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ORIGINAL ARTICLE

Effect of the darkforce on the extra-anomalous apsidal precession of solar planets

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Abstract We demonstrate that, while the proposed *Gravitational Dark-force Theory* (of Nyambuya (New Astron. 67:1, 2019b) here-in Paper II) predicts an extra-anomalous apsidal precession for Solar planets due to the gravitational dark-force on the orbits of these planets, the predicted extra-anomalous apsidal precession is so small—*so much that*—it can not account for the observed extra-anomalous apsidal precession of Solar planets. This null result is important in that it informs us that whatever may be the cause of the extra-anomalous apsidal precession, it is not the proposed gravitational dark-force.

Keywords Anomalous apsidal precession · Darkmatter · Gravitatomagnetism · Fisher-Tully relation · Galaxy rotation curves

1 Introduction

This reading is the last in our four part series where we develop a new model of gravitation whose endeavour is to explain the *Flat Rotation Curve Problem of Spiral Galaxies via* the path of *Modified Gravity* (MoG) where the hypothesis of dark-matter is not assumed. In the first part of the series *i.e.*, in the reading Nyambuya (2019a) [hereafter Paper (I)], we start off by justifying the Nordström gravitomagnetic theory that we use to suggest our solution to the flat rotation curve problem of spiral galaxies. In the second part (*i.e.*, in the reading; Nyambuya 2019b, hereafter Paper II), we consider one of the five gravitational potentials (Φ_4) that emerge from Nordström (1913)'s *Relativistic Theory of Gravitation* (as presented in the reading Nyambuya 2015); and from this potential, we harness an *inverse distance law of gravitation* and with it, we make the *temerarious endeavour* to suggest the resulting MoG theory as an alternative explanation to the long standing flat rotation curves problem of spiral galaxies. We have coined this inverse distance law ($F_{\mathrm{D}} \propto 1/r$) the *gravitational darkforce*. In a pre-sequential reading (*i.e.*, in the reading Nyambuya 2018, hereafter Paper III), we demonstrated how one can explain—from this gravitational darkforce—the shape of spiral galaxies. In the present reading, we will consider the effects of this new gravitational darkforce on the orbits of Solar planet. The aim of which is—*if needs be*—to constrain this gravitational darkforce so that its affects are in-accord with experience on both the Solar and galactic scales.

Obviously, any non-Newtonian addition to the Newton model of gravitation is going to bring in new effects that must be observable should this modification be correct description of physical and natural reality as we know it. One of these effects is the anomalous apsidal precession of Solar planets. The issue of the anomalous apsidal precession of Solar planets is an issue that began with planet Mercury more than a century ago. When Newton's theory of gravitation was (is) applied to the case of the planet Mercury where the extra tugging from other planets is taken into account, perturbative Newtonian theory of gravitation (Le Verrier 1859) gave (gives) $\sim 5557'' \,\mathrm{cy}^{-1}$ for the apsidal precession of the planet Mercury (cf., Clemence 1947). In 1846, the consummate French mathematician—Urbain Jean Joseph Le Verrier (1811–1877), accurately measured Mercury's apsidal precession whereby he obtained $\sim 5600'' \,\mathrm{cy}^{-1}$, which is a difference of $\sim +43'' \,\mathrm{cy}^{-1}$ when compared to the expected Newtonian value obtaining from theory.

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An explanation of this unaccounted $\sim +43'' \, \mathrm{cy}^{-1}$ presented a challenge for theoretical physicists even to this day it is still a problem of much interest (*e.g.*, Park et al. 2017; Roy 2015; Cornejo 2014; Hu et al. 2014; Lo et al. 2013; Stewart 2005; Campbell et al. 1983). Various theoretical solutions were proposed in the order to explain this unaccounted $\sim +43'' \, \mathrm{cy}^{-1}$, but, nobody was able to provide a widely satisfactory explanation. Having carefully applied himself to this problem and after the construction of his great master piece (Einstein 1915a,b,c,d)—the *General Theory of Relativity* (GTR), Albert Einstein (1915a) was the first to successfully solve this problem—by successful, it is here meant that Einstein (1915a) was the first to provide a solution—to this problem of the anomalous apsidal precession of the planet Mercury—that was accepted by the majority of physicists. For his solution, Einstein (1915a) obtained the exact same formula as that obtained by the German high school teacher—Mr. Paul Gerber (1854–1909). Gerber (1898a)'s solution is based on the Newtonian theory of gravitation in which he (Gerber 1917, 1898a,b) makes an *ad hoc* hypothesis that the gravitational potential depends on the speed of the moving test body while Einstein (1915a)'s solution is based on the effect of curvature of spacetime. In Sect. 3, we will discuss briefly, Einstein (1915a)'s solution.

As it turns out, not only is the orbit of Mercury experiencing an anomalous apsidal precession that is not explicable from the confines of Newtonian gravitation, all the planets are—*i.e.*, this is true for at least Venus, Earth, Mars, Jupiter, Saturn. The other planets—Uranus, Pluto, *etc.*, have orbital periods that are much longer than an average human lifetime and as such, direct measurements are not available. The observed anomalous apsidal precessions of these planets are in satisfactory—*if not good*—agreement with Einstein (1915a)'s GTR.

However, present day Solar astrometric has achieved such level of accuracy that it is now possible to measure down to a µ-*arcsecond/yr* for the apsidal precessions of these planets. What this means is that, these measurement now put stringent constraints on theory. As it turns out, after all the known effects are taken into account, there remains some very tiny extra-anomalous apsidal precession that are not accounted for. These obviously point to horizons beyond the GTR. It is in this domain that MoG theories enter. If they are anything to go by, they must at least stand up to these data of the observed extra-anomalous apsidal precessions. The present theory obviously fall in the domain of MoG theories. So, it is not only important, but very important to know this theory's prediction regarding these extra-anomalous apsidal precession of Solar planets.

Because of the high level precession in the measurement and all this being a direct result advanced modern technology, contemporary Solar astrometry studies (*e.g.*, Pitjeva and Pitjev 2013; Pitjev and Pitjeva 2013; Pitjeva 2013; Fienga et al. 2009, 2011) now place stringent constrains on MoG theories *via e.g.*, the measurement of apsidal precession that are not accounted for by the GTR and these are typically refereed to as '*extra-anomalous apsidal precession*' (e.g., Iorio 2009).

Now, in-closing this introductory section, we shall give the synopsis of the reminder of the reading. For instructive, completeness and self-containment purposes, in Sect. 2 and Sect. 3, we discuss the Newtonian solution of the orbit and the GTR's solution to the apsidal precession of test bodies, respectively. In Sect. 4, for the purposes of this reading, we give an exposition of the gravitational dark-force theory. In Sect. 5, we derive the solution to the apsidal precession in-accordance with the gravitational dark-force theory and in Sect. 6, we discuss this solution with regard to Solar planets. In Sect. 7, we give a brief general discussion and the conclusion drawn thereof.

2 Newtonian orbit

As is well known, the inverse square law of Newtonian gravitation F_{N}, leads to the following second order differential equation of motion:

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{\ell} = 0, \tag{1}$$

where $(u = 1/r)$ with r being the radial distance of the test particle from the centre of mass of the massive central gravitating body about which it orbits and φ is the angular displacement. The solution u_{N} to this (1), is:

$$u_{\mathrm{N}} = \frac{1 + e\cos\varphi}{\ell}, \tag{2}$$

where e is the eccentricity of the orbit and:

$$\ell = \frac{J_\varphi^2}{G\mathcal{M}}, \tag{3}$$

is *semi-latus rectum* of the orbit, G is the usual Newtonian gravitational constant, $\mathcal{M}$ is the mass of the central massive gravitating body and $(J_\varphi = r^2\dot{\varphi})$ is specific orbital angular momentum of a test body about the $\hat{\boldsymbol{\varphi}}$-direction and this specific orbital angular momentum is such that:

$$\frac{dJ_\varphi}{dt} = 0. \tag{4}$$

Having presented the traditional Newtonian solution of an orbiting test particle, we shall in the next section present the traditional Einsteinian modification to the Newtonian solution, which, we shall thereafter modify to include the gravitational dark-force.

3 General relativistic apsidal precession

As already stated in the introduction section: no sooner had Einstein (1915b,c,d) discovered his GTR did he (Einstein 1915a) apply this theory to the problem of the anomalous apsidal precession of the planet Mercury. In his application of the GTR to the mercurial problem, he obtained that the trajectory of solar planets must be described by the equation:

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{\ell} = \left(\frac{3G\mathcal{M}}{c^2}\right)u^2, \tag{5}$$

where c is the speed of Light in *vacuo*. From this equation, Einstein (1915a) obtained that the average the rate of precession of the perihelion is given by:

$$\left\langle \frac{\Delta\varphi}{\mathcal{T}_\varphi} \right\rangle_{\mathrm{E}} = \frac{6\pi G\mathcal{M}}{\mathcal{T}_\varphi c^2(1-e^2)a} = \dot{\varpi}_{\mathrm{E}}, \tag{6}$$

where $\mathcal{T}_\varphi$ is the time period of revolution for the orbital motion under consideration. As aforestated: although this formula (6) had been derived earlier by Gerber (1898a), the credit for the 'correct' derivation goes to Einstein (1915a) because he derived it from an acceptable theory, while Gerber (1898a)'s MoG theory is considered physically incorrect. Einstein (1915a)'s formula (6) was unprecedented in its prediction of the anomalous apsidal precession of the planet Mercury which—*well*—before then was known to be $\sim 43.2''$, and, this value is the value that Einstein obtained from the formula (6). Given that this anomalous apsidal precession of the planet Mercury had become a well recognised problem that required a solution, Einstein's explanation of it led to the instant recognition that—*one way or the other*—Einstein's GTR, must be an acceptable description of physical and natural reality. The predicted value for the apsidal precession for the other planets (Venus, Earth, Mars, Jupiter and Saturn) are given column (5) of Table 1.

Now, in the next section before we can add the *Gravitational Darkforce Correction* to the anomalous apsidal precession, we shall formally (*i.e.*, for the completeness and self-containment purposes of present reading) define the *Gravitational Darkforce*. For a full lay out of the *Gravitational Darkforce Theory*, one can visit Paper (II).

4 Gravitational darkforce

The gravitational darkforce F_{D} comes in as an addition to the Newtonian gravitational force ($F_{\mathrm{N}} = -G\mathcal{M}_{\mathrm{gal}}m/r^2$), that is to say, if F_{res} is the resultant gravitational force acting on a test particle orbiting the galactic bulge, then, in accordance with *Newton's Second Law of Motion*, we know that ($F_{\mathrm{res}} = F_{\mathrm{N}} + F_{\mathrm{D}}$), where:

$$F_{\mathrm{D}} = -\frac{G\mathcal{M}m}{r\mathscr{R}_{\mathrm{D}}}. \tag{7}$$

In (7), m is the mass of a test particle orbiting the central mass at the radial distance r and, $\mathscr{R}_{\mathrm{D}}$ is the darkforce scale-length which according to Paper (II), is defined:

$$\mathscr{R}_{\mathrm{D}} = a_{\mathrm{D}}^{\mathrm{RM}}\left(\frac{\mathcal{M}}{\mathcal{M}_\odot}\right)^{1/2}\mathscr{R}_{\mathrm{kpc}}, \tag{8}$$

where:

$$a_{\mathrm{D}}^{\mathrm{RM}} = (3.00 \pm 0.20) \times 10^{-5}, \tag{9}$$

and ($\mathscr{R}_{\mathrm{kpc}} = 1.00$ kpc). As stated in Paper (II), we have in (8) a mass dependent scale-length $\mathscr{R}_{\mathrm{D}}$ which is similar to what Moffat et al. (2018) have suggested in their MoG theory. It strongly appears that such a scale-length that is mass-dependent is a kind of *sine-quo-non* for any MoG theory to reproduce the Tully and Fisher (1977) relation. In the next section, we shall now proceed to add the *Gravitational Darkforce Correction* to the anomalous apsidal precession formula (6) of Einstein (1915a).

5 Gravitational darkforce induced apsidal precession

Viewed from a Newtonian gravitational standpoint, Einstein (1915a)'s equation (5) adds a correction term $F_{\mathrm{E}}(u)$ to the Newtonian gravitational paradigm: (*i.e.*, $F_{\mathrm{E}}(u) = -3\mathrm{m}c^2\ell(G\mathcal{M}/c^2)^2u^4$). If we in-cooperate the darkforce field as an additional force to the Newton-Einstein gravitational paradigm, then, the resultant equation for the orbit is:

$$\frac{d^2u}{d\varphi^2} + u = -\frac{F_{\mathrm{N}}(u) + F_{\mathrm{E}}(u) + F_{\mathrm{D}}(u)}{\mathrm{m}u^2J_\varphi^2}. \tag{10}$$

The equation of motion that results from (10), is:

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{\ell} = \left(\frac{3G\mathcal{M}}{c^2}\right)u^2 + \left(\frac{G\mathcal{M}}{J_\varphi^2\mathscr{R}_{\mathrm{D}}}\right)\frac{1}{u}. \tag{11}$$

This (11), can be re-written as:

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{\ell} = \left(\frac{3G\mathcal{M}}{\ell c^2}\right)\frac{(\ell u)^2}{\ell} + \left(\frac{G\mathcal{M}\ell}{J_\varphi^2\mathscr{R}_{\mathrm{D}}}\right)\frac{1}{\ell u}. \tag{12}$$

Since ($J_\varphi^2 = G\mathcal{M}\ell$), it follows that:

$$\frac{d^2u}{d\varphi^2} + u - \frac{1}{\ell} = \frac{\eta_{\mathrm{E}}(\ell u)^2}{\ell} + \frac{\eta_{\mathrm{D}}(\ell u)^{-1}}{\ell}, \tag{13}$$

where:

$$\eta_{\mathrm{D}} = \frac{\ell}{\mathscr{R}_{\mathrm{D}}} \quad \text{and} \quad \eta_{\mathrm{E}} = \frac{3G\mathcal{M}}{\ell c^2}. \tag{14}$$

For the Solar system (*i.e.*, the Sun in particular), the dark-force scale-length $\mathscr{R}_{\mathrm{D}}$, is such that:

$$\mathscr{R}_{\mathrm{D}}^{\odot} = 6100 \pm 400 \text{ AU}. \tag{15}$$

What this (15) is telling us is that, if the *Gravitational Dark-force Theory*, is an acceptable description of physical and natural reality, then, this dark-force should—in the Solar system, dominate the Newtonian force at the distance of ~ 6000 AU. This is about six times the size of the Solar system—*i.e.*, if we take the size of Solar system to be defined at least by the furthest hypothesized object—dubbed *Planet Nine* which is predicted to be orbiting with a semi-major axis of about 1200 AU (Batygin and Brown 2016a,b; Gomes et al. 2016; Trujillo and Sheppard 2014). Within the region $\lesssim 6000$ AU of the orbits around the Sun, while the dark-force may not dominate in this region, it might have some minute effects on the Solar orbits. It is these minute effects that we here seek—*i.e.*, *Does the dark-force have any significant toll on the apsidal precession of Solar orbits?*

Now—in-order for a solution to (13), we shall proceed to assume a solution of the form $(u = u_N + u_p)$, where u_p is the particular solution and u_N the Newtonian solution. Further, we shall assume that for this solution $(e\cos\varphi + \ell u_p \ll 1)$, so that to first order:

$$\begin{aligned} (\ell u)^2 &\simeq 1 + 2e\cos\varphi \\ (\ell u)^{-1} &\simeq 1 - e\cos\varphi - u_p\ell, \end{aligned} \tag{16}$$

hence:

$$\frac{d^2u_p}{d\varphi^2} + (1+\eta_{\mathrm{D}})u_p = \frac{\eta_{\mathrm{E}}(1+2e\cos\varphi)}{\ell} + \frac{\eta_{\mathrm{D}}(1-e\cos\varphi)}{\ell}. \tag{17}$$

If we are to obtain a solution for (17), we surely must come up with a strategy to solve it. That strategy is as follows. We shall assume a solution of the form: $(u_p = u_{ap} + u_{pp})$, where u_{ap} is the auxiliary part of u_p and u_{pp} is particular solution of u_p. For the solution u_p, the idea is to obtain two differential equations in-terms of u_{ap} and u_{pp}. In-order for us to do that, we will need to linearise the term: $(1+\eta_{\mathrm{D}})u_p$, *i.e.*, obtain a first order expression that is linear in u_{ap} and u_{pp}.

$$(1+\eta_{\mathrm{D}})u_p \simeq (1+\eta_{\mathrm{D}})u_{ap} + u_{pp}. \tag{18}$$

In (18), we have dropped the term $\eta_{\mathrm{D}}u_{pp}$ on the assumption that it is too small, so small so much that, it can be neglected without altering in any significant way, the content of the final results.

Now, with this term having been linearised, we can now write (17), as:

$$\frac{d^2u_{ap}}{d\varphi^2} + (1+\eta_{\mathrm{D}})u_{ap} + \frac{d^2u_{pp}}{d\varphi^2} + u_{pp} \simeq \frac{\eta_{\mathrm{E}}(1+2e\cos\varphi)}{\ell} + \frac{\eta_{\mathrm{D}}(1-e\cos\varphi)}{\ell}. \tag{19}$$

Further, we can split this (19) into two equations with one for u_{ap} and the other for u_{pp}, as follows:—we set:

$$\frac{d^2u_{ap}}{d\varphi^2} + (1+\eta_{\mathrm{D}})u_{ap} = 0, \tag{20}$$

so that:

$$\frac{d^2u_{pp}}{d\varphi^2} + u_{pp} = \frac{\eta_{\mathrm{E}}(1+2e\cos\varphi)}{\ell} + \frac{\eta_{\mathrm{D}}(1-e\cos\varphi)}{\ell}. \tag{21}$$

The solution to (21), is:

$$u_{ap} = \frac{\eta_{\mathrm{E}}(1+e\varphi\sin\varphi)}{\ell} + \frac{\eta_{\mathrm{D}}[1-(e/2)\varphi\sin\varphi]}{\ell}, \tag{22}$$

while that (20), is:

$$u_{ap} = \frac{e\alpha_0\sin(k\varphi+\beta_0)}{\ell} \tag{23}$$

where $(k = \sqrt{1+\eta_{\mathrm{D}}} \simeq 1 + \eta_{\mathrm{D}}/2)$ and (α_0, β_0) are constants of integration where we shall set $(\beta_0 = 0)$ while $(\alpha_0 \neq 0)$, hence the full solution $(u_p = u_{ap} + u_{pp})$ is given by:

$$u = \frac{1 + e\cos\varphi + e\eta\varphi\sin\varphi + e\alpha_0\sin(k\varphi)}{\ell}, \tag{24}$$

where $(\eta = \eta_{\mathrm{E}} - \eta_{\mathrm{D}}/2)$. In writing this solution [*i.e.*, (24)], we have dropped the term $(\eta_{\mathrm{E}} + \eta_{\mathrm{D}})$, because it is negligibly small, so small so much that, dropping it will not any way significantly alter the final desired and sought for result.

Now, for small η, we know that to first order approximation:

$$\begin{aligned} \cos\varphi + \eta\varphi\sin\varphi &\simeq \cos(\eta\varphi)\cos\varphi + \sin(\eta\varphi)\sin\varphi \\ &= \cos(\eta\varphi + \varphi) \\ &= \cos\big[(1+\eta)\varphi\big] \\ &= \cos(\gamma_{\mathrm{ED}}\varphi), \end{aligned} \tag{25}$$

where $(\gamma_{\mathrm{ED}} = 1 + \eta)$; therefore:

$$u = \frac{1 + e[\cos(\gamma_{\mathrm{ED}}\varphi) + \alpha_0\cos(k\varphi)]}{\ell}. \tag{26}$$

At the perihelion, we will have:

$$\cos(\gamma_{\mathrm{ED}}\varphi) + \alpha_0\cos(k\varphi) = 1. \tag{27}$$

For our convenience, we shall set ($\gamma_{\mathrm{ED}}\varphi = \varphi + \delta_1$) and ($k\varphi = \varphi + \delta_2$) where ($\delta_1 = \eta\varphi$) and ($\delta_2 = \eta_{\mathrm{D}}\varphi/2$), so that to first order approximation, we will have:

$$\cos\varphi - \delta_1 \sin\varphi + \alpha_0 \cos\varphi - \alpha_0\delta_2 \sin\varphi = 1, \tag{28}$$

hence:

$$(1+\alpha_0)\cos\varphi - (\delta_1 + \alpha_0\delta_2)\sin\varphi = 1, \tag{29}$$

thus, we can write the left hand-side of this equation as:

$$(1+\alpha_0)\cos\varphi - (\delta_1 + \alpha_0\delta_2)\sin\varphi \equiv R\cos[\varphi - \alpha_*], \tag{30}$$

where:

$$\begin{aligned} R &= \sqrt{(1+\alpha_0)^2 + (\delta_1+\alpha_0\delta_2)^2} \simeq 1 \\ \alpha_* &= \arctan\left(\frac{\delta_1+\alpha_0\delta_2}{1+\alpha_0}\right) \simeq \frac{\delta_1+\alpha_0\delta_2}{1+\alpha_0}. \end{aligned} \tag{31}$$

From all this, it follows that:

$$\cos\left(\varphi + \frac{\delta_1+\alpha_0\delta_2}{1+\alpha_0}\right) \simeq 1. \tag{32}$$

Substituting back ($\delta_1 = \eta\varphi$) and ($\delta_2 = \eta_{\mathrm{D}}\varphi/2$), we will have:

$$\cos\left[\varphi\left(1 + \frac{\eta+\alpha_0\eta_{\mathrm{D}}/2}{1+\alpha_0}\right)\right] \simeq 1, \tag{33}$$

hence:

$$\varphi\left(1 + \frac{\eta+\alpha_0\eta_{\mathrm{D}}/2}{1+\alpha_0}\right) = 2\pi n, \tag{34}$$

where ($n = 0, 1, 2,$ *etc.*). Since φ is a function of n, we can write (34) with φ having a subscript n, as follows:

$$\varphi_n\left(1 + \frac{\eta+\alpha_0\eta_{\mathrm{D}}/2}{1+\alpha_0}\right) = 2\pi n, \tag{35}$$

so that:

$$(\varphi_{n+1} - \varphi_n)\left(1 + \frac{\eta+\alpha_0\eta_{\mathrm{D}}/2}{1+\alpha_0}\right) = 2\pi, \tag{36}$$

where ($\delta\varphi = \varphi_{n+1} - \varphi_n$). Since $[(\eta + \alpha_0\eta_{\mathrm{D}}/2)/(1+\alpha_0)]$ is very small, much smaller than unity, it follows from this that, to first order approximation:

$$\varphi_{n+1} - \varphi_n = 2\pi - \underbrace{\frac{\pi(2\eta+\alpha_0\eta_{\mathrm{D}})}{(1+\alpha_0)}}_{\Delta\varphi}, \tag{37}$$

where:

$$\Delta\varphi = \frac{\pi(2\eta+\alpha_0\eta_{\mathrm{D}})}{(1+\alpha_0)}, \tag{38}$$

is the apsidal advance of that occurs every revolution of the orbit. From all these computations, it is clear that the resulting apsidal precession rate of advance will be given by:

$$\frac{\Delta\varphi}{\mathcal{T}_\varphi} = \frac{1}{1+\alpha_0}\frac{2\pi\eta_{\mathrm{E}}}{\mathcal{T}_\varphi} + \frac{\alpha_0}{1+\alpha_0}\frac{\pi\eta_{\mathrm{D}}}{\mathcal{T}_\varphi}, \tag{39}$$

and this equation can be written as:

$$\frac{\Delta\varphi}{\mathcal{T}_\varphi} = \frac{1}{1+\alpha_0}\left\langle\frac{\Delta\varphi}{\mathcal{T}_\varphi}\right\rangle_{\mathrm{E}} + \frac{\alpha_0}{1+\alpha_0}\frac{\pi\eta_{\mathrm{D}}}{\mathcal{T}_\varphi}. \tag{40}$$

The unknown parameter, α_0, is expected to be extremely small ranging in order of magnitude perhaps to η if not far less. If α_0 is small as afore-stated, it follows that to first order approximation, we must have:

$$\left\langle\frac{\Delta\varphi}{\mathcal{T}_\varphi}\right\rangle_{\mathrm{ED}} \simeq \left\langle\frac{\Delta\varphi}{\mathcal{T}_\varphi}\right\rangle_{\mathrm{E}} + \left\langle\frac{\Delta\varphi}{\mathcal{T}_\varphi}\right\rangle_{\mathrm{D}}, \tag{41}$$

where:

$$\left\langle\frac{\Delta\varphi}{\mathcal{T}_\varphi}\right\rangle_{\mathrm{D}} = -\frac{\pi\alpha_0\eta_{\mathrm{D}}}{\mathcal{T}_\varphi} = -\frac{\pi(1-e^2)a\alpha_0}{\mathcal{T}_\varphi\mathscr{R}_{\mathrm{D}}} = \Delta\dot{\varpi}_{\mathrm{D}}, \tag{42}$$

is the gravitational dark-force field contribution to the anomalous apsidal precession rate of planetary orbits. Now having derived the formula that gives us the apsidal precession due to the gravitational darkforce, we shall in Sect. 6, discuss the dark 'anomalous' apsidal precession, *i.e.*, how it fairs with observational evidence.

6 Apsidal precession of the solar planets

Table 1 gives that anomalous apsidal precession of the Solar planets *i.e.*, the predicted Einstein anomalous apsidal precession $\dot{\varpi}_{\mathrm{E}}$, the predicted gravitational dark-force extra-anomalous apsidal precession $\Delta\dot{\varpi}_{\mathrm{D}}/\alpha_0$, and the measured anomalous apsidal precession from the $\Delta\dot{\varpi}_{\mathrm{EPM2011}}$ (Pitjeva 2013) and $\Delta\dot{\varpi}_{\mathrm{INPOPa}}$ (Fienga et al. 2011, 2009) ephemerides respectively. From the measurements, the extra-anomalous apsidal precession are of the order of mas/cy, while $\Delta\dot{\varpi}_{\mathrm{D}}/\alpha_0$ is of the order of µas/cy. Given that α_0 is small, it follows from this that even if α_0 where known with whatever accuracy—for so long as it is a small quantity, the extra-anomalous apsidal precession, $\Delta\dot{\varpi}_{\mathrm{D}}$, due to the gravitational dark-force is much smaller that a µas/cy, hence, this force is not expected to be the cause of the extra-anomalous apsidal precession detected by *e.g.* in the $\Delta\dot{\varpi}_{\mathrm{EPM2011}}$ and $\Delta\dot{\varpi}_{\mathrm{INPOPa}}$ ephemerides. This is not an unexpected result. Naturally and logically, we expect the dark-force to be significant on the scale-length $\mathscr{R}_{\mathrm{D}}$, which (according to (15)) for the Sun—is, ~ 6000 AU.

Table 1 *Predicted Anomalous Apsidal Precession due to the Gravitational Darkforce Field.* Column (1) gives the name of the planet, columns (2)–(4), the semi-major axis 'a' of the planet's orbits, its orbital period and the eccentricity the planet's orbits, respectively. Column (5) gives the predicted Einstein anomalous apsidal precession $\dot{\varpi}_{\mathrm{E}}$, column (6) the predicted gravitational dark-force extra-anomalous apsidal precession $\Delta\dot{\varpi}_{\mathrm{D}}/\alpha_0$, columns (7) and (8), the measured anomalous apsidal precession from the $\Delta\dot{\varpi}_{\mathrm{EPM2011}}$ and $\Delta\dot{\varpi}_{\mathrm{INPOPa}}$ ephemerides, respectively

Planet	a (AU)	$\mathcal{T}_\varphi$ (yr)	e	$\dot{\varpi}_{\mathrm{E}}$ ($1''$/cy)	$\Delta\dot{\varpi}_{\mathrm{D}}/\alpha_0$ (µas/cy)	$\Delta\dot{\varpi}_{\mathrm{EPM2011}}$ (mas/cy)	$\Delta\dot{\varpi}_{\mathrm{INPOPa}}$ (mas/cy)
Mercury	0.39	0.24	0.206	43.29	1.60 ± 0.10	$+2.00 \pm 3.00$	$+0.40 \pm 0.60$
Venus	0.72	0.62	0.007	8.59	1.20 ± 0.08	$+2.60 \pm 1.60$	$+0.20 \pm 1.50$
Earth	1.00	1.00	0.017	3.85	1.10 ± 0.07	$+0.19 \pm 0.19$	-0.20 ± 0.90
Mars	1.52	1.88	0.093	1.36	0.80 ± 0.06	-0.02 ± 0.04	-0.04 ± 0.10
Jupiter	5.20	11.86	0.048	0.06	0.50 ± 0.03	$+58.70 \pm 28.30$	-41.00 ± 42.00
Saturn	9.54	29.46	0.054	0.01	0.30 ± 0.02	-0.32 ± 0.47	$+0.15 \pm 0.65$

7 General discussion and conclusion

The '*dark-matter*' phenomenon has manifested itself on the grandest of scales—*i.e.*, on the galactic scale. If this dark-matter phenomenon is really a result of our gravitational models lacking, then, it is expected that the new terms arising in the new MoG theory should have be negligible for non-colossal bodies such as our Sun, the meaning of which is that one would naturally expect not to pick-up any signals associated with the new terms in the MoG. This is different from dark-matter theories—*i.e.*, theories that hold that our gravitational theories are not lacking and go to postulate the existence of some—*perhaps*—invisible, weak interacting and non-luminous matter, these theories (*e.g.*, Pitjeva and Pitjev 2013; Pitjev and Pitjeva 2013) do expect the effects of this dark-matter to manifest in the Solar system. The simple message coming out of this reading that the present MoG theory predicts that the extra gravitational poles that come into play to explain the flat rotation curves of spiral galaxies, these terms are negligible on smaller scales (*e.g.*, stars, molecular core, molecular clouds, *etc.*) where the effects dark-matter have not been dictated.

8 Conclusion

In conclusion, while the gravitational darkforce does contribute to the apsidal precession of Solar planetary orbits, the contribution of this force is negligibly small.

Dedications As is the case with Papers (II) & (III), this paper is dedicated to the memory, illustrious life and enduring works the brilliant *Nobel Laurent* that was not afforded to the *World*—Professor Dr. Vera Cooper Rubin (July 23, 1928–December 25, 2016). May Her Dear Soul Rest In Peace. Despite the fact this work being submitted for publication more than a year latter, this work was completed just before the passing on of Professor Dr. Vera Cooper Rubin and prior to her passing on, the work was already dedicated to her—the hope of which that she would one-day read this dedication, thus, we felt it proper to maintain this dedication as our own lasting tribute to her unparalleled devotion on this issue of dark-matter. We have kept this work unpublished because, related work—that would support the main theory out of which the present dark-matter model has been build, was still ongoing.

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