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ORIGINAL ARTICLE

Solution of the radiative transfer equation for Rayleigh scattering using the infinite medium Green's function

M. Biçer[1] · A. Kaşkaş[1]

Abstract The infinite medium Green's function is used to solve the half-space albedo, slab albedo and Milne problems for the unpolarized Rayleigh scattering case; these problems are the most classical problems of radiative transfer theory. The numerical results are obtained and are compared with previous ones.

Keywords Rayleigh scattering · Infinite medium Green's function · Albedo problem · Slab albedo problem · Milne problem

1 Introduction

Radiative transfer theory can be applied to many different areas in physics, for example, oceanography, tissue optics, and astrophysics. The radiative transfer equation defines the photon distribution in a scattering, emitting and absorbing medium. The time-independent, monoenergetic radiative transfer equation in plane geometry for the unpolarized case can be written as (Pomraning 1970)

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} + I(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{+1} f(\mu, \mu') I(\tau, \mu') d\mu' + S, \tag{1}$$

where $I(\tau, \mu)$ defines the angular intensity, w is the single scattering albedo, $f(\mu, \mu')$ is the scattering phase func-

✉ A. Kaşkaş
  kaskas@science.ankara.edu.tr

  M. Biçer
  bicer@ankara.edu.tr

[1] Department of Physics, Faculty of Science, Ankara University, Ankara, Turkey

tion, μ is the cosine of the direction, τ is the optical variable and S is the source function. In the literature, many methods, such as Case (which was developed to solve the linear transport equation), F_N, C_N, P_N and H_N, have been used to solve this equation for different problems, such as the half-space albedo, slab albedo and Milne problems (Case and Zweifel 1967; Siewert and Beonist 1979; Grandjean and Siewert 1979; Beonist and Kavenoky 1968; Kavenoky 1978; Case et al. 1953; Tezcan et al. 2003).

Rayleigh scattering occurs when the incident wavelength is greater than the size of the scattering particles. Lord Rayleigh explained why the sky is blue and sunset is red by using this scattering function. This scattering function can be used in astrophysics, oceanography and tissue optics. Chandrasekhar formulated the scattering function for the polarized case using the Stokes parameters (Chandrasekhar 1950). The scattering function can be written for the unpolarized Rayleigh case in the following form:

$$f(\mu, \mu') = a + b(3\mu^2 - 1)(3\mu'^2 - 1). \tag{2}$$

This scattering function has the same form as the quadratic scattering kernel. This kernel has been used to solve some physical problems in the neutron transport equation by using the H_N method (Türeci and Türeci 2007, 2017; Türeci and Güleçyüz 2008; Türeci 2010a, 2010b; Türeci et al. 2012). If we choose $a = 1$ and $b = 1/8$, then we can obtain the unpolarized Rayleigh scattering function (Pomraning 1969, 1970). Substituting this scattering function in Eq. (1) for $S = 0$, the solution of this equation can be found in terms of Case's singular eigenfunctions. It is assumed that

$$I(\tau, \mu) = e^{-\frac{\tau}{\nu}} \phi(\nu, \mu). \tag{3}$$

Thus, the discrete and continuous eigenfunctions are obtained as

$$\phi(\pm\nu_0, \mu) = \frac{\pm w \nu_0}{2} \frac{1}{\pm\nu_0 - \mu} \left[1 + \alpha_0(3\mu^2 - 1)\right],$$

$$\phi(\nu, \mu) = \frac{w\nu}{2} P \frac{1}{\nu - \mu} \left[1 + \alpha(3\mu^2 - 1)\right] + \lambda(\nu)\delta(\nu - \nu'), \tag{4}$$

where P is the Cauchy principle value, and α_0, α and $\lambda(\nu)$ have the following forms:

$$\alpha_0 = \frac{1}{8}\left[3\nu_0^2(1 - w) - 1\right],$$

$$\alpha = \frac{1}{8}\left[3\nu^2(1 - w) - 1\right], \tag{5}$$

$$\lambda(\nu) = 1 + 3w\nu^2\alpha - w\nu \tanh^{-1}(\nu)\left(1 + 3\nu^2\alpha - \alpha\right).$$

The full-range orthogonality relations for these eigenfunctions are

$$\int_{-1}^{+1} \mu\phi(\pm\nu_0, \mu)\phi(\pm\nu_0, \mu)d\mu = N(\pm\nu_0),$$

$$\int_{-1}^{+1} \mu\phi(\nu, \mu)\phi(\nu', \mu)d\mu = N(\nu)\delta(\nu - \nu'). \tag{6}$$

Here,

$$N(\nu_0) = -N(-\nu_0)$$
$$= \frac{w\nu_0}{2(\nu_0^2 - 1)}\left(1 - \alpha_0 + \nu_0^2(w - 1\right.$$
$$\left. + \alpha_0(16 - 11w + 15\nu_0^2(w - 1)) + 4w\alpha_0^2)\right),$$

$$N(\nu) = -N(-\nu) = \frac{\pi^2 w^2 \nu^3}{4}\left[1 + \alpha(3\nu^2 - 1)\right]^2 + \nu\lambda^2(\nu). \tag{7}$$

In this paper, the half-space albedo and slab albedo problems are solved by using the H_N method, and the Milne problem is solved using the C_N equations (Beonist and Kavenoky 1968; Kavenoky 1978; Tezcan et al. 1999).

2 Half-space albedo problem

One of the well known problems is the calculation of half-space albedo. The half-space albedo and slab albedo problems can be solved by using the F_N method. In the F_N method, the Placzek lemma defines that any finite medium problem can be converted to the infinite medium problem by inserting a proper source function at the boundaries (Siewert and Beonist 1979; Grandjean and Siewert 1979; Case et al. 1953). In this manner, the infinite medium Green's function can be used to find the solution of the problems. Applying the Placzek lemma to Eq. (1) for half-space,

we obtain (Siewert and Beonist 1979; Grandjean and Siewert 1979)

$$\mu\frac{\partial I_1(\tau, \mu)}{\partial\tau} + I_1(\tau, \mu) = \frac{\omega}{2}\int_{-1}^{+1} f(\mu, \mu')I_1(\tau, \mu')d\mu'$$
$$+ S(\tau, \mu)H(\tau) + \mu I(0, \mu)\delta(\tau), \tag{8}$$

$$I_1(\tau, \mu) = H(\tau)I(\tau, \mu),$$

$$H(\tau) = \begin{cases} 1, & \text{for } \tau \in (0, \infty), \\ 0, & \text{otherwise.} \end{cases}$$

Then, the solution of Eq. (8) is

$$I_1(\tau, \mu) = \int_{-1}^{+1} d\mu_0 \int_{-\infty}^{+\infty} d\tau_0 G(\tau_0 \to \tau; \mu_0 \to \mu)$$
$$\times \left[H(\tau_0)S(\tau_0, \mu_0) + \mu_0 I(0, \mu_0)\delta(\tau_0)\right]. \tag{9}$$

The infinite medium Green's function for the isotropic scattering case is defined by Case and Zweifel (1967):

$$G^{\pm}(\tau_0 \to \tau; \mu_0 \to \mu)$$
$$= \frac{\phi(\pm\nu_0, \mu)\phi(\pm\nu_0, \mu_0)}{N(\nu_0)}e^{-\frac{|\tau - \tau_0|}{\nu_0}}$$
$$+ \int_0^{+1} \frac{\phi(\pm\nu, \mu)\phi(\pm, \mu_0)}{N(\nu)}e^{-\frac{|\tau - \tau_0|}{\nu}}d\nu, \tag{10}$$

where the upper signs apply for $\tau > \tau_0$, and the lower signs apply for $\tau < \tau_0$. Here, if we use the discrete and continuous eigenfunctions given in Eq. (4), then we obtain the infinite medium Green's functions for the Rayleigh scattering case.

In the half-space albedo problem, a scattering/absorbing medium is at $\tau > 0$, and vacuum is at $\tau < 0$. The boundary conditions are

$$I(0, \mu) = \mu^\beta, \quad \beta = 0, 1, 2, \ldots$$

$$I(0, -\mu) = \sum_{l=0}^{N} a_l\mu^l, \quad \mu > 0. \tag{11}$$

We can write the exit distributions of the photons by using these conditions in Eq. (9) for $S(\tau_0, \mu_0) = 0$:

$$I(0, -\mu) = \int_0^{+1} d\mu_0 G^+(0 \to 0^+; \mu_0 \to -\mu)\mu_0 I(0, \mu_0)$$
$$- \int_0^{+1} d\mu_0 G^+(0 \to 0^+; -\mu_0 \to -\mu)$$
$$\times \mu_0 I(0, -\mu_0), \quad \mu > 0. \tag{12}$$

Table 1 The convergence of half-space albedo values for $\omega = 0.8$ and $\beta = 0$

N	0	1	2	3	4	5	10	20	30	Schnatz and Siewert (1971)	Şenyiğit (2016) ($N = 10$)
A^*	0.34375	0.34291	0.34288	0.34288	0.34288	0.34288	0.34288	0.34288	0.34287	0.34378	0.343784

Table 2 The eigenvalues and the half-space albedo results for selected values of ω for $N = 3$

ω	v_0	$\beta = 0$	Schnatz and Siewert (1971)	Türeci (2010b) ($N = 6$)	Şenyiğit (2016) ($N = 10$)	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$
0.1	1.000000656297	0.022038	0.02206	–	0.022056	0.020499	0.019746	0.019308	0.019024
0.2	1.000563217394	0.046906	0.04698	–	0.046981	0.043700	0.042115	0.041188	0.040584
0.3	1.006003984463	0.075348	0.07552	0.07534792	0.075517	0.070330	0.067825	0.066349	0.065384
0.4	1.022321209266	0.10843	0.10873	–	0.108732	0.10144	0.097916	0.095824	0.094449
0.5	1.055542695062	0.14777	0.14824	0.14777108	0.148237	0.13864	0.13398	0.13119	0.12935
0.6	1.114950198229	0.19599	0.19664	–	0.196638	0.18455	0.17863	0.17506	0.17268
0.7	1.219536479551	0.25777	0.25859	–	0.258587	0.24389	0.23659	0.23213	0.22915
0.8	1.418784755734	0.34288	0.34378	0.34287736	0.343784	0.32659	0.31786	0.31246	0.30881
0.9	1.911313421782	0.47864	0.47940	–	0.479395	0.46080	0.45097	0.44479	0.44055
0.95	2.640900251692	0.59700	0.59747	–	0.597474	0.58001	0.57046	0.56437	0.56015

Multiplying both sides of Eq. (12) by μ^{m+1} and integrating over $\mu \in [0, 1]$, we can obtain

$$\sum_{l=0}^{N} a_l T_{ml} = R_{m\beta}, \tag{13}$$

where T_{ml} and $R_{m\beta}$ are defined as

$$
\begin{aligned}
T_{ml} &= \frac{1}{l+m+2} + \left(\frac{wv_0}{2}\right)^2 \frac{A_m(v_0)A_l(v_0)}{N(v_0)} \\
&\quad + \left(\frac{w}{2}\right)^2 \int_0^{+1} \frac{v^2 A_m(v)A_l(v)}{N(v)} dv, \\
R_{m\beta} &= \left(\frac{wv_0}{2}\right)^2 \frac{A_m(v_0)B_\beta(v_0)}{N(v_0)} \\
&\quad + \left(\frac{w}{2}\right)^2 \int_0^{+1} \frac{v^2 A_m(v)B_\beta(v)}{N(v)} dv,
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
A_m(\xi) &= \frac{2}{w\xi} \int_0^{+1} \mu^{m+1} \phi(\xi, -\mu) d\mu, \\
A_0(\xi) &= 1 - \frac{3}{2}\alpha\xi + 3\alpha\xi^2 - \left(\xi + 3\alpha\xi^3 - \alpha\xi\right) \ln\left(\frac{\xi+1}{\xi}\right), \\
A_m(\xi) &= \frac{1}{m+1} + \alpha\left(\frac{3}{m+3} - \frac{1}{m+1}\right) - \xi A_{m-1}(\xi); \\
& m \geq 1,
\end{aligned}
\tag{15}
$$

$$B_m(\xi) = \frac{2}{w\xi} \int_0^{+1} \mu^{m+1} \phi(\xi, \mu) d\mu,$$

$$
\begin{aligned}
B_0(\xi) &= \frac{2}{w} - 1 + 3\alpha\xi^2 - \frac{3}{2}\alpha\xi \\
&\quad - \xi\left(1 + 3\alpha\xi^2 - \alpha\right) \ln\left(\frac{\xi+1}{\xi}\right),
\end{aligned}
\tag{16}
$$

$$B_m(\xi) = \xi B_{m-1}(\xi) - \frac{1-\alpha}{m+1} - \frac{3\alpha}{m+3}, \quad m \geq 1,$$

$$m \geq 1, \ \xi = v_0 \text{ or } \xi = v, \ v \in (0, 1).$$

Albedo is given by

$$A^* = \frac{\int_0^{+1} \mu I(0, -\mu) d\mu}{\int_0^{+1} \mu I(0, \mu) d\mu} = (\beta + 2) \sum_{l=0}^{N} \frac{a_l}{l+2}. \tag{17}$$

The unknown a_l expansion coefficients in Eq. (17) are obtained from Eq. (13).

3 Slab albedo problem

Let us consider a plane-parallel medium surrounded by vacuum on both sides at the boundaries with $[-\tau_0, \tau_0]$. In the F_N method, Siewert and Beonist (1979) defined the following equation:

$$\mu \frac{\partial I_1(\tau, \mu)}{\partial \tau} + I_1(\tau, \mu)$$

$$= \frac{\omega}{2} \int_{-1}^{+1} f(\mu, \mu') I_1(\tau, \mu') d\mu'$$
$$+ H^*(\tau) S(\tau, \mu) + \mu I(\tau, \mu) [\delta(\tau + \tau_0) - \delta(\tau - \tau_0)],$$

$$I_1(\tau, \mu) = H^*(\tau) I(\tau, \mu), \tag{18}$$

$$H^*(\tau) = \begin{cases} 1, & \text{for } \tau \in [-\tau_0, \tau_0], \\ 0, & \text{otherwise.} \end{cases}$$

For $\tau \in [-\tau_0, +\tau_0]$, $I(\tau, \mu)$ can be written as

$$I(\tau, \mu)$$
$$= \int_{-1}^{+1} d\mu_0 \int_{-\tau_0}^{+\tau_0} d\tau_0 G(\tau_0 \to \tau; \mu_0 \to \mu) S(\tau_0, \mu_0)$$
$$+ \int_{-1}^{+1} d\mu_0 [G(-\tau_0 \to \tau; \mu_0 \to \mu) \mu_0 I(-\tau_0, \mu_0)$$
$$- G(\tau_0 \to \tau; \mu_0 \to \mu) \mu_0 I(\tau_0, \mu_0)] \tag{19}$$

and can be rearranged into the following form:

$$I(\tau, \mu)$$
$$= \int_{-1}^{+1} d\mu_0 \int_{-\tau_0}^{+\tau_0} d\tau_0 G(\tau_0 \to \tau; \mu_0 \to \mu) S(\tau_0, \mu_0)$$
$$+ \int_{0}^{+1} d\mu_0 G(-\tau_0 \to \tau; \mu_0 \to \mu) \mu_0 I(-\tau_0, \mu_0)$$
$$- \int_{0}^{+1} d\mu_0 G(-\tau_0 \to \tau; -\mu_0 \to \mu) \mu_0 I(-\tau_0, -\mu_0)$$
$$+ \int_{0}^{+1} d\mu_0 G(\tau_0 \to \tau; -\mu_0 \to \mu) \mu_0 I(\tau_0, -\mu_0)$$
$$- \int_{0}^{+1} d\mu_0 G(\tau_0 \to \tau; \mu_0 \to \mu) \mu_0 I(\tau_0, \mu_0).$$

Then, for $S(\tau_0, \mu_0) = 0$, the exit distributions for $\tau = \mp\tau_0$ are

$$I(-\tau_0, -\mu)$$
$$= \int_{0}^{+1} d\mu_0 G^+(-\tau_0 \to -\tau_0^+; \mu_0 \to -\mu) \mu_0 I(-\tau_0, \mu_0)$$
$$- \int_{0}^{+1} d\mu_0 G^+(-\tau_0 \to -\tau_0^+; -\mu_0 \to -\mu) \mu_0$$
$$\times I(-\tau_0, -\mu_0)$$
$$+ \int_{0}^{+1} d\mu_0 G^-(\tau_0 \to -\tau_0; -\mu_0 \to -\mu) \mu_0 I(\tau_0, -\mu_0)$$
$$- \int_{0}^{+1} d\mu_0 G^-(\tau_0 \to -\tau_0; \mu_0 \to -\mu) \mu_0 I(\tau_0, \mu_0),$$
$$\mu > 0, \tag{20}$$

$$I(\tau_0, \mu)$$
$$= \int_{0}^{+1} d\mu_0 G^+(-\tau_0 \to \tau_0; \mu_0 \to \mu) \mu_0 I(-\tau_0, \mu_0)$$
$$- \int_{0}^{+1} d\mu_0 G^+(-\tau_0 \to \tau_0; -\mu_0 \to \mu) \mu_0 I(-\tau_0, -\mu_0)$$
$$+ \int_{0}^{+1} d\mu_0 G^-(\tau_0 \to \tau_0^-; -\mu_0 \to \mu) \mu_0 I(\tau_0, -\mu_0)$$
$$- \int_{0}^{+1} d\mu_0 G^-(\tau_0 \to \tau_0^-; \mu_0 \to \mu) \mu_0 I(\tau_0, \mu_0),$$
$$\mu > 0. \tag{21}$$

Assume that photons enter the medium from the left boundary. The boundary conditions are (Siewert and Beonist 1979; Grandjean and Siewert 1979)

$$I(-\tau_0, \mu) = \mu^k,$$
$$I(\tau_0, -\mu) = 0,$$
$$I(-\tau_0, -\mu) = \sum_{l=0}^{N} a_l \mu^l, \tag{22}$$
$$I(\tau_0, \mu) = \sum_{l=0}^{N} b_l \mu^l.$$

By substituting these boundary conditions and the infinite medium Green's function given by Eq. (10) into Eqs. (20), (21), multiplying each equation by μ^{m+1} and then integrating over $\mu \in [0, 1]$, we can obtain the matrix equation for a_l and b_l expansion coefficients:

$$\sum_{l=0}^{N} a_l T_{ml} + \sum_{l=0}^{N} b_l H_{ml} = S_{mk},$$
$$\sum_{l=0}^{N} a_l H_{ml} + \sum_{l=0}^{N} b_l T_{ml} = R_{mk}, \tag{23}$$

where

$$T_{ml} = \frac{1}{l+m+2} + \left(\frac{wv_0}{2}\right)^2 \frac{A_m(v_0) A_l(v_0)}{N(v_0)}$$
$$+ \left(\frac{w}{2}\right)^2 \int_0^{+1} \frac{v^2 A_m(v) A_l(v)}{N(v)} dv,$$

$$H_{ml} = \left(\frac{wv_0}{2}\right)^2 \frac{B_m(v_0) A_l(v_0)}{N(v_0)} e^{-\frac{2\tau_0}{v_0}}$$
$$+ \left(\frac{w}{2}\right)^2 \int_0^{+1} \frac{v^2 B_m(v) A_l(v)}{N(v)} e^{-\frac{2\tau_0}{v}} dv, \tag{24}$$

$$S_{mk} = \left(\frac{wv_0}{2}\right)^2 \frac{A_m(v_0) B_k(v_0)}{N(v_0)}$$

Table 3 The convergence of reflection and transmission coefficients for $\omega = 0.8$ and $\beta = 0$, $2\tau_0 = 1$

N	0	1	2	3	4	5	10	20	30	Türeci (2010b) (N = 7)
A^*	0.28011	0.28066	0.28063	0.28063	0.28063	0.28063	0.28063	0.28063	0.28063	0.28063
B^*	0.41577	0.41654	0.41653	0.41653	0.41653	0.41653	0.41653	0.41653	0.41653	0.41653

Table 4 The reflection and transmission coefficients for selected values of τ_0, $k = 0$ and $N = 3$

w	$2\tau_0 = 1$				$2\tau_0 = 2$		$2\tau_0 = 3$			
	A^*	B^*	A^{*a}	B^{*a}	A^*	B^*	A^*	B^*	A^{*a}	B^{*a}
0.1	0.02101	0.23211	0.0210	0.2321	0.02196	0.06610	0.02203	0.02016	0.0220	0.0202
0.2	0.04442	0.24674	–	–	0.04669	0.07321	0.04689	0.02309	–	–
0.3	0.07073	0.26373	0.0707	0.2637	0.07491	0.08206	0.07530	0.02692	0.0753	0.0269
0.4	0.10060	0.28369	–	–	0.10760	0.09332	0.10833	0.03210	–	–
0.5	0.13496	0.30746	0.1349	0.3075	0.14622	0.10803	0.14757	0.03939	0.1475	0.0394
0.6	0.17506	0.33623	–	–	0.19301	0.12793	0.19554	0.05015	–	–
0.7	0.22272	0.37172	0.2227	0.3717	0.25165	0.15602	0.25664	0.06714	0.2566	0.0671
0.8	0.28063	0.41653	–	–	0.32873	0.19799	0.33952	0.09647	–	–
0.9	0.35301	0.47473	0.3530	0.4747	0.43756	0.26586	0.46456	0.15459	0.4646	0.1546
0.95	0.39663	0.51083	–	–	0.51236	0.31747	0.55900	0.20862	–	–

[a] Abdel Krim et al. (1992)

Table 5 The reflection and transmission coefficients for various selected values of k ($N = 3$)

w	$k = 1$		$k = 2$		$k = 3$		$k = 4$	
	A^*	B^*	A^*	B^*	A^*	B^*	A^*	B^*
0.1	0.01932	0.27104	0.01847	0.29466	0.01797	0.31023	0.01764	0.32116
0.2	0.04085	0.28572	0.03906	0.30929	0.03799	0.32480	0.03730	0.33567
0.3	0.06509	0.30269	0.06222	0.32615	0.06052	0.34154	0.05940	0.35232
0.4	0.09265	0.32253	0.08857	0.34579	0.08613	0.36102	0.08452	0.37166
0.5	0.12440	0.34604	0.11894	0.36901	0.11564	0.38399	0.11346	0.39444
0.6	0.16154	0.37436	0.15445	0.39689	0.15015	0.41153	0.14729	0.42171
0.7	0.20576	0.40913	0.19677	0.43102	0.19128	0.44517	0.18761	0.45499
0.8	0.25963	0.45281	0.24835	0.47379	0.24141	0.48726	0.23674	0.49657
0.9	0.32713	0.50931	0.31303	0.52896	0.30428	0.54146	0.29836	0.55004
0.95	0.34566	0.55625	0.32891	0.57544	0.31843	0.58756	0.31129	0.59585

$$+ \left(\frac{w}{2}\right)^2 \int_0^{+1} \frac{\nu^2 A_m(\nu) B_k(\nu)}{N(\nu)} d\nu,$$

$$R_{mk} = \left(\frac{w\nu_0}{2}\right)^2 \frac{B_m(\nu_0) B_k(\nu_0)}{N(\nu_0)} e^{-\frac{2\tau_0}{\nu_0}}$$

$$+ \left(\frac{w}{2}\right)^2 \int_0^{+1} \frac{\nu^2 B_m(\nu) B_k(\nu)}{N(\nu)} e^{-\frac{2\tau_0}{\nu}} d\nu.$$

The reflection and transmission coefficients are given by

$$A^* = \frac{\int_0^{+1} \mu I(-\tau_0, -\mu) d\mu}{\int_0^{+1} \mu I(-\tau_0, \mu) d\mu} = (k+2) \sum_{l=0}^{N} \frac{a_l}{l+2}, \quad (25)$$

$$B^* = \frac{\int_0^{+1} \mu I(\tau_0, \mu) d\mu}{\int_0^{+1} \mu I(-\tau_0, \mu) d\mu} = (k+2) \sum_{l=0}^{N} \frac{b_l}{l+2}. \quad (26)$$

4 Milne problem

The Milne problem is a half-space problem of determining the photon distribution everywhere in a source-free medium. The photons are emitted from a source at infinity ($\tau = +\infty$) into the medium. Vacuum is at $\tau < 0$. At the boundary $\tau = 0$, no photons are entering from $\tau < 0$ inside the

Table 6 The convergence of the results for extrapolation endpoint ($\omega = 0.8$)

N	0	1	2	3	4	5	10	20	30	Türeci (2010b) ($N = 5$)
τ_0 (Eq. (31a))	0.90557	0.89631	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622
τ_0 (Eq. (31b))	0.87540	0.89724	0.89619	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622

medium. Thus,

$$I(0, \mu) = 0, \quad \mu > 0. \tag{27}$$

The third form of the transport equation, also known as the C_N equations, is solved by using the C_N method (Beonist and Kavenoky 1968; Kavenoky 1978) as follows:

$$0 = \int_0^{+1} \mu' I(0, \mu') G^-(\mu, \mu') d\mu'$$
$$+ \int_{-1}^{0} \mu' I(0, -\mu') G^-(\mu, \mu') d\mu', \quad \mu > 0,$$
$$I(0, -\mu) = \int_0^{+1} \mu' I(0, \mu') G^-(\mu, \mu') d\mu'$$
$$+ \int_{-1}^{0} \mu' I(0, -\mu') G^+(\mu, \mu') d\mu', \quad \mu \le 0. \tag{28}$$

The angular flux of a source at infinity is proportional to $\frac{v_0}{v_0 + \mu} + \frac{v_0}{v_0 + \mu} \alpha_0 (3\mu^2 - 1)$, and using this with the boundary condition in Eq. (27), we can rewrite the previous equations as

$$\int_{-1}^{0} \mu' I(0, -\mu') G^-(\mu, \mu') d\mu' + \frac{v_0}{v_0 + \mu}$$
$$+ \frac{v_0}{v_0 + \mu} \alpha_0 (3\mu^2 - 1) = 0, \quad \mu > 0,$$
$$\int_{-1}^{0} \mu' I(0, -\mu') G^+(\mu, \mu') d\mu' + \frac{v_0}{v_0 + \mu}$$
$$+ \frac{v_0}{v_0 + \mu} \alpha_0 (3\mu^2 - 1) = I(0, -\mu), \quad \mu \le 0. \tag{29}$$

The exit distribution can be defined by

$$I(0, -\mu) = \sum_{l=0}^{N} a_l \mu^l, \quad \mu > 0. \tag{30}$$

Multiplying both sides of Eqs. (29) by μ^{m+1} and then integrating over $\mu \in [0, 1]$ and $\mu \in [-1, 0]$, we can obtain the following equations (Tezcan et al. 1999):

$$\sum_{l=0}^{N} a_l (-1)^l \left\{ \left(\frac{wv_0}{2} \right)^2 \frac{A_m(v_0) B_l(v_0)}{N(v_0)} \right.$$
$$\left. + \left(\frac{w}{2} \right)^2 \int_0^{+1} \frac{v^2 A_m(v) B_l(v)}{N(v)} dv \right\} = v_0 A_m(v_0), \tag{31a}$$

$$\sum_{l=0}^{N} a_l (-1)^l \left\{ \left(\frac{wv_0}{2} \right)^2 \frac{A_m(v_0) A_l(v_0)}{N(v_0)} \right.$$
$$+ \left(\frac{w}{2} \right)^2 \int_0^{+1} \frac{v^2 A_m(v) A_l(v)}{N(v)} dv$$
$$\left. + \frac{1}{l + m + 2} \right\} = v_0 B_m(v_0). \tag{31b}$$

The asymptotic flux can be written as a summation of the source at infinity and the source at the surface:

$$\phi_{as}(\tau) = v_0 e^{\frac{\tau}{v_0}} \left\{ \ln \left(\frac{v_0 + 1}{v_0 - 1} \right) [1 + \alpha_0 (3v_0^2 - 1)] - 6\alpha v_0 \right\}$$
$$+ \int_{-1}^{+1} d\mu \int_{-1}^{0} G(\tau, \mu, \mu') \mu' I(0, -\mu') d\mu'$$
$$= \phi_{\infty}(v_0) e^{\frac{\tau}{v_0}} + \phi_N(v_0) e^{-\frac{\tau}{v_0}}, \tag{32}$$

where

$$\phi_{\infty}(v_0) = v_0 \left\{ \ln \left(\frac{v_0 + 1}{v_0 - 1} \right) [1 + \alpha_0 (3v_0^2 - 1)] - 6\alpha v_0 \right\}$$
$$= \frac{2}{w},$$
$$\phi_N(v_0) = \frac{wv_0}{2N(v_0)} \sum_{l=0}^{N} (-1)^{l+1} A_l(v_0) a_l. \tag{33}$$

The extrapolated endpoint is obtained from the relation (Case and Zweifel 1967)

$$\phi_{as}(-\tau_0) = 0 \tag{34}$$

and can be written as

$$\tau_0 = -\frac{v_0}{2} \ln \left[-\frac{\phi_N(v_0)}{\phi_{\infty}(v_0)} \right]. \tag{35}$$

The unknown coefficients in this equation, a_l, are obtained from Eqs. (31a), (31b). The extrapolated endpoints have the same numerical results for both equations.

5 Conclusions

F_N and C_N equations were used to solve the half-space albedo, slab albedo and Milne problems for the unpolarized

Table 7 The extrapolation endpoint results for selected values of ω from Eq. (31a)

ω/N	0	1	2	3	4	5	6	7	8	9	10	20	30	a	$N=5$[b]
0.1	6.32428	6.20515	6.19330	6.19243	6.19236	6.19236	6.19236	6.19236	6.19236	6.19236	6.19236	6.19236	6.19236	6.059782	–
0.2	3.28259	3.19089	3.18284	3.18232	3.18229	3.18229	3.18229	3.18229	3.18229	3.18229	3.18229	3.18229	3.18229	3.084199	–
0.3	2.28942	2.22200	2.21706	2.21679	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.147636	–
0.4	1.77334	1.72530	1.72251	1.72239	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.677234	–
0.5	1.44452	1.41119	1.40973	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.384414	1.40968
0.6	1.21252	1.19001	1.18933	1.18932	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.024846	–
0.7	1.03935	1.02463	1.02436	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	0.901861	–
0.8	0.90557	0.89631	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.799784	0.89622
0.9	0.79975	0.79420	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79413	0.754441	0.99519
0.95	0.75495	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.74461	0.720357	–

[a]Schnatz and Siewert (1971)

[b]Türeci (2010b)

Table 8 The extrapolation endpoint results for selected values of ω from Eq. (31b)

ω/N	0	1	2	3	4	5	6	7	8	9	10	20	30	a	$N=5$[b]
0.1	6.22467	6.21291	6.18865	6.19303	6.19224	6.19238	6.19235	6.19236	6.19236	6.19236	6.19236	6.19236	6.19236	6.059782	–
0.2	3.19316	3.19891	3.17955	3.18274	3.18221	3.18230	3.18228	3.18229	3.18229	3.18229	3.18229	3.18229	3.18228	3.084199	–
0.3	2.21094	2.22941	2.21496	2.21704	2.21674	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.21678	2.147636	–
0.4	1.70563	1.73149	1.72127	1.72251	1.72237	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.72238	1.677234	–
0.5	1.38702	1.41590	1.40907	1.40974	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.40968	1.384414	1.40968
0.6	1.16462	1.19326	1.18902	1.18933	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.18931	1.024846	–
0.7	1.00050	1.02658	1.02423	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	1.02435	0.901861	–
0.8	0.87540	0.89724	0.89619	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.89622	0.799784	0.89622
0.9	0.77852	0.79444	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79418	0.79417	0.754441	0.99519
0.95	0.73909	0.75076	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75071	0.75045	0.87200	0.720357	–

[a]Schnatz and Siewert (1971)

[b]Türeci (2010b)

Rayleigh scattering case. The H_N method was applied to these equations by using the infinite medium Green's function. The numerical results were compared with previous results. Some results were found to be in good agreement with Türeci (2010b); these results were obtained for the maximum eighth degree of approximation. In this reference, the quadratic scattering function was used to solve the neutron transport equation for different values of the scattering coefficient, f_2. Here, $f_2 = 0.1$ corresponds to Rayleigh scattering. The numerical results of half-space albedo for the unpolarized case were also compared with the results of Schnatz and Siewert (1971) and Şenyiğit (2016). Our numerical results were found to be slightly different from their numerical results (Tables 1 and 2). The reflection and transmission coefficients were found to be in good agreement with the results of Abdel Krim et al. (1992) (Tables 3, 4 and 5). The numerical results of the Milne problem were obtained by using Eqs. (31a), (31b) and were compared with the results of Türeci (2010b) for $f_2 = 0.1$. However, to our knowledge, no work in the literature considered the unpolarized case of Rayleigh scattering. Thus, the numerical results of the unpolarized case were compared with the results of the polarized case to gain insight into its behavior. We obtained numerical results up to the 30th approximation to test the convergence (Tables 6, 7, and 8).

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