamic properties of the cosmic radiations using *COBE* FIRAS instrument data: IV. Sunyaev-Zel'dovich (μ-type) distortion effect

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Abstract The Sunyaev-Zel'dovich (SZ) effect represents a small spectral distortion to the cosmic microwave background (CMB) radiation, caused by the Compton scattering of CMB photons by the hot gas of galaxy clusters. In an early stage of universe, the SZ effect generates μ-type of distortions for the CMB spectrum. A μ-type distortion is created between the double Compton scattering decoupling ($z \sim 10^6$) and the thermalization decoupling by the Compton scattering ($z \sim 10^5$). In this case, to describe the small spectral distortion of the CMB spectrum, we use the Bose-Einstein (μ-type) distribution with a non-zero chemical potential. At present, it is interesting to investigate the effect of this spectral distortion on the integral characteristics of the Bose-Einstein (μ-type) spectrum. The thermal radiative and thermodynamic functions are such integral characteristics. These functions are as follows: a) the total radiation power per unit area; b) total energy density; c) number density of photons; d) grand potential density; e) Helmholtz free energy density; f) entropy density; g) heat capacity at constant volume; h) enthalpy density; and i) pressure. Precise analytical expressions are obtained for the temperature dependences of these functions. Using the observational data obtained by the *COBE* FIRAS, PIXIE, PRISM, and *Planck* missions, the thermal radiative and thermodynamic functions are calculated. A comparative analysis of the results obtained with the results for the same functions of the CMB spectrum at $T = 2.72548$ K is carried out. Very small distortions are observed for the thermal radiative and thermodynamic functions. In the redshift range $10^5 < z < 3 \times 10^6$, these functions are calculated. The expressions are obtained for new astrophysical parameters, such as the entropy density/Boltzmann constant and number density, created by the Bose-Einstein (μ-type) spectrum.

Keywords Bose-Einstein distribution · μ-Type distortion · The thermal radiative and thermodynamic functions · COBE/FIRAS · PIXIE · PRISM · *Planck* · Cosmology: cosmic background radiation · Cosmology: theory

1 Introduction

It is well-known that the Sunyaev-Zel'dovich effect (Zel'dovich and Sunyaev 1969; Sunyaev and Zel'dovich 1970, 1980) is a small spectral distortion of the cosmic microwave background (CMB) spectrum caused by the Compton scattering of the CMB photons on high energy electrons in hot gases of galaxy clusters (Carlstrom et al. 2002; Rephaeli 1995; Birkinshaw 1999). This mechanism is also important when we consider the early stage of universe with the epoch between $z = 10^5$ and $z = 3 \times 10^6$ (Mather et al. 2013). After many Compton scattering, photons and electrons reach statistical equilibrium, which leads to the use of the Bose-Einstein distribution with a non-zero chemical potential.

It is well known that in order to obtain a clear picture of the thermal history of the universe, we must have information about the thermal radiative and thermodynamic properties of cosmic radiation for each epoch. This article is one of a set of articles associated with the study of the thermal radiative and thermodynamic properties of the cosmic radiations using the *COBE* FIRAS, PIXIE, PRISM, and *Planck* observation data. In previous articles, these properties have been discussed in detail for the cosmic microwave background radiation (Fisenko and Lemberg 2014a, 2016),

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the extragalactic far-infrared background radiation (Fisenko and Lemberg 2014b), and the galactic far-infrared radiation (Fisenko and Lemberg 2015). The exact analytical expressions for temperature and redshift dependences of the thermal radiative and thermodynamic functions of the cosmic radiations have been obtained in each cases. These functions have been calculated using the *COBE* FIRAS observation data. New astrophysical parameters, such as the entropy density/Boltzmann constant and the number density of photons have been constructed.

This article is devoted to the study of the thermal history of the universe in the epoch between $z = 10^5$ and $z = 3 \times 10^6$. The main idea is to obtain the analytical expressions for the integral characteristics of the Bose-Einstein (μ-type) spectrum. These integral characteristics include the thermal radiative and thermodynamic functions of a system. The latter functions are: a) the total radiation power per unit area; b) total energy density; c) number density of photons; d) grand potential density; e) Helmholtz free energy density; f) entropy density; g) heat capacity at constant volume; h) enthalpy density; and i) pressure. Using *COBE* FIRAS, PIXIE, PRISM, and *Planck* observations, the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum are calculated at $z = 0$ and at the monopole temperature $T = 2.72548$ K. A comparative analysis of the results obtained with the results for the same functions of the CMB radiation is carried out. The same functions are calculated at a redshift from $z = 10^5$ to $z = 3 \times 10^6$. New astrophysical parameters, such as the entropy density/Boltzmann constant and number density of photons, are constructed.

2 General relationships for the Bose-Einstein (μ-type) distribution

According to Fixsen et al. (1996), the Planck function in the redshift range $10^5 < z < 3 \times 10^6$ at a given temperature T is represented by the Bose-Einstein (μ-type) function

$$I(\nu, T) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{\frac{h\nu}{k_B T} + \mu} - 1}, \tag{1}$$

where μ is the dimensionless chemical potential with the value $|\mu| < 9 \times 10^{-5}$ (95% CL), T is the temperature.

Let us construct the integral functions of Bose-Einstein (μ-type) distribution. Using Eq. (1), the total energy density of the Bose-Einstein (μ-type) distribution in the frequency domain is defined as (Landau and Lifshitz 1980)

$$I_0(T) = \int_0^\infty I_\nu(T) \mathrm{d}\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{k_B T} + \mu} - 1} \mathrm{d}\nu. \tag{2}$$

Using relationship between the total energy density and the total radiation power per unit area $I_0^{SB}(T) = \frac{c}{4} I_0(T)$ for the Stefan-Boltzmann law can be defined as follows:

$$I_0^{SB}(0, \infty, T) = \frac{2\pi h}{c^2} \int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{k_B T} + \mu} - 1} \mathrm{d}\nu. \tag{3}$$

The number density of photons of the Bose-Einstein distribution has the following form (Landau and Lifshitz 1980),

$$n = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^2}{e^{\frac{h\nu}{k_B T} + \mu} - 1} \mathrm{d}\nu. \tag{4}$$

The thermodynamic functions of the Bose-Einstein (μ-type) distribution can be presented as (Landau and Lifshitz 1980):

1) The grand potential density Ω:

$$\Omega(T) = \frac{8\pi k_B T}{c^3} \int_0^\infty \nu^2 \ln\left(1 - e^{\frac{h\nu}{k_b T} + \mu}\right) d\nu. \tag{5}$$

Here $\Omega = I_o - Ts - \mu N$, where I_o, S, and μ are the total energy density, the entropy density, and chemical potential density of a system.

2) Helmholtz free energy density $f = \frac{F}{V}$:

$$f = \Omega - \mu \left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V}. \tag{6}$$

3) Entropy density $s = \frac{S}{V}$:

$$s = -\left(\frac{\partial \Omega(T)}{\partial T}\right)_V. \tag{7}$$

4) Heat capacity at constant volume per unit volume $c_V = \frac{C_V}{V}$:

$$c_V = \left(\frac{\partial I_0(T)}{\partial T}\right)_V. \tag{8}$$

5) Pressure p:

$$p = -\Omega. \tag{9}$$

6) Enthalpy density $h = u + p$

$$h = -\mu \left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} - T\left(\frac{\partial \Omega}{\partial T}\right)_{\mu,V}. \tag{10}$$

3 Results for the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum

According to Eqs. (2)–(4), after computing the integrals in Eq. (2) and Eq. (4), the exact expressions for the total energy density, the total radiation power per unit area, and the number density of photons have the following forms:

a) The total energy density

$$I_0(T) = \frac{48\pi (k_B T)^4}{c^3 h^3} \mathrm{Li}_4\big(e^{-\mu}\big), \tag{11}$$

where $\Gamma(x)$ is the gamma function and $\mathrm{Li}_n(x)$ is the polylogarithm function (Abramowitz and Stegun 1972).

b) The Stefan-Boltzmann law

$$I_0^{\mathrm{SB}}(T) = \sigma_0 T^4, \tag{12}$$

where $\sigma_0 = \frac{12\pi k_B^4}{c^2 h^3} \mathrm{Li}_4(e^{-\mu})$.

c) The number density of photons

$$n = \frac{16\pi (k_B T)^3}{c^3 h^3} \mathrm{Li}_3\big(e^{-\mu}\big). \tag{13}$$

Using Eqs. (5)–(10), the exact expressions for the thermodynamic functions of the Bose-Einstein (μ-type) spectrum have the following structures:

1) Grand potential density

$$\Omega = -\frac{16\pi (k_B T)^4}{c^3 h^3} \mathrm{Li}_4\big(e^{-\mu}\big). \tag{14}$$

2) Helmholtz free energy density:

$$f = -\frac{16\pi (k_B T)^4}{c^3 h^3} \mathrm{Li}_4\big(e^{-\mu}\big)\left[1 + \frac{\mathrm{Li}_3(e^{-\mu})}{\mathrm{Li}_4(e^{-\mu})}\mu\right]. \tag{15}$$

3) Entropy density

$$s = \frac{64\pi k_B^4 T^3}{c^3 h^3} \mathrm{Li}_4\big(e^{-\mu}\big)\left[1 + \frac{\mathrm{Li}_3(e^{-\mu})}{4\mathrm{Li}_4(e^{-\mu})}\mu\right]. \tag{16}$$

4) Heat capacity at constant volume per unit volume

$$c_V = \frac{192\pi k_B^4}{c^3 h^3} T^3 \mathrm{Li}_4\big(e^{-\mu}\big)\left[1 + \frac{\mathrm{Li}_3(e^{-\mu})}{4\mathrm{Li}_4(e^{-\mu})}\mu\right]. \tag{17}$$

5) Pressure of photons

$$p = \frac{16\pi k_B^4}{c^3 h^3} T^4 \mathrm{Li}_4\big(e^{-\mu}\big). \tag{18}$$

6) Enthalpy density

$$h = \frac{16\pi k_B^4}{c^3 h^3} T^4 \mathrm{Li}_4\big(e^{-\mu}\big). \tag{19}$$

Table 1 shows the calculated values of thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum and the monopole spectrum at $T = 2.72548$ K at the present day $z = 0$. The calculated values are obtained for the lower and upper limits of the distortion $|\mu| < 9 \times 10^{-5}$ (95% CL). Here we use the COBE/FIRAS observational data.

As can be seen from Table 1, the number density of photons produced by the μ-type distortion of the CMB spectrum for the upper limit of distortion $\mu = 9 \times 10^{-5}$ is less than for the number density of photons of the CMB monopole spectrum. The percentage of photon destruction is $\tilde{n} = \frac{n^m - n^\mu}{n^m} \times 100\% = 0.015\%$. In this case, for the distortions generated by electron heating, the spectrum has a deficit of photons in comparison with a black-body spectrum at the equilibrium electronic temperature (De Zotti et al. 2016). In the opposite case, the situation with photon injections is different for the lower limit of distortion $\mu = -9 \times 10^{-5}$. As a result, the number density of photons, produce by the Bose-Einstein (μ-type) radiation is greater than for one of the CMB radiation (Chluba 2015). This means that photons are injected into a system. The percentage of injected photons is $|\tilde{n}'| \approx 0.01\%$. As you can see, this percentage is small. The same situation arises for other thermal radiative and thermodynamic functions.

According to COBE/FIRAS observation data, the possible spectral derivation from black-body spectrum is limited to $\frac{\Delta I(\nu)}{I(\nu)} \sim 10^{-5}$–$10^{-4}$ (Fixsen et al. 1996; Mather et al. 1994). Let us calculate the integral derivations. As an example, let us consider the total energy density. According to Table 1, when $\mu = 9 \times 10^{-5}$, we obtain $\frac{I_0^M - I_0^\mu}{I_0^M} \approx 1.2 \times 10^{-5}$. For the entropy density we have the following value $\frac{s^M - s^\mu}{s^M} \approx 9.7 \times 10^{-5}$. As can be seen from Table 1, the possible derivations from the total energy density, as well as for other thermal radiative thermodynamic functions are in the same order of magnitude as for the average spectral distortion $\frac{\Delta I(\nu)}{I(\nu)} \sim 10^{-5}$–$10^{-4}$.

It is important to note that COBE/FIRAS was designed to measure the CMB spectrum and to detect the deviation of the observed spectrum from the black-body spectrum. However, according to Fixsen et al. (1996), the COBE/FIRAS observation does not detect distortions from the blackbody spectrum. This means that integral distortions, such as thermal radiative and thermodynamic functions, can not be detected either.

To detect the μ-type of distortion from black-body spectrum, missions such as the Primordial Inflation Explorer (PIXIE) (Kogut et al. 2011) and the Polarized Radiation Imaging and Spectroscopy Mission (PRISM) (Andre et al. 2013) were proposed. These missions make it possible to measure the absolute frequency spectrum with high degree of accuracy than COBE/FIRAS. Their sensitivity is 10^3–10^4 times better than that of COBE/FIRAS. Some of the main conclusions of these missions can be summaries as follows. PIXIE and PRISM are capable of detecting the CMB distortions with: a) $\mu \sim 5 \times 10^{-8}$ for PIXIE; and b) $\mu \sim 10^{-9}$ for PRISM.

In Khatri and Sunyaev (2015), the *Planck* HFI channel maps were using to make μ-type distortion map. From these

Table 1 Calculated values of the radiative and thermodynamic functions for the Bose-Einstein (μ-type) spectrum using COBE/FIRAS data and monopole spectrum data at present day $z = 0$ and $T = 2.72548$ K

Quantity	μ-Distortion spectrum $\mu = -9 \times 10^{-5}$	μ-Distortion spectrum $\mu = 9 \times 10^{-5}$	Monopole spectrum $\mu = 0$
$I_0(T)$ [J m^{-3}]	4.1751×10^{-14}	4.1742×10^{-14}	4.1747×10^{-14}
	–	–	
$I_0^{SB}(T)$ [W m^{-2}]	3.1291×10^{-6}	3.1285×10^{-6}	3.1288×10^{-6}
	–	–	
$\Omega(T)$ [J m^{-3}]	-1.3917×10^{-14}	-1.3914×10^{-14}	–
f [J m^{-3}]	$-1.391561 \times 10^{-14}$	$-1.391559 \times 10^{-14}$	$-1.391560 \times 10^{-14}$
s [J m^{-3} K^{-1}]	2.0424×10^{-14}	2.0421×10^{-14}	2.0423×10^{-14}
P [J m^{-3}]	1.3917×10^{-14}	1.3914×10^{-14}	1.3915×10^{-14}
c_V [J m^{-3} K^{-1}]	6.1273×10^{-14}	6.1264×10^{-14}	6.1269×10^{-14}
n [m^{-3}]	4.1076×10^{8}	4.1066×10^{8}	4.1072×10^{8}
$h(T)$ [J m^{-3}]	1.3917×10^{-14}	1.3914×10^{-14}	1.3916×10^{-14}

maps the limit was obtained for root mean square (rms) value of $\mu_{\rm rms}^{10'}$. This limit value is $\mu_{\rm rms}^{10'} \sim 6.4 \times 10^{-6}$ and 14 times stronger than the COBE-FIRAS limit on the mean of $\mu_{\rm rms}^{10'} \sim 9 \times 10^{-5}$ at 95% confidence level.

Now it is interesting to obtain the values of the integral characteristic of the system, such as the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum for various dimensionless chemical potentials: a) $\mu \sim 5 \times 10^{-8}$ for PIXIE; b) $\mu \sim 10^{-9}$ for PRISM and c) $\mu_{\rm rms}^{10'} \sim 6.4 \times 10^{-6}$ for *Planck* HFI. These functions are calculated at $T = 2.72548$ K and at present day $z = 0$. The results are shown in Table 2. As can be seen from Table 2, the thermal radiative and thermodynamic functions are very close to each other for different observation data. The integral distortion of the total energy densities for the monopole spectrum of CMB and the PRISM data ($\mu \sim 10^{-9}$) is $\frac{I_0^M - I_0^\mu}{I_0^M} \approx 1.2 \times 10^{-7}$. The total energy density, as well as other functions in Table 2 can be determined using observational data. For example, the total energy density is zero moment of the Bose-Einstein (μ-type) distribution. The zeroth moment is equal to the area of the distribution and can be obtain from the Bose-Einstein (μ-type) spectrum.

Now let's calculate the thermal radiative and thermodynamic functions in the redshift range $10^5 < z < 3 \times 10^6$. To do this, we must transform Eqs. (11)–(19) to the z-dependence of the redshift. For temperature we have $T(z) = T_0(1+z)$. We also need to know the z-dependence of the dimensionless chemical potential μ. Using the COBE/FIRAS data at $z = 0$, let us represent the function $\mu(z)$ in the form:

$$|\mu| < 9 \times 10^{-5}(1+z). \tag{20}$$

According to Eq. (20), at the redshift $z = 10^5$, we obtain $\mu = 270$. Using Eq. (11), the calculated value for the total energy density at $T = 2.72550 \times 10^5$ K is as follows: $I_0(T) = 4.7603 \times 10^2$ J m^{-3}. The same function for CMB radiation at $T = 2.72550 \times 10^5$ K has the following value of $I_0(T) = 4.1748 \times 10^6$ J m^{-3}. Comparing these values, we can see that when $z = 10^5$, CMB radiation strongly dominates in comparison with Bose-Einstein (μ-type) radiation. So, the integral distortion of the total energy density, when $\mu = 270$ is $\frac{I_0^M - I_0^\mu}{I_0^M} \approx 0.999$. But this fact contradicts the physical picture. The same situation arises when we consider the epoch at $z = 3 \times 10^6$. Therefore, in what follows we shall assume that dimensionless chemical potential μ does not depend on z.

Table 3 and Table 4 present a comparative analysis of the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum with analogous functions of the CMB monopole spectrum for: a) 2.72548×10^5 K, $z = 10^5$ and b) $T = 8.176442 \times 10^6$ K, $z = 3 \times 10^6$. As can be seen from Table 2 and Table 3, the integral distortion of the total energy densities for the CMB monopole spectrum and PRISM data spectrum ($\mu \sim 10^{-9}$) are: a) at $z = 10^5$, $\frac{I_0^M - I_0^\mu}{I_0^M} \approx 1.1 \times 10^{-7}$; and b) at $z = 3 \times 10^6$, $\frac{I_0^M - I_0^\mu}{I_0^M} \approx 2.9 \times 10^{-7}$. According to Table 3 and Table 4, the total energy density for PRISM data ($\mu \sim 10^{-9}$) in the redshift interval $10^5 < z < 3 \times 10^6$ take the following values:

$$4.1746770509 \times 10^6 \ \mathrm{J\,m^{-3}} \le I_0(T) \le 3.38149171 \times 10^{12} \ \mathrm{J\,m^{-3}}. \tag{21}$$

For the number density of photons, we have

$$4.107175237 \times 10^{23} \ \mathrm{m^{-3}} \le n \le 1.1089381277 \times 10^{28} \ \mathrm{m^{-3}} \tag{22}$$

Table 2 Calculated values of the radiative and thermodynamic functions for the Bose-Einstein (μ-type) spectrum using: a) *Planck* HFI channel data; b) PIXIE data; c) PRISM data; and d) COBE/FIRAS data for the monopole spectrum. $T = 2.72548$ K and $z = 0$—present day

Quantity	μ-Distortion *PLANCK data* $\mu_{\text{rms}}^{10'} \sim 6.4 \times 10^{-6}$	μ-Distortion *PIXIE data* $\mu \approx 5 \times 10^{-8}$	μ-Distortion *PRISM data* $\mu \sim 10^{-9}$	Monopole spectrum $\mu = 0$
$I_0(T) \times 10^{14}$ [J m^{-3}]	4.17464731	4.1746761	4.174677050	4.174677055
$I_0^{\text{SB}}(T) \times 10^{6}$ [W m^{-2}]	3.1288194	3.1288415	3.128841736	3.128841739
$\Omega(T) \times 10^{14}$ [J m^{-3}]	−1.3915491	−1.39155894	−1.391559016	–
$f \times 10^{14}$ [J m^{-3}]	−1.39155901847	−1.39155901851	−1.39155901851701	−1.39155901851702
$S \times 10^{14}$ [J m^{-3} K^{-1}]	2.04228480	2.04229561	2.042295694	2.042295696
$P \times 10^{14}$ [J m^{-3}]	1.39155901847	1.39155901851	1.39155901851701	1.39155901851702
$c_V \times 10^{14}$ [J m^{-3} K^{-1}]	6.1268544	6.1268868	6.126887083	6.126887088
$n \times 10^{-8}$ [m^{-3}]	4.107139	4.1071749	4.107175237	4.107175242
$h(T) \times 10^{14}$ [J m^{-3}]	1.391549	1.391558	1.391559016	1.391559018

Table 3 Calculated values of the radiative and thermodynamic functions for the Bose-Einstein (μ-type) spectrum using: a) *Planck* HFI channel data; b) PIXIE data; c) PRISM data; and d) COBE/FIRAS data for the monopole spectrum. $T = 2.72548 \times 10^{5}$ K and $z = 10^{5}$

Quantity	μ-Distortion *PLANCK data* $\mu_{\text{rms}}^{10'} \sim 6.4 \times 10^{-6}$	μ-Distortion *PIXIE data* $\mu \approx 5 \times 10^{-8}$	μ-Distortion *PRISM data* $\mu \sim 10^{-9}$	Monopole spectrum $\mu = 0$
$I_0(T) \times 10^{-6}$ [J m^{-3}]	4.17464738	4.17467682	4.1746770509	4.1746770555
$I_0^{\text{SB}}(T) \times 10^{-14}$ [W m^{-2}]	3.12881949	3.12884156	3.128841736	3.128841739
$\Omega(T) \times 10^{-6}$ [J m^{-3}]	−1.39154912	−1.39155894	−1.39155901	–
$f \times 10^{-6}$ [J m^{-3}]	−1.3915590184	−1.39155901851701	−1.39155901851702	−1.39155901851703
S [J m^{-3} K^{-1}]	20.42284808	20.42295611	20.42295694	20.42295696
$P \times 10^{-6}$ [J m^{-3}]	1.39154912735	1.3915589412	1.3915590169	1.3915590185
c_V [J m^{-3} K^{-1}]	61.26854426	61.26886833	61.26887083	61.26887088
$n \times 10^{-23}$ [m^{-3}]	4.107139296	4.107174961	4.107175237	4.107175242
$h(T) \times 10^{-6}$ [J m^{-3}]	1.39154912	1.39155894	1.391559016	1.391559018

Table 4 Calculated values of the radiative and thermodynamic functions for the Bose-Einstein (μ-type) spectrum using: a) *Planck* HFI channel data; b) PIXIE data; c) PRISM data; and d) COBE/FIRAS data for the monopole spectrum. $T = 8.176442 \times 10^{6}$ K and $z = 3 \times 10^{6}$

Quantity	μ-Distortion *PLANCK data* $\mu_{\text{rms}}^{10'} \sim 6.4 \times 10^{-6}$	μ-Distortion *PIXIE data* $\mu \approx 5 \times 10^{-8}$	μ-Distortion *PRISM data* $\mu \sim 10^{-9}$	Monopole spectrum $\mu = 0$
$I_0(T) \times 10^{-12}$ [J m^{-3}]	3.3814676	3.38149153	3.38149171	3.38149172
$I_0^{\text{SB}}(T) \times 10^{-20}$ [W m^{-2}]	2.53434627	2.53436414	2.534364285	2.534364288
$\Omega(T) \times 10^{-12}$ [J m^{-3}]	−1.12715589	−1.12716382	−1.12716390	–
$f \times 10^{-12}$ [J m^{-3}]	−1.1271639078038	−1.127163907838865	−1.1271639078388694	−1.1271639078388699
$S \times 10^{-6}$ [J m^{-3} K^{-1}]	0.5514173029	0.5514202196	0.55142024215	0.55142024261
$P \times 10^{-12}$ [J m^{-3}]	1.1271639078038	1.127163907838865	1.1271639078388694	1.1271639078388699
$c_V \times 10^{-6}$ [J m^{-3} K^{-1}]	1.6542519089	1.6542606589	1.6542607264	1.6542607278
$n \times 10^{-28}$ [m^{-3}]	1.10892842	1.1089380534	1.1089381277	1.1089381292
$h(T) \times 10^{-12}$ [J m^{-3}]	1.127155895	1.1271638452	1.1271639065	1.1271639078

4 Astrophysical constants and parameters

The expressions obtained for the thermal radiative and thermodynamic functions are important for constructing new astrophysical parameters. Indeed, according to the table of astrophysical constants and parameters (Groom 2013), two fundamental parameters for the CMB monopole spectrum, such as the number density of CMB photons and the entropy density/Boltzmann are of interest.

Let us construct the same astrophysical parameters for the Bose-Einstein (μ-type) spectrum. According to Tables 1–4 and using Eq. (13) and Eq. (16), we obtain the following expressions for new astrophysical parameters:

1. Entropy density/Boltzmann constant

$$\frac{s}{k_{\mathrm{B}}} = A\left(\frac{T}{T_0}\right)^3 \mathrm{cm}^{-3}, \tag{23}$$

where

$$A = \frac{64\pi k_{\mathrm{B}}^3 T_0^3 \mathrm{Li}_4(e^{-\mu_0})}{c^3 h^3}\left[1 + \frac{\mathrm{Li}_3(e^{-\mu_0})}{4\mathrm{Li}_4(e^{-\mu_0})}\mu_0\right] = 2891.12. \tag{24}$$

2. Number density of the photons for the Bose-Einstein (μ-type) spectrum

$$n = B\left(\frac{T}{T_0}\right)^3 \mathrm{cm}^{-3}, \tag{25}$$

where

$$B = \frac{16\pi (k_{\mathrm{B}} T_0)^3 \mathrm{Li}_3(e^{-\mu_0})}{c^3 h^3} = 410.66. \tag{26}$$

Here $T_0 = 2.72548$ K. The values of the dimensionless chemical potential (μ_{o}) at present day are the follows: a) $|\mu| < 9 \times 10^{-5}$ (95% CL)—COBE/FIRAS; b) $\mu \sim 5 \times 10^{-8}$—PIXIE mission; c) $\mu \sim 10^{-9}$—PRISM missions; d) $\mu_{\mathrm{rms}}^{10'} \sim 6.4 \times 10^{-6}$—*Planck* missions. The range of the redshift varies from $z = 10^5$ to $z = 3 \times 10^6$.

5 Conclusions

In the present work, exact expressions are obtained for calculating the temperature dependences of integral characteristics, such as the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum. Such functions are: a) the total radiation power per unit area; b) total energy density; c) number density of photons; d) grand potential density; e) Helmholtz free energy density; f) entropy density; g) heat capacity at constant volume; h) enthalpy density; and i) pressure.

Utilizing the observational data obtained by the *COBE* FIRAS, PIXIE, PRISM, and *Planck* missions, the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum are calculated at present day $z = 0$ at the monopole temperature $T = 2.72548$ K. The results are shown in Table 1 and Table 2. A comparative analysis of the results obtained with the results of the monopole spectrum of the CMB radiation is carried out. Very small distortions are observed for the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum. We believe that these distortions can be detected.

Knowing the dependence of the temperature T on the redshift z we can study the state of the universe many years ago at the redshifts from $z = 10^5$ to $z = 3 \times 10^6$. Using the different values of the dimensionless chemical potential for the PIXIE, PRISM, and *Planck* missions, the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum are calculated at $T = 2.72550 \times 10^5$ K and $T = 8.176442 \times 10^6$ K. The results of these values are presented in Table 3 and Table 4. A comparative analysis of the results obtained with the results for the monopole spectrum is carried out. For example, it is shown that the total energy densities for the PRISM data ($\mu \sim 10^{-9}$) in the redshift interval $10^5 < z < 3 \times 10^6$ can take the following values:

$$4.1746770509 \times 10^6\ \mathrm{J\,m^{-3}} \le I_0(T) \le 3.38149171 \times 10^{12}\ \mathrm{J\,m^{-3}}.$$

According to the Table 3 and Table 4, we can obtain ranges of values for the thermal radiative and thermodynamic functions for PIXIE, PRISM, and *Planck* missions in the early epoch from $z = 10^5$ to $z = 3 \times 10^6$.

The analytical expressions for the thermal radiative and thermodynamic functions of the Bose-Einstein (μ-type) spectrum make it possible to construct new astrophysical parameters, such as the entropy density/Boltzmann constant, and number density of photons.

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