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Dark energy cosmological models with cosmic string

T. Vinutha[1] · V.U.M. Rao[1] · Bekele Getaneh[1] · Molla Mengesha[1]

Abstract In this paper we have studied the anisotropic Kantowski-Sachs, locally rotationally symmetric (LRS) Bianchi type-I and LRS Bianchi type-III geometries filled with dark energy and one dimensional cosmic string in the Saez-Ballester theory of gravitation. To get physically valid solution we take hybrid expansion law of the average scale factor which is a product of power and exponential type of functions that results in time dependent deceleration parameter (q). The equation of state parameter of dark energy (ω_{de}) has been discussed and we have observed that for the three models it crosses the phantom divide line ($\omega_{de} = -1$) and shows quintom like behavior. The density of dark energy (ρ_{de}) is an increasing function of redshift and remains positive throughout the evolution of the universe for the three models. Moreover in Kantowski-Sachs and LRS Bianchi type-I geometries the dark energy density dominates the string tension density (λ) and proper density (ρ) throughout the evolution of the universe. The physical and geometrical aspects of the statefinder parameters (r, s), squared speed of sound (v_s^2) and ω_{de}–ω'_{de} plane are also discussed.

Keywords Kantowski-Sachs space-time · LRS Bianchi type-I space-time · LRS Bianchi type-III space time · Dark energy · String tension density · Scalar tensor theory of gravitation

1 Introduction

Recently, there has been a considerable interest in cosmological models with dark energy due to the accelerated expansion of the universe. Astronomical observations indicate that our universe is flat and currently consists of about 70% dark energy and 30% in the form of non relativistic matter (including both baryons and dark matter). Cosmological observations such as type Ia supernovae (SNe Ia) indicate that the universe is undergoing accelerated expansion as described by Riess et al. (1998), Perlmutter et al. (1999), Spergel et al. (2007), Spergel (2003), Bennett et al. (2003), Tegmark et al. (2004), Abazajian et al. (2003). Cosmological models of the universe are central in understanding the mysteries of the early stages of its evolution. Dark energy (DE) is a component to the matter distribution of the universe with a large negative pressure. The nature of dark energy as well as dark matter is unknown. Recent studies to extract the properties of a dark energy component of the universe from observational data focus on the determination of its equation of state (ω_{de}), which is the ratio of the dark energy's pressure (p_{de}) to its energy density (ρ_{de}), $\omega_{de} = \frac{p_{de}}{\rho_{de}}$, which is not necessarily constant. Sahni and Starobinsky (2006) have developed a method for revival of the quantity ω_{de} from experimental data. Also Sahni et al. (2008) analyzed to get ω_{de} as a function of cosmological time. The simplest dark energy candidate is the vacuum energy ($\omega_{de} = -1$), which is mathematically equivalent to the cosmological constant (Λ). The other conventional alternatives, which can be described by minimally coupled scalar fields, are quintessence ($-1 < \omega_{de} \leq 0$), phantom energy ($\omega_{de} < -1$) and quintom (that can cross from phantom region to quintessence region as evolved) and have time dependent EoS parameter. Due to lack of observational evidence in making a distinction between constant and variable ω_{de}, usually the equation of state parameter is considered as a constant, Bartelmann et al. (2005), Kujat et al. (2002) described ω_{de} with phase wise values $-1, 0, 1/3$ and 1 for vacuum fluid, dust fluid, radiation and stiff matter dominated universe, respectively. But in

✉ T. Vinutha
vinuthatummala@gmail.com

1 Department of Applied Mathematics, Andhra University, Visakhapatnam 530003, India

general ω_{de} is a function of time or redshift Das et al. (2005), Jimenez (2003).

Thorne (1967) has described that Kantowski-Sachs class of metric represents homogeneous but anisotropically expanding (contracting) cosmologies and provides models where the effect of anisotropy can be estimated and compared with FRW class of cosmologies. Rahaman et al. (2002), Rao and Neelima (2013) and Santhi et al. (2016) are some authors who worked on Kantowski-Sachs cosmological model.

The spatially homogeneous and anisotropic Bianchi type space-time presents a "middle way" between the FRW model and inhomogeneous anisotropic universe. It thus plays an important role in modern cosmology. Some works on LRS Bianchi type-I and LRS Bianchi type-III models can be found in Rao et al. (2012), Adhav et al. (2013), Bali and Upadhaya (2003), Deo et al. (2016).

Alternative theories of gravitation become important in the field of cosmology, this is because of the fact that general relativity does not fully incorporate Mach's principle. To do this Brans and Dicke (1961) scalar–tensor theory of gravitation has been formulated which develops Mach's principle by assuming interaction of inertial masses of fundamental particles with some cosmic scalar field coupled with the large scale distribution of matter in motion. Following Saez and Ballester (1986) developed a scalar–tensor theory of gravitation in order to solve dark matter problem in which the metric is coupled with a dimensionless scalar field ϕ in a simple manner. Rao and Prasanthi (2017), Rao et al. (2011), Rao et al. (2007), Rao et al. (2008), Ramesh and Umadevi (2016) have studied some Bianchi type cosmological models in Saez-Ballester theory of gravitation. Some works on Saez-Ballester theory can be found in Pradhan et al. (2012), Naidu et al. (2012) and Pawar and Solanke (2016). Inspired by the above in the present paper we have studied Kantowski-Sachs, LRS Bianchi type-I and LRS Bianchi type-III cosmological models in Saez-Ballester theory of gravitation with dark energy and one dimensional cosmic string.

In our work the present value of the deceleration parameter is calculated. The plots of some of the cosmological parameters versus red shift are presented to show their physical properties. Equations of state parameter for the dark energy, the string and isotropic fluid has been considered. In addition to the anisotropic DE fluid, cosmic strings assigned along x-direction are also considered to comprise some anisotropic effect (Mishra et al. 2017).

The paper is organized as follows. In Sect. 2, we discuss about the basic formulation of field equations followed by the solution of the field equations of the three models. In Sect. 3, we discussed about exact solution of scalar field (ϕ), in Sect. 4 we discuss about some important properties of the models and we summarize the results of the three models in the last section.

2 Basic formalism

The field equations of Saez and Ballester scalar–tensor theory ($8\pi G = 1$ and $c = 1$) (Saez and Ballester 1986) are

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij} + \overline{\omega}\phi^n\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,\beta}\phi^{,\beta}\right) \quad (1)$$

and the scalar field ϕ satisfies the equation

$$2\phi^n\phi^i_{;i} + n\phi^{n-1}\phi_{,\beta}\phi^{,\beta} = 0, \quad (2)$$

where R_{ij} is the Ricci tensor, R is the scalar curvature, the energy momentum tensor for a given environment of two non interacting fluids is given by

$$T_{ij} = T^{de}_{ij} + T^{\lambda}_{ij}, \quad (3)$$

where, T^{de}_{ij} is the stress energy tensor of the dark energy and T^{λ}_{ij} is the stress energy tensor of one dimensional cosmic string. ϕ is a dimensionless scalar field which is a function of cosmic time t alone. $\overline{\omega}$ and n are constants. Comma and semicolon denote partial and covariant differentiation, respectively.

Also, we have energy conservation equation as

$$T^{ij}_{;j} = 0. \quad (4)$$

The energy-momentum tensor of the dark energy is given by

$$T^{de}_{ij} = \text{diag}[1, -\omega_{de}, -\omega_{de}, -\omega_{de}]\rho_{de}, \quad (5)$$

where $\omega_{de} = \frac{p_{de}}{\rho_{de}}$ is the dark energy equation of state parameter (EoS) and p_{de} and ρ_{de} are pressure and density of the dark energy, respectively.

For fluid containing one dimensional cosmic string, the energy momentum tensor is given by Mishra et al. (2017), Letelier (1980) and Stachel (1980)

$$T^{\lambda}_{ij} = (\rho + p)u_iu_j - pg_{ij} - \lambda x_ix_j. \quad (6)$$

Here $u_iu_j = 1$, $x_ix_j = -1$ (along x-direction). In the co-moving coordinate system u^i is the four velocity vector and p is the isotropic pressure of the fluid. ρ is the proper density and is composed of energy density due to massive particle and string tension density λ. In the absence of any string phase, the total contribution to the baryonic energy density comes from particles only. In contrast to isotropic pressure of usual cosmic fluid, we wish to incorporate some degree of anisotropy in the dark energy pressure.

Now we consider the metric

$$ds^2 = dt^2 - D^2(t)\,dr^2 - E^2(t)\left[d\theta^2 + F_l^2(\theta)\,d\psi^2\right], \quad (7)$$

where

$$
F_l^2(\theta) = \begin{cases}
\sin^2\theta, & \text{when } l=1 \\
& \text{(Kantowski-Sachs model)}, \\
\theta^2, & \text{when } l=0 \\
& \text{(LRS Bianchi type-I model)}, \\
\sinh^2\theta, & \text{when } l=-1 \\
& \text{(LRS Bianchi type-III model)}.
\end{cases}
\tag{8}
$$

Here l is the spatial curvature index and the above three models are closed, Euclidean and semi-closed respectively (Shamir 2010).

2.1 Case (i): $l = 1$, Kantowski-Sachs space-time

We consider the spatially homogeneous and anisotropic Kantowski-Sachs space-time given by Kantowski and Sachs (1966), equations (7) and (8) in the form

$$
ds^2 = dt^2 - D^2(t)\,dr^2 - E^2(t)\big[d\theta^2 + \sin^2\theta\,d\psi^2\big],
\tag{9}
$$

where D and E are the functions of cosmic time t only.

For the metric given by (9), using (3), (5) and (6), the field equations (1) and (2) can be written as

$$
\frac{\dot{E}^2}{E^2} + 2\frac{\ddot{E}}{E} + \frac{1}{E^2} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = \lambda - p - \omega_{de}\rho_{de},
\tag{10}
$$

$$
\frac{\ddot{D}}{D} + \frac{\ddot{E}}{E} + \frac{\dot{D}\dot{E}}{DE} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = -p - \omega_{de}\rho_{de},
\tag{11}
$$

$$
2\frac{\dot{D}\dot{E}}{DE} + \frac{\dot{E}^2}{E^2} + \frac{1}{E^2} + \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = \rho + \rho_{de},
\tag{12}
$$

$$
\ddot{\phi} + \dot{\phi}\left(\frac{\dot{D}}{D} + 2\frac{\dot{E}}{E}\right) + \frac{n}{2}\frac{\dot{\phi}^2}{\phi} = 0.
\tag{13}
$$

The energy conservation equation (4), yields

$$
\dot{\rho} + 3(p+\rho)H - \lambda H_1 + \dot{\rho}_{de} + 3\rho_{de}(\omega_{de}+1)H = 0,
\tag{14}
$$

where H is Hubble's parameter, H_1 is directional Hubble's parameter and dot (.) represents derivative with respect to cosmic time t.

2.1.1 Solutions of the field equations

From the field equations (10) to (13) we have four independent equations with eight unknowns D, E, ϕ, ρ_{de}, ω_{de}, ρ, p and λ. In order to find a deterministic solution we take the following two physically valid conditions.

(i) We consider the cosmological scale factor as a hybrid expansion law (Saha et al. 2012)

$$
R(t) = me^{at}t^b,
\tag{15}
$$

where $a \geq 0$, $b \geq 0$ and $m > 0$ are constants.

(ii) We take the shear scalar σ in the model to be proportional to the expansion scalar θ, this condition leads to (Collins et al. 1980)

$$
D = E^k,
\tag{16}
$$

where $k > 0$ and $k \neq 1$ is a constant.

Using (16) we get,

$$
\frac{\dot{D}}{D} = k\frac{\dot{E}}{E}.
\tag{17}
$$

The directional Hubble parameters are,

$$
H_1 = \frac{\dot{D}}{D}, \qquad H_2 = H_3 = \frac{\dot{E}}{E}.
\tag{18}
$$

The mean Hubble parameter is,

$$
H = \frac{1}{3}(H_1 + 2H_2).
\tag{19}
$$

Then from (17)–(19) it follows that

$$
H_1 = \frac{3k}{k+2}H, \qquad H_2 = H_3 = \frac{3}{k+2}H.
\tag{20}
$$

The Hubble parameter H is given by

$$
H = \frac{\dot{R}}{R}.
\tag{21}
$$

From (15) and (21) the Hubble parameter H is obtained as

$$
H = a + \frac{b}{t}.
\tag{22}
$$

From (17)–(19) and (21) we get,

$$
D = \left(me^{at}t^b\right)^{\frac{3k}{k+2}},
\tag{23}
$$

$$
E = \left(me^{at}t^b\right)^{\frac{3}{k+2}}.
\tag{24}
$$

Now the metric with (9) with the help of (23) and (24) can be written as

$$
ds^2 = dt^2 - \left(me^{at}t^b\right)^{\frac{6k}{k+2}} dr^2
$$
$$
- \left(me^{at}t^b\right)^{\frac{6}{k+2}}\big[d\theta^2 + \sin^2\theta\,d\psi^2\big].
\tag{25}
$$

From (13), (18), (19), (21), (23) and (24) we find Saez-Ballester scalar field as

$$
\phi = \left[\frac{n+2}{2}\int \left(ce^{at}t^b\right)^{-3} dt + \phi_0\right]^{\frac{2}{n+2}},
\tag{26}
$$

where c and ϕ_0 are constants of integration.

From (10), (11), (13), (23) and (24) the string tension density λ is obtained as

$$\lambda = \frac{9(1-k)}{k+2}\left(a+\frac{b}{t}\right)^2 + \frac{3b(k-1)}{t^2(k+2)} + \frac{1}{(me^{at}t^b)^{\frac{6}{k+2}}}. \quad (27)$$

We consider,

$$\lambda = \alpha\rho, \qquad p = \omega\rho \quad (28)$$

where α and ω are assumed to be non evolving state parameters.

The proper density ρ is given by

$$\rho = \frac{9(1-k)}{\alpha(k+2)}\left(a+\frac{b}{t}\right)^2 + \frac{3b(k-1)}{\alpha t^2(k+2)} + \frac{1}{\alpha(me^{at}t^b)^{\frac{6}{k+2}}}. \quad (29)$$

The isotropic pressure of the fluid p is given by

$$p = \frac{9\omega(1-k)}{\alpha(k+2)}\left(a+\frac{b}{t}\right)^2 + \frac{3b\omega(k-1)}{\alpha t^2(k+2)} + \frac{\omega}{\alpha(me^{at}t^b)^{\frac{6}{k+2}}}. \quad (30)$$

From equations (12), (13), (23), (24), (27)–(29) the dark energy density (ρ_{de}) is obtained as,

$$\rho_{de} = 9\left[\frac{2\alpha k + \alpha + k^2 + k - 2}{\alpha(k+2)^2}\right]\left(a+\frac{b}{t}\right)^2 + \frac{\alpha-1}{\alpha(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{3b(k-1)}{\alpha(k+2)t^2} + \frac{\overline{\omega}}{2(me^{at}t^b)^6}. \quad (31)$$

Using (10), (11), (13), (23), (24), (27)–(31) the EoS parameter (ω_{de}) for the dark energy is calculated as

$$\omega_{de} = \frac{18[\frac{\alpha(k^2+k+1)-\omega(k^2+k-2)}{\alpha(k+2)^2}](a+\frac{b}{t})^2 + (\omega(k-1)-\alpha(k+1))(\frac{6b}{\alpha t^2(k+2)}) + \frac{2\omega}{\alpha(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{\overline{\omega}}{(me^{at}t^b)^6}}{-2([\frac{18\alpha k + 9\alpha - 9k^2 - 9k + 18}{\alpha(k+2)^2}](a+\frac{b}{t})^2 + \frac{\alpha-1}{\alpha(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{3b(k-1)}{\alpha(k+2)t^2} + \frac{\overline{\omega}}{2(me^{at}t^b)^6})}. \quad (32)$$

Using (31), (32) the pressure of dark energy (p_{de}) is given by

$$p_{de} = -9\left[\frac{\alpha(k^2+k+1)-\omega(k^2+k-2)}{\alpha(k+2)^2}\right]\left(a+\frac{b}{t}\right)^2$$
$$- (\omega(k-1)-\alpha(k+1))\left(\frac{3b}{\alpha t^2(k+2)}\right)$$
$$- \frac{\omega}{\alpha(me^{at}t^b)^{\frac{6}{k+2}}} + \frac{\overline{\omega}}{2(me^{at}t^b)^6}. \quad (33)$$

Average density parameters of the dark energy (Ω_{de}), dark matter (Ω_m) are given by

$$\Omega_{de} = \frac{\rho_{de}}{3H^2} = 9\left[\frac{2\alpha k + \alpha + k^2 + k - 2}{3\alpha(k+2)^2}\right]$$
$$+ \frac{\alpha-1}{3\alpha(me^{at}t^b)^{\frac{6}{k+2}}(a+\frac{b}{t})^2} - \frac{b(k-1)}{\alpha(k+2)t^2(a+\frac{b}{t})^2}$$
$$+ \frac{\overline{\omega}}{6(me^{at}t^b)^6(a+\frac{b}{t})^2}, \quad (34)$$

$$\Omega_m = \frac{\rho}{3H^2} = \frac{3(1-k)}{\alpha(k+2)} + \frac{b(k-1)}{\alpha t^2(k+2)(a+\frac{b}{t})^2}$$
$$+ \frac{1}{3\alpha(me^{at}t^b)^{\frac{6}{k+2}}(a+\frac{b}{t})^2}. \quad (35)$$

2.2 Case (ii): $l = 0$, LRS Bianchi type-I space-time

We consider the spatially homogeneous and anisotropic LRS Bianchi type-I space-time given by (7) and (8) in the form

$$ds^2 = dt^2 - D^2(t)\,dr^2 - E^2(t)\left[d\theta^2 + \theta^2 d\psi^2\right], \quad (36)$$

where D and E are the functions of cosmic time t only.

For the metric given by (36), using (3), (5) and (6), the field equations (1) and (2) takes the form

$$\frac{\dot{E}^2}{E^2} + 2\frac{\ddot{E}}{E} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = \lambda - p - \omega_{de}\rho_{de}, \quad (37)$$

$$\frac{\ddot{D}}{D} + \frac{\ddot{E}}{E} + \frac{\dot{D}\dot{E}}{DE} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = -p - \omega_{de}\rho_{de}, \quad (38)$$

$$2\frac{\dot{D}\dot{E}}{DE} + \frac{\dot{E}^2}{E^2} + \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = \rho + \rho_{de}, \quad (39)$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{D}}{D} + 2\frac{\dot{E}}{E}\right) + \frac{n}{2}\frac{\dot{\phi}^2}{\phi} = 0. \quad (40)$$

The energy conservation equation (4), yields

$$\dot{\rho} + 3(p+\rho)H - \lambda H_1 + \dot{\rho}_{de} + 3\rho_{de}(\omega_{de}+1)H = 0. \quad (41)$$

2.2.1 Solutions of the field equations

From the field equations (37) to (40) we have four independent equations with eight unknowns D, E, ϕ, ρ_{de}, ω_{de}, ρ, p and λ. In order to find a deterministic solution we take the following two physically valid conditions.

(i) We consider the cosmological scale factor given by (15)
(ii) We take the relation given by (16).

Now the metric with equation (36) with the help of (17)–(24) can be written as

$$ds^2 = dt^2 - \left(me^{at}t^b\right)^{\frac{6k}{k+2}}(t)\, dr^2$$
$$- \left(me^{at}t^b\right)^{\frac{6}{k+2}}\left[d\theta^2 + \theta^2\, d\psi^2\right]. \tag{42}$$

The Saez-Ballester scalar field is given by (26).

From (23), (24), (37), (38) and (40) the string tension density λ is obtained as

$$\lambda = \frac{9(1-k)}{k+2}\left(a + \frac{b}{t}\right)^2 + \frac{3b(k-1)}{t^2(k+2)}. \tag{43}$$

Using the relations given by (28), the proper density ρ is given by

$$\rho = \frac{9(1-k)}{\alpha(k+2)}\left(a + \frac{b}{t}\right)^2 + \frac{3b(k-1)}{\alpha t^2(k+2)}, \tag{44}$$

and the isotropic pressure of the fluid p is given by

$$p = \frac{9\omega(1-k)}{\alpha(k+2)}\left(a + \frac{b}{t}\right)^2 + \frac{3\omega b(k-1)}{\alpha t^2(k+2)}. \tag{45}$$

From equations (23), (24), (39), (40) and (44) the dark energy density (ρ_{de}) is obtained as,

$$\rho_{de} = 9\left[\frac{2\alpha k + \alpha + k^2 + k - 2}{\alpha(k+2)^2}\right]\left(a + \frac{b}{t}\right)^2$$
$$- \frac{3b(k-1)}{\alpha(k+2)t^2} + \frac{\overline{\omega}}{2(me^{at}t^b)^6}. \tag{46}$$

Using (23), (24), (37)–(40), (43)–(46) the EoS parameter (ω_{de}) for the dark energy is calculated as

$$\omega_{de} = \frac{18\left[\frac{\alpha(k^2+k+1)-\omega(k^2+k-2)}{\alpha(k+2)^2}\right](a+\frac{b}{t})^2 - (\alpha(k+1)+\omega(k-1))(\frac{6b}{\alpha t^2(k+2)}) - \frac{\overline{\omega}}{(me^{at}t^b)^6}}{-2([9\frac{2\alpha k+\alpha+k^2+k-2}{\alpha(k+2)^2}](a+\frac{b}{t})^2 - \frac{3b(k-1)}{\alpha(k+2)t^2} + \frac{\overline{\omega}}{2(me^{at}t^b)^6})}. \tag{47}$$

Using (46) and (47) the pressure of dark energy (p_{de}) is given by

$$p_{de} = -9\left[\frac{\alpha(k^2+k+1)+\omega(k^2+k-2)}{\alpha(k+2)^2}\right]\left(a + \frac{b}{t}\right)^2$$
$$+ (\alpha(k+1)+\omega(k-1))\left(\frac{3b}{\alpha t^2(k+2)}\right)$$
$$+ \frac{\overline{\omega}}{2(me^{at}t^b)^6}. \tag{48}$$

Average density parameters of the dark energy (Ω_{de}), dark matter (Ω_m) are given by

$$\Omega_{de} = 9\left[\frac{2\alpha k + \alpha + k^2 + k - 2}{3\alpha(k+2)^2}\right] - \frac{b(k-1)}{\alpha(k+2)t^2(a+\frac{b}{t})^2}$$
$$+ \frac{\overline{\omega}}{6(me^{at}t^b)^6(a+\frac{b}{t})^2}, \tag{49}$$

$$\Omega_m = \frac{3(1-k)}{\alpha(k+2)} + \frac{b(k-1)}{\alpha t^2(k+2)(a+\frac{b}{t})^2}. \tag{50}$$

2.3 Case (iii): $l = -1$, LRS Bianchi type-III space-time

We consider the spatially homogeneous and anisotropic LRS Bianchi type-III space-time given by (7) and (8) in the form

$$ds^2 = dt^2 - D^2(t)dr^2 - E^2(t)\left[d\theta^2 + \sinh^2\theta d\psi^2\right], \tag{51}$$

where D and E are the functions of cosmic time t only. For the metric given by (51), using (3), (5) and (6), the field equations (1) and (2) takes the form

$$\frac{\dot{E}^2}{E^2} + 2\frac{\ddot{E}}{E} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = \lambda - p - \omega_{de}\rho_{de}, \tag{52}$$

$$\frac{\ddot{D}}{D} + \frac{\ddot{E}}{E} + \frac{\dot{D}\dot{E}}{DE} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = -p - \omega_{de}\rho_{de}, \tag{53}$$

$$\frac{\ddot{D}}{D} + \frac{\ddot{E}}{E} + \frac{\dot{D}\dot{E}}{DE} - \frac{1}{D^2} - \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = -p - \omega_{de}\rho_{de}, \tag{54}$$

$$2\frac{\dot{D}\dot{E}}{DE} + \frac{\dot{E}^2}{E^2} - \frac{1}{D^2} + \frac{1}{2}\overline{\omega}\phi^n\dot{\phi}^2 = \rho + \rho_{de}, \tag{55}$$

$$\frac{\dot{D}}{D} - \frac{\dot{E}}{E} = 0, \tag{56}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{\dot{D}}{D} + 2\frac{\dot{E}}{E}\right) + \frac{n}{2}\frac{\dot{\phi}^2}{\phi} = 0. \tag{57}$$

The energy conservation equation (4), yields

$$\dot{\rho} + 3(p+\rho)H - \lambda H_1 + \dot{\rho}_{de} + 3\rho_{de}(\omega_{de}+1)H = 0. \tag{58}$$

2.3.1 Solutions of the field equations

From the field equations (52)–(57) we have six independent equations with eight unknowns D, E, ϕ, ρ_{de}, ω_{de}, ρ, p and λ. In order to find a deterministic solution we take the cosmological scale factor given by (15)

Using (56) we get,

$$D = c_1 E, \tag{59}$$

where c_1 is constant of integration. The directional Hubble parameters are,

$$H_1 = H_2 = H_3 = \frac{\dot{E}}{E}. \tag{60}$$

The mean Hubble parameter is,

$$H = \frac{1}{3}(H_1 + H_2 + H_3) = \frac{\dot{E}}{E} = \frac{\dot{R}}{R}. \tag{61}$$

Hence taking $c_1 = 1$ it follows that

$$D = E = me^{at}t^b. \tag{62}$$

Now the metric with equation (51) with the help of (62) can be written as

$$ds^2 = dt^2 - \left(me^{at}t^b\right)^2 dr^2 - \left(me^{at}t^b\right)^2\left[d\theta^2 + sinh^2\theta d\psi^2\right]. \tag{63}$$

From (57), (61) and (62) we find Saez-Ballester scalar field as

$$\phi = \left[\frac{n+2}{2}\int \left(ce^{at}t^b\right)^{-3}dt + \phi_0\right]^{\frac{2}{n+2}}, \tag{64}$$

where c and ϕ_0 are constants of integration.

From (52)–(54) and (57), (62) the string tension density λ is obtained as

$$\lambda = 0 \tag{65}$$

Using (28) and (65), the proper density ρ is given by

$$\rho = 0. \tag{66}$$

The isotropic pressure of the fluid p is given by

$$p = 0. \tag{67}$$

From (55), (57) and (62) the dark energy density (ρ_{de}) is obtained as

$$\rho_{de} = 3\left(a + \frac{b}{t}\right)^2 + \frac{\overline{\omega}}{2(me^{at}t^{bt})^6} - \frac{1}{(me^{at}t^{bt})^2}. \tag{68}$$

Using (52)–(54), (57), (62), (65)–(68), the EoS parameter (ω_{de}) for the dark energy is calculated as

$$\omega_{de} = \frac{-3(a + \frac{b}{t})^2 + \frac{2b}{t^2} + \frac{\overline{\omega}}{2(me^{at}t^{bt})^6} + \frac{1}{3(me^{at}t^{bt})^2}}{3(a + \frac{b}{t})^2 + \frac{\overline{\omega}}{2(me^{at}t^{bt})^6} - \frac{1}{(me^{at}t^{bt})^2}}. \tag{69}$$

Using (68) and (69) the pressure of dark energy (p_{de}) is given by

$$p_{de} = -3\left(a + \frac{b}{t}\right)^2 + \frac{2b}{t^2} + \frac{\overline{\omega}}{2(me^{at}t^{bt})^6} + \frac{1}{3(me^{at}t^{bt})^2}. \tag{70}$$

Average density parameters of the dark energy (Ω_{de}) is given by

$$\Omega_{de} = 1 + \frac{\overline{\omega}}{6(me^{at}t^{bt})^6(a + \frac{b}{t})^2} - \frac{1}{3(me^{at}t^{bt})^2(a + \frac{b}{t})^2}. \tag{71}$$

3 Exact solutions

In this section we search to find exact solution of scalar field following the work of Liddle and Scherrer (1998). In our choice of scale factor we assumed hybrid-expansion law where $R \propto e^{at}t^b$, now in order to find exact solution let us consider two cases

Case (I) Power law where,

$$R \propto t^b. \tag{72}$$

In this case the scalar field ϕ which is the solution of (13), (40) and (57) is obtained as

$$\phi = \left[\frac{n+2}{2(1-3b)}t^{1-3b}\right]^{\frac{2}{n+2}}. \tag{73}$$

Case (II) Exponent law where

$$R \propto e^{at}. \tag{74}$$

In this case the scalar field ϕ which is the solution of (13), (40) and (57) is obtained as

$$\phi = \left[\frac{n+2}{-6a}e^{-3at}\right]^{\frac{2}{n+2}}. \tag{75}$$

The solutions that we get for ϕ are particular solutions containing no arbitrary constants and known as exact (singular) solutions.

4 Some other important properties of the models

The spatial volume of the three models with (25), (42) and (63) is given by

$$V = R^3 = \left(me^{at}t^b\right)^3. \tag{76}$$

The expansion scalar θ for the three models is

$$\theta = u^i_{;i} = 3H = 3\left(a + \frac{b}{t}\right). \tag{77}$$

From (77) we observe that when $t \to 0$, $\theta \to \infty$ and this indicates the inflationary scenario at early stages of the universe.

The shear scalar σ for the Kantowski-Sachs and LRS Bianchi type-I models with (25) and (42) is

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{2}\left(\frac{\dot{D}^2}{D^2} + 2\frac{\dot{E}^2}{E^2}\right) - \frac{1}{6}\theta^2$$

$$= 3\left(\frac{k^2 - 2k + 1}{(k+2)^2}\right)\left(a + \frac{b}{t}\right)^2. \tag{78}$$

The shear scalar σ for the LRS Bianchi type-III model with (63) is given by

$$\sigma = 0. \tag{79}$$

The average anisotropic parameter $\mathscr{A}$ for the Kantowski-Sachs and LRS Bianchi type-I models with (25) and (42) is given by

$$\mathscr{A} = \frac{1}{3}\sum_{i=1}^{3}\frac{(H_i - H)^2}{H^2} = 2\frac{(k-1)^2}{(k+2)^2}. \tag{80}$$

The value of the average anisotropic parameter $\mathscr{A}$ is positive constant for $k \neq 1$ which shows that Kantowski-Sachs and LRS Bianchi type-I models are anisotropic through out the evolution of the universe.

The value of average anisotropic parameter $\mathscr{A}$ for the LRS Bianchi type-III model with equation (63) is given by

$$\mathscr{A} = 0, \tag{81}$$

which shows that LRS Bianchi type-III model is isotropic throughout the evolution of the universe.

In all the discussions of the models and graphical representations of physical parameters we constrain the constants as: $a = 0.055$, $b = 0.29$, $k = 0.8$, $\alpha = 0.9$, $m = 0.3$, $\omega = 0.8$, $\overline{\omega} = 50$ and the cosmic time t in billion years.
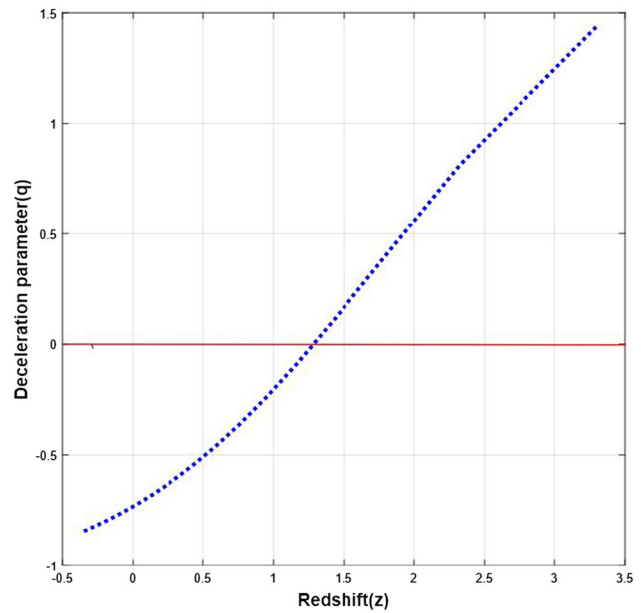

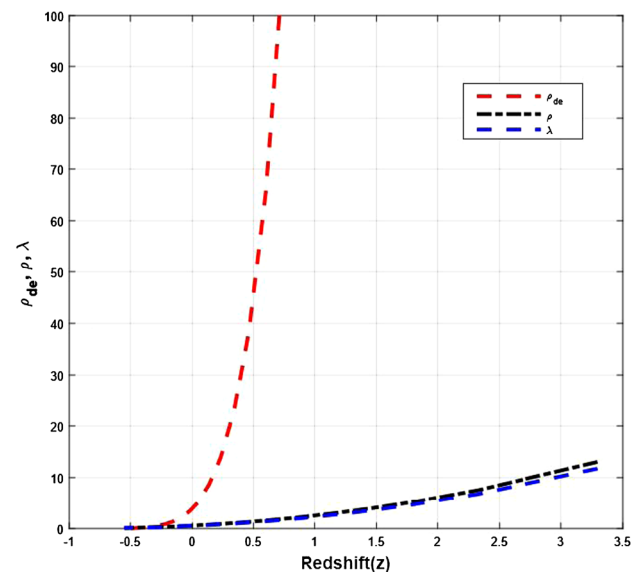**Fig. 1** Plot of q versus red shift (z) for the three models

Fig. 2 Plot of ρ_{de}, λ and ρ versus red shift (z) for Kantowski-Sachs model

The deceleration parameter (DP) q which measure the rate of slowing down of the expansion factor, for the three models is given by

$$q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) = -1 + \frac{b}{(at+b)^2}. \tag{82}$$

Positive value of q indicates the standard decelerating model, while negative value of q indicates inflation (Riess et al. 1998; Bennett et al. 2003). From Fig. 1 we observed that the values of the deceleration parameter $q < 0$ for $z < 1.25$ indicating that the universe appears to be expand-
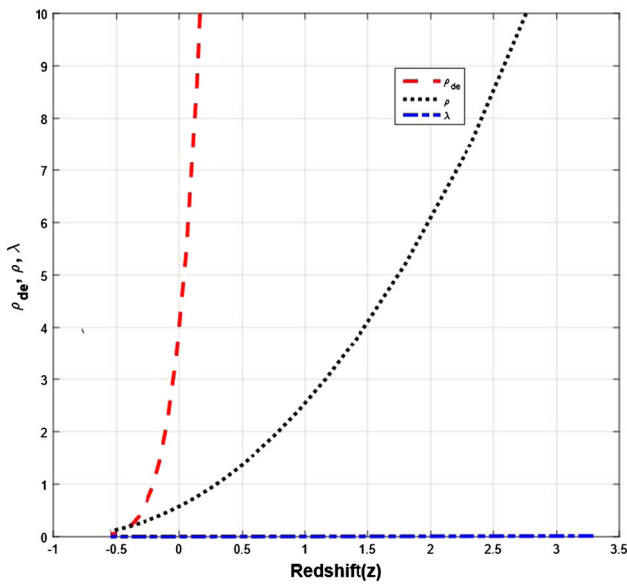
Fig. 3 Plot of ρ_{de}, λ and ρ versus red shift (z) for LRS B-I model
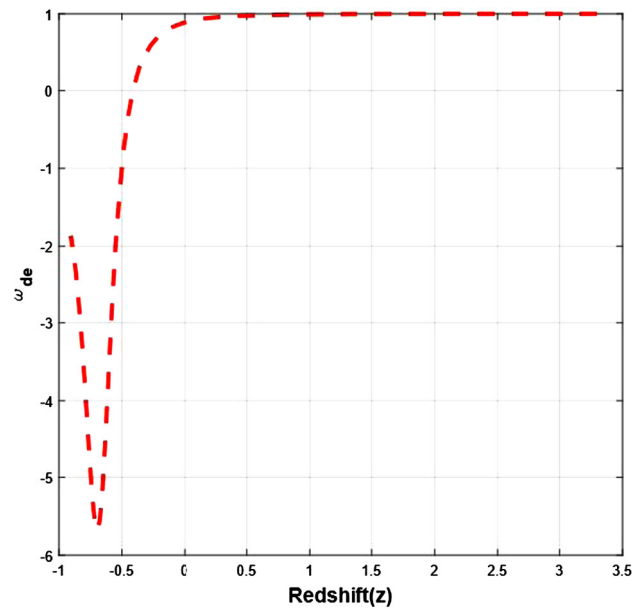
Fig. 5 Plot of ω_{de} versus red shift (z) for Kantowski-Sachs model
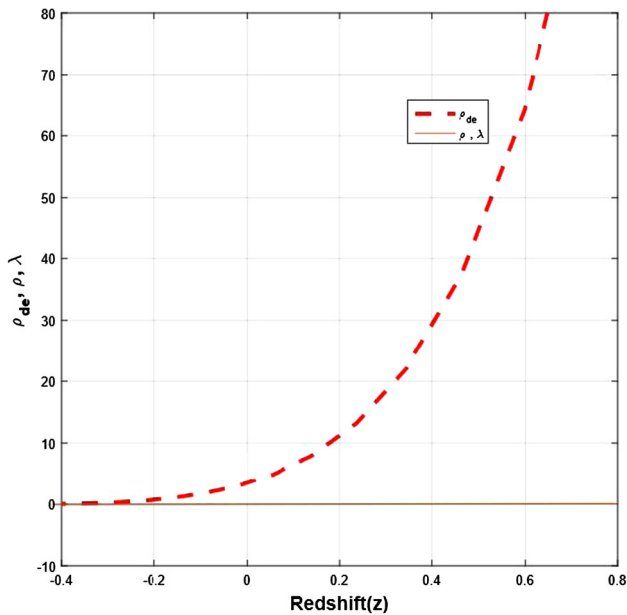
Fig. 4 Plot of ρ_{de}, ρ, λ versus red shift (z) for LRS B-III model
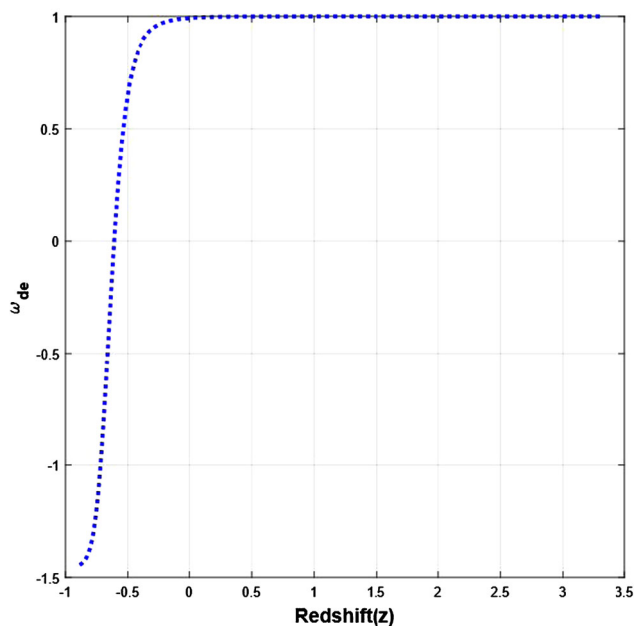

Fig. 6 Plot of ω_{de} versus red shift (z) for LRS B-I model

ing in accelerating rate at present epoch and late time and $q > 0$ for $z > 1.25$ indicating that the models were decelerating at early time. From Fig. 1 we can also observe that $q = 0$ when $z \approx 1.25$, therefore transition from early deceleration to late time inflation of the universe in our models occurs at $z \approx 1.25$. In our models the present value of deceleration parameter $q_0 \approx -0.73$, which match the observed value (Cunha et al. 2009).

From Figs. 2, 3 and 4 we observed that the density of dark energy (ρ_{de}) for the three models is increasing functions of redshift and remains positive throughout the evo-

lution of the universe. From Fig. 2 and 3 we have also observed that for Kantowski-Sachs and Bianchi type-I geometries the dark energy density dominates the string tension density and the proper density throughout the evolution of the universe. From Fig. 3 we observed that the value of λ is very small for LRS Bianchi type-I model. Form Fig. 4 we also observed that for LRS Bianchi type-III model λ and ρ vanishes for all t.

From (32), (47) and (69) it is clear that the equation of state parameter of dark energy ω_{de} is a function of time and
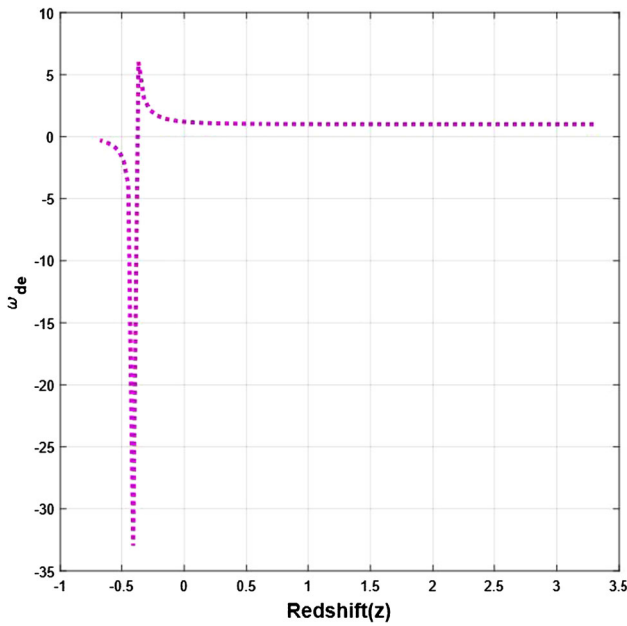
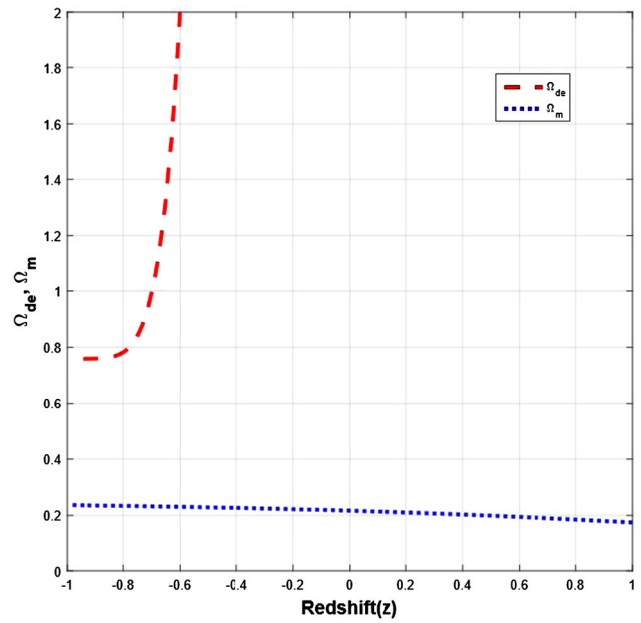**Fig. 7** Plot of ω_{de} versus red shift (z) for LRS B-III model

Fig. 9 Plot of Ω_{de}, Ω_m versus red shift (z) for LRS B-I model
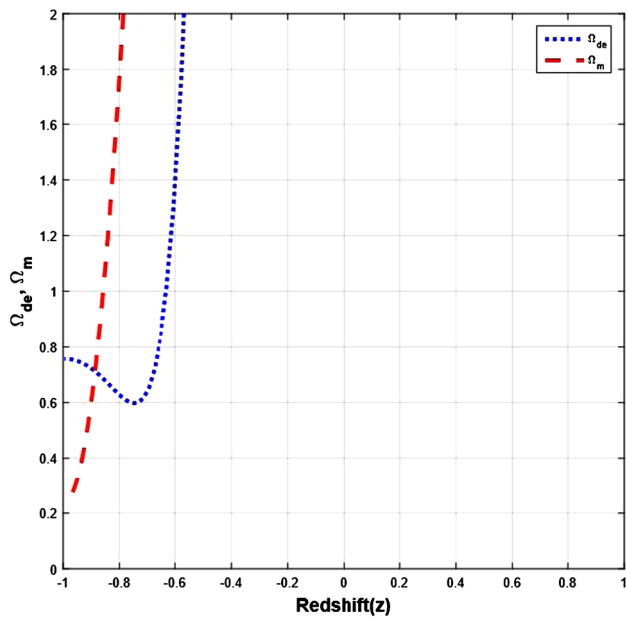
Fig. 8 Plot of Ω_{de}, Ω_m versus red shift (z) for Kantowski-Sachs model
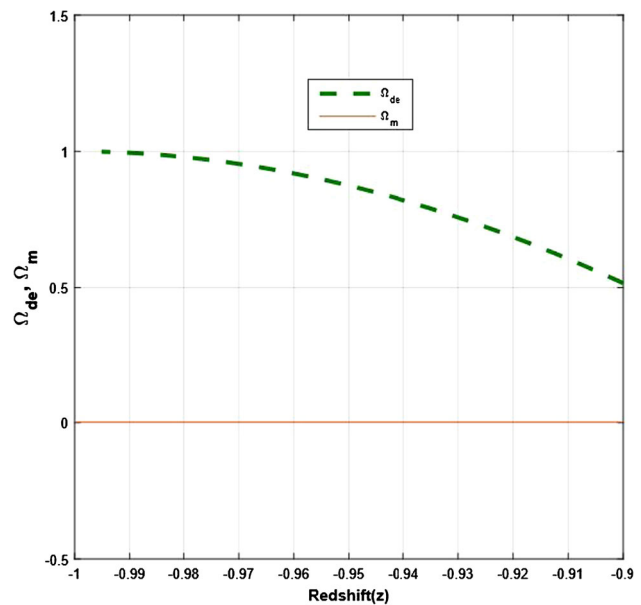

Fig. 10 Plot of Ω_{de}, Ω_m versus red shift (z) for LRS B-III model

in Figs. 5, 6 and 7 the equation of state of dark energy (ω_{de}) is plotted as function of redshift (z) and it is observed that it crosses the phantom divide line $\omega_{de} = -1$. Thus ω_{de} transits from quintessence to phantom so it has quintom like behavior and can explain the acceleration of the universe.

From Fig. 8 and 9 we observed that the average density parameters, Ω_{de} and Ω_m are positive throughout the evolution of the universe and approaches to a number less than one at late time. From Fig. 10 we observed that the average density parameter, Ω_{de} is positive throughout the evolution

of the universe and approaches to 1 at late time since $\Omega_m = 0$ for all t.

4.1 Classical stability of the models

In order to forecast the final evolution of the universe, it is important to study classical stability of the models. Points which are classically stable are exciting from cosmological point of view.

4.1.1 Squared speed of the sound

We now consider and study an important quantity considered in cosmology in order to check the stability of any DE model and it is known as squared speed of sound, it is denoted with v_s^2. The models with $v_s^2 > 0$ are stable where as models with $v_s^2 < 0$ are unstable. The squared speed of the sound is defined as follows (Myung 2007):

$$v_s^2 = \frac{\dot{p}_{de}}{\dot{\rho}_{de}} \tag{83}$$

where $\dot{p}_{de}$ and $\dot{\rho}_{de}$ are cosmic time derivatives of pressure and density of dark energy, respectively. Using (31) and (33), v_s^2 for Kantowski-Sachs model with (25) is given by

$$v_s^2 = \frac{[\frac{18b(\alpha(k^2+k+1)-\omega(k^2+k-2))}{\alpha t^2(k+2)^2} + \frac{6\omega}{\alpha(k+2)(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{3\overline{\omega}}{(me^{at}t^b)^6}](a + \frac{b}{t}) - (k - \frac{\omega}{\alpha}(k-1) + 1)(\frac{6b}{t^3(k+2)})}{[\frac{-18b(2k\alpha+\alpha+k^2+k-2)}{\alpha t^2(k+2)^2} - \frac{6(\alpha-1)}{\alpha(k+2)(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{3\overline{\omega}}{(me^{at}t^b)^6}](a + \frac{b}{t}) + \frac{6b(k-1)}{\alpha(k+2)t^3}}. \tag{84}$$

Using (46) and (48), v_s^2 for LRS Bianchi type-I model with (42) is given by

$$v_s^2 = \frac{[\frac{18b(\alpha(k^2+k+1)-\omega(k^2+k-2))}{\alpha t^2(k+2)^2} - \frac{3\overline{\omega}}{(me^{at}t^b)^6}](a + \frac{b}{t}) - (k - \frac{\omega}{\alpha}(k-1) + 1)(\frac{6b}{t^3(k+2)})}{[\frac{-18b(2k\alpha+\alpha+k^2+k-2)}{\alpha t^2(k+2)^2} - \frac{3\overline{\omega}}{(me^{at}t^b)^6}](a + \frac{b}{t}) + \frac{6b(k-1)}{\alpha(k+2)t^3}}. \tag{85}$$

Using equations (68) and (70), v_s^2 for LRS Bianchi-III model with (63) is given by

$$v_s^2 = \frac{[\frac{6b}{t^2} - \frac{3\overline{\omega}}{(me^{at}t^b)^6} - \frac{2}{3(me^{at}t^b)^2}](a + \frac{b}{t}) - \frac{4b}{t^3}}{[\frac{2}{(me^{at}t^b)^2} - \frac{6b}{t^2} - \frac{3\overline{\omega}}{(me^{at}t^b)^6}](a + \frac{b}{t})}. \tag{86}$$

The plot of squared speed of sound for three models with (25), (42) and (63) is displayed against redshift in Figs. 11, 12 and 13 respectively and we have observed that the squared speed of sound is positive throughout the evolution of the universe and hence the three models are stable, moreover the Kantowski-Sachs and LRS Bianchi type-I models satisfies the inequality $0 \le v_s^2 < 1$ at throughout the evolution of the universe.

4.1.2 Testing the non-negativity of the squared sound speed against small perturbations

Assuming a small perturbation in the background energy density, we need to observe if the perturbation grows or collapse. In the linear perturbation theory, the perturbed energy density of the background can be written as

$$\rho_{de}(t, x) = \rho_{de}(t) + \delta\rho_{de}(t, x) \tag{87}$$

where $\rho_{de}(t)$ is unperturbed background energy density. The energy conservation equation (4) yields (Peebles and Ratra 2003)

$$\delta\ddot{\rho}_{de} = v_s^2 \Delta^2 \rho_{de}(t, x). \tag{88}$$

Solutions of (88) include two cases of interest. First when v_s^2 is positive (88) becomes an ordinary wave equation whose solutions would be oscillatory waves of the form

$$\delta\rho_{de} = \rho_0 e^{-i\gamma t + i\mathbf{k}.\mathbf{x}} \tag{89}$$

which indicates a propagation mode for the density perturbations. The second is when v_s^2 is negative. In this case the
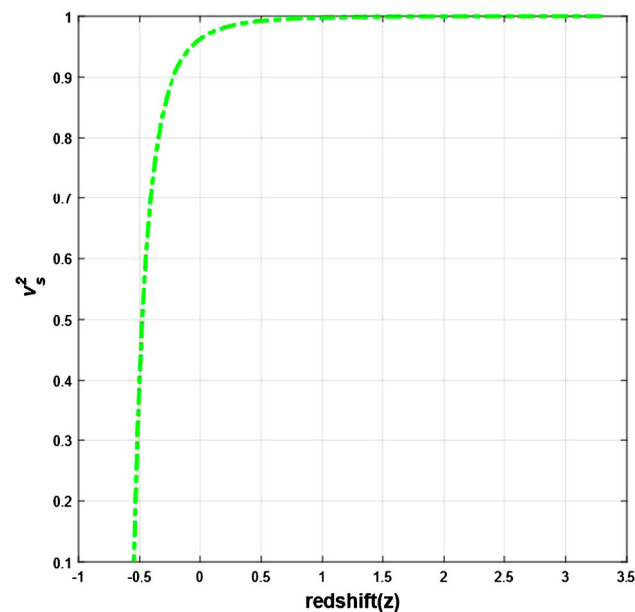

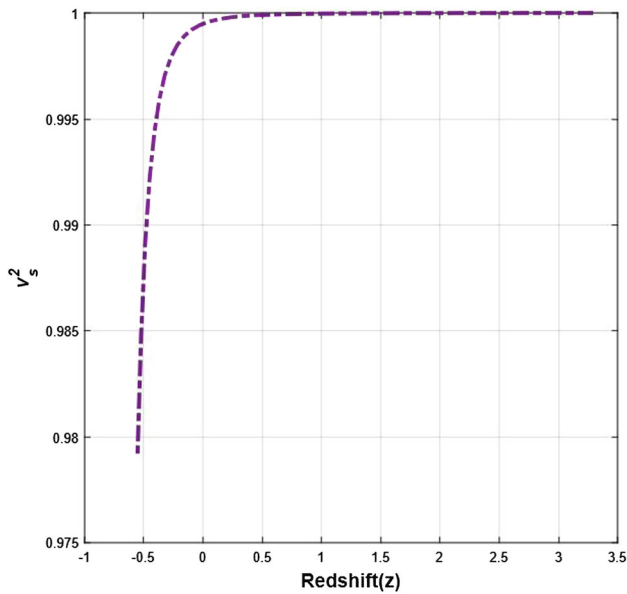**Fig. 11** Plot of v_s^2 versus red shift (z) for Kantowski-Sachs model

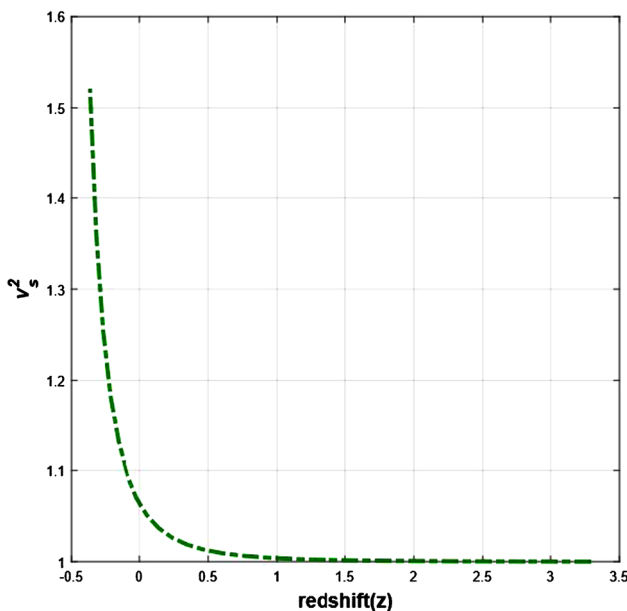**Fig. 12** Plot of v_s^2 versus red shift (z) for LRS B-I model

Fig. 13 Plot of v_s^2 versus red shift (z) for LRS B-III model

frequency of the oscillations becomes pure imaginary and the density perturbations will grow with time as

$$\delta\rho_{de} = \rho_0 e^{i\gamma t + i\mathbf{k}.\mathbf{x}} \qquad (90)$$

where ρ_0 is constant of integration. Here Thus the growing perturbation with time indicates a possible emergency of instabilities in the background. From Figs. 11, 12 and 13 it is clear that $v_s^2 > 0$ hence the three models are stable within small perturbations.

4.1.3 Possible violations of causality associated with superluminal propagation of small perturbations

One of the criteria for cosmological models to survive is that the speed of the sound is less than the local light speed, i.e. $v_s < 1$ (Garcia-Salcedo et al. 2014). From Figs. 11 and 12 we observed that the inequality $v_s < 1$ occurs at present and early times. Thus Kantowski-Sachs, and LRS Bianchi type-I models did not admit superluminal fluctuations and meets the required bound $v_s < 1$. The requirement that the energy density perturbations do not uncontrollably grow leads to a classical stability $v_s^2 < 1$. But this does not happen for LRS Bianchi type-III model, indicating violation of causality.

4.1.4 Existence/absence of curvature singularities of sudden and big rip types

The existence of sudden singularities is stable when the density is bounded near the singularity except for some cases with special parameter choices. This result holds regardless of whether the background metric is spatially flat, closed or open. The scale factor of our three models of the universe can be written as

$$R(t) = mt^b + mat^{b+1} + m\frac{a^2}{4}t^{b+2} + m\frac{a^3}{6}t^{b+3} + \cdots, \quad (91)$$

hence by definition (Visser and Cattoën 2005) a sudden singularity will be said to occur at time $t = 0$ so that some derivative of $R(t)$ blows up near the singularity.

From (91) we observed that a sudden singularity for three models occurs at $t = 0$, where the scale factor of the three models is discontinuous so that some derivative of $R(t)$ blows up near $t = 0$.

We also observed that our model shows phantom property and hence big rip singularity will takes place, which is curvature singularity at which both the scale factor and the energy density diverge.

4.2 State finder parameters

The investigation and the study of some important cosmological quantities like the EoS parameter (ω_{de}), the Hubble parameter (H) and the deceleration parameter (q) have attracted a lot of attention in modern cosmology. In cosmology different DE models usually lead to a positive value of the Hubble parameter H and a negative value of deceleration parameter q at the present epoch for ($z = 0$). For this reason we can conclude that the Hubble and the deceleration parameters H and q can not effectively distinguish between the various DE models taken into account. Thus, a higher order of derivatives with respect to the cosmic time t of the scale factor R must be taken in order to have a better and deeper study of the DE models. Sahni et al. (2003) and Alam et al.
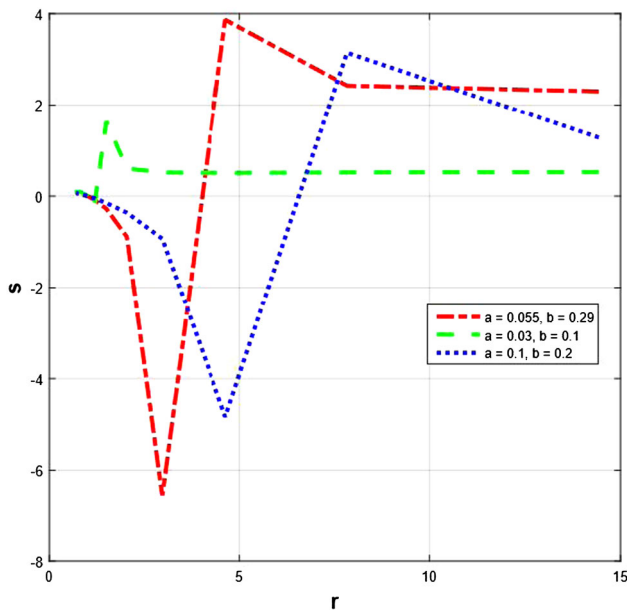
Fig. 14 Plot of r versus s for three models

(2003), considered the third derivative with respect to the cosmic time t of the scale factor R and introduced the state finder pair $\{r, s\}$ to remove the degeneracy of H and q at the present epoch of the universe. The state finder parameters are defined as

$$r = \frac{\dddot{R}}{RH^3}, \qquad s = \frac{r-1}{3(q-\frac{1}{2})}. \tag{92}$$

Using (15) and (22) in (92) we get

$$r = 1 - \frac{b(3at + 3b - 2)}{(at + b)^3}. \tag{93}$$

From (82), (92) and (93) we get

$$s = \frac{2b(2 - 3at - 3b)}{(6b - 9(at+b)^2)(at+b)}. \tag{94}$$

In r–s plane $s > 0$ implies a quintessences-like model where as $s < 0$ indicate a phantom like model. Furthermore an evolution from quintessence to phantom (or vice-versa) is obtained when the point with coordinate $\{r, s\} = \{1, 0\}$ is crossed (Wu and Yu 2010). One of the most important properties of the statefinder parameters r and s is that the point $\{r, s\} = \{1, 0\}$ in the r–s plane indicates the point corresponding to the flat ΛCDM model (Huang et al. 2008). From (88) and (89) we observed that $r \to 1$, $s \to 0$, when $t \to \infty$. From Fig. 14 we have observed that the values of state finder pair becomes $r = 1$, $s = 0$ at late time and consistent with standard ΛCDM model.

4.3 ω_{de}–ω'_{de} plane analysis

The ω_{de}–ω'_{de} plane analysis is proposed by Caldwell and Linder (2005) which is very useful tool in our modern days cosmological analysis. Basically, it has been used to distinguish different DE models through trajectories on its plane. Initially, this method has been applied on quintessence DE model which leads to two classes of its plane the one with $\omega_{de} < 0$ and $\omega'_{de} < 0$ is called freezing region and the other with the property $\omega_{de} < 0$ and $\omega'_{de} > 0$ is known as thawing region.

Differentiating EoS parameter $\omega_{de} = \frac{p_{de}}{\rho_{de}}$ with respect to $(\ln R)$ we get,

$$\omega'_{de} = \frac{\dot{p}_{de}\rho_{de} - p_{de}\dot{\rho}_{de}}{H\rho_{de}^2}. \tag{95}$$

For model with (25)

$$\dot{p}_{de} = \left[\frac{18b(\alpha(k^2+k+1) - \omega(k^2+k-2))}{\alpha t^2 (k+2)^2} \right. \\ \left. + \frac{6\omega}{\alpha(k+2)(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{3\overline{\omega}}{(me^{at}t^b)^6} \right] \left(a + \frac{b}{t} \right) \\ + \frac{6b(\omega(k-1) - \alpha(k+1))}{\alpha t^3 (k+2)}, \tag{96}$$

$$\dot{\rho}_{de} = \left[\frac{-18b(2k\alpha + \alpha + k^2 + k - 2)}{\alpha t^2 (k+2)^2} \right. \\ \left. - \frac{6(\alpha-1)}{\alpha(k+2)(me^{at}t^b)^{\frac{6}{k+2}}} - \frac{3\overline{\omega}}{(me^{at}t^b)^6} \right] \left(a + \frac{b}{t} \right) \\ + \frac{6b(k-1)}{\alpha(k+2)t^3}, \tag{97}$$

and ρ_{de} and p_{de} is given by (31) and (33).
For model with (42)

$$\dot{p}_{de} = \left[\frac{18b(\alpha(k^2+k+1) - \omega(k^2+k-2))}{\alpha t^2 (k+2)^2} \right. \\ \left. - \frac{3\overline{\omega}}{(me^{at}t^b)^6} \right] \left(a + \frac{b}{t} \right) \\ + \frac{6b(\omega(k-1) - \alpha(k+1))}{\alpha t^3 (k+2)}, \tag{98}$$

$$\dot{\rho}_{de} = \left[\frac{-18b(2k\alpha + \alpha + k^2 + k - 2)}{\alpha t^2 (k+2)^2} \right. \\ \left. - \frac{3\overline{\omega}}{(me^{at}t^b)^6} \right] \left(a + \frac{b}{t} \right) + \frac{6b(k-1)}{\alpha(k+2)t^3}, \tag{99}$$
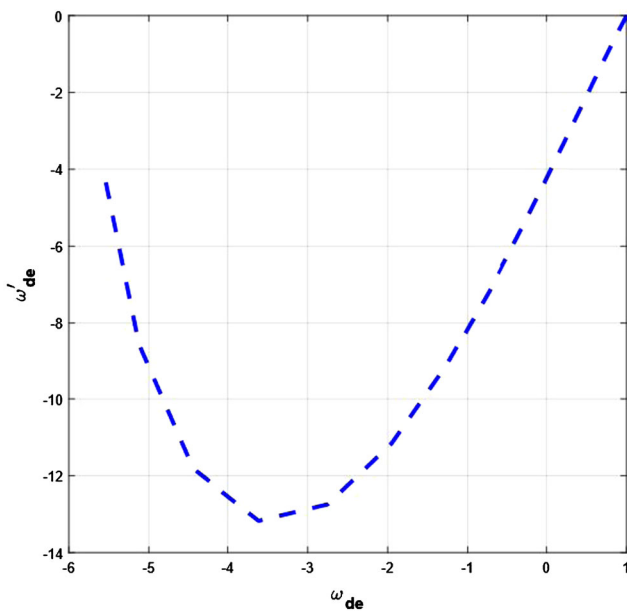
and ρ_{de} and p_{de} are given by (46) and (48)

Fig. 15 Plot of ω_{de} versus ω'_{de} for Kantowski-Sachs model
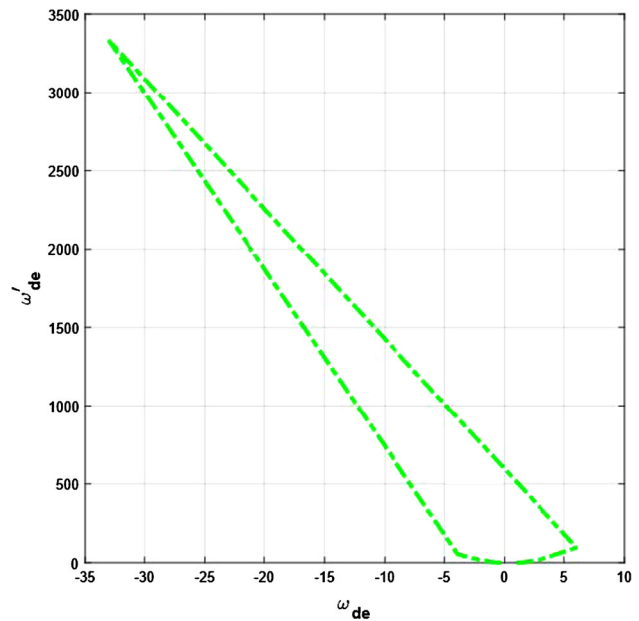
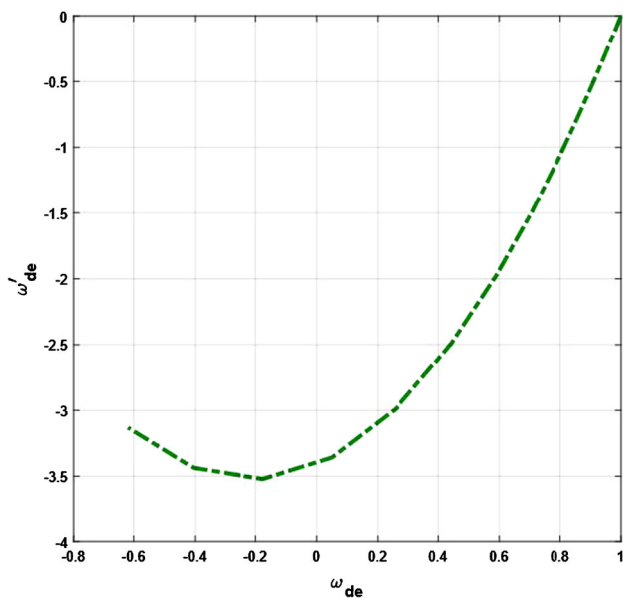**Fig. 16** Plot of ω_{de} versus ω'_{de} for LRS B-I model

For the model with (63)

$$\dot{p}_{de} = \left[\frac{6b}{t^2} - \frac{3\overline{\omega}}{(me^{at}t^b)^6} - \frac{2}{3(me^{at}t^b)^2}\right]\left(a + \frac{b}{t}\right) - \frac{4b}{t^3},$$

(100)

$$\dot{\rho}_{de} = \left[\frac{2}{(me^{at}t^b)^2 - \frac{6b}{t^2}} - \frac{3\overline{\omega}}{(me^{at}t^b)^6}\right]\left(a + \frac{b}{t}\right),$$

(101)

and p_{de} and ρ_{de} are given by (68) and (70) respectively.

From Fig. 15 and Fig. 16 we have observed that Kantowski-Sachs and LRS Bianchi type-I models lie in freezing region

Fig. 17 Plot of ω_{de} versus ω'_{de} for LRS B-III model

and from Fig. 17 we have observed that LRS Bianchi type-III model exhibit a thawing region.

5 Conclusion

In this paper we have presented a spatially homogeneous anisotropic Kantowski-Sachs, locally rotationally symmetric (LRS) Bianchi type-I and LRS Bianchi type-III space times filled with dark energy and one dimensional string in the framework of Saez-Ballester scalar tensor theory of gravitation. The spacial volume V is an increasing function of time. The time dependent DP (q) is positive at early age of the universe and becomes negative at present and late time, showing that our models evolves from early decelerating phase to late time accelerating phase. We have found the present value of deceleration parameter as $q_0 \approx -0.73$, which match the observed value. We have also observed that the dark energy density (ρ_{de}) is increasing with respect to red shift for the three models and dominates the string tension density (λ) and the proper density (ρ) throughout the evolution of the universe in the case of LRS Bianchi type-I and Kantowski-Sachs space time. The EoS parameter (ω_{de}) for three models crosses the phantom divide line $\omega_{de} = -1$, thus it has quintom-like behavior. Since the squared speed of sound v_s^2 is positive for the three models we have observed that the three models are stable through the evolution of the universe. In addition for Kantowski-Sachs and LRS Bianchi type-I models it is observed that the values of v_s^2 satisfy the inequality $0 < v_s^2 < 1$ at present and early time so that they do not admit superluminal fluctuations. We have observed that the values of state finder pair becomes $r = 1$, $s = 0$ at

late time and consistent with standard ΛCDM model. We have observed that the Kantowski-Sachs and LRS Bianchi type-I models lies in freezing region and LRS Bianchi type-III model lies in thawing region. The models obtained and presented here represents accelerating and expanding cosmological models of the universe. Thus our models are in harmony with present cosmological observations.

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