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Regularization of circular restricted three-body problem accounting radiation pressure and oblateness

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Abstract In this paper, a time- and space-coordinate transformation, commonly known as the Kustaanheimo–Stiefel (KS)-transformation, is applied to reduce the order of singularities arising due to the motion of an infinitesimal body in the vicinity of a smaller primary in the three-body system. In this work, the Sun–Earth system is considered assuming the Sun to be a radiating body and the Earth as an oblate spheroid. The study covers motion around collinear Lagrangian L_1 and L_2 points. Numerical computations are performed for both regularized and non-regularized equations of motion and results are compared for non-periodic as well as periodic motion. In the transformed space, time is also computed as a function of solar radiation pressure (q) and oblateness of the Earth (A_2). The two parameters (q, A_2) have a significant impact on time in the transformed space. It is found that KS-regularization reduces the order of the pole from five to three at the point of singularity of the governing equations of motion.

Keywords Three-body problem · Solar radiation · Oblate spheroid · Regularization

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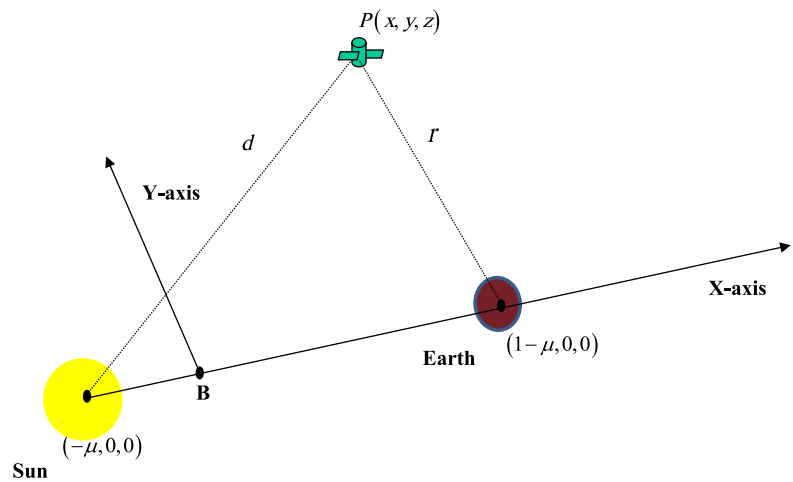
1 Introduction

In recent years, regularization has become popular for long term studies of motion of celestial bodies. At collision the equations of motion possess singularities. Indeed, a collision between any two objects is marked by the fact that their distance becomes zero. The problem of singularities plays an important role under conceptual, computational and physical aspects. The singularities occurring at collisions can be eliminated by using proper choice of the independent variable. The basic idea of the regularization theory is to compensate for the infinite increase of the velocity at collision. For this purpose, a new independent fictitious time variable is chosen to slow the motion near the singularities.

In 1913, Sundman (Szebehely 1967) proposed regularization of the two-body problem using a transformation of the independent variable, time. Levi–Civita, in 1903, demonstrated a coordinate transformation in addition to the independent variable transformation for the 2-dimensional equation of motion. This coordinate transformation was from one 2-dimensional vector to another more effective 2-dimensional vector using complex variables. In 1933, Hurwitz described that direct generalization of Levi–Civita transformation to a 3-dimensional vector was impossible; however, he advised that the transformation may be performed by transforming the original 3-dimensional vector to a 4-dimensional vector. Kustaanheimo and Stiefel (1965) generalized the Levi–Civita transformation to 3-dimensional motion and this generalized transformation is commonly known as a KS-transformation. Both Levi–Civita and KS regularizations are local transformations in the sense that their application allows one to regularize collisions with only one of the two primaries. A suitable extension of such transformations allows for obtaining a simultaneous regularization with both primaries, thus obtaining a global

2 Mathematical formulation

Let $m_\odot$, $m_\oplus$, and m be the masses of the Sun, the Earth, and an infinitesimal body (spacecraft), respectively, in the restricted three-body problem. We assume that the infinitesimal body is moving in the gravitational field of the Sun and the Earth (refer to Fig. 1) called primaries, which revolve around their common center of mass on circular orbits. The Sun is taken as a radiating body and the Earth as an oblate spheroid.

2.1 Mass reduction factor

In this study, the Sun is assumed as a radiating body and the radiation force always acts on the spacecraft away from the Sun in the Sun and spacecraft line, which effectively causes the reduction in the gravitational force due to the Sun on the spacecraft (if radiation force is merged with the gravitational force). This causes a reduction in the mass of the Sun.

Let F_p denote the radiation force and F_g denote the gravitational force due to the Sun, then the resultant force, F, on the infinitesimal body is given by

$$F = F_g - F_p = F_g\left(1 - \frac{F_p}{F_g}\right) = qF_g \tag{1}$$

Here $q = 1 - \frac{F_p}{F_g}$ is the mass reduction factor (Tiwary and Kushvah 2015).

2.2 Potential field due to an oblate body

The gravitation potential per unit mass at an exterior point at a distance ρ from the center of an oblate body having mean radius R_m and mass distributed symmetrically about its equator can be expressed as (McCueskey 1963; Peter and Lissauer 2001; Fitzpatrick 2012; Singh and Taura 2014)

$$V_r = -\frac{G}{\rho}\left[1 - \sum_{l=1}^{\infty} J_{2l}\left(\frac{R_m}{\rho}\right)^{2l} P_{2l}(\cos\theta)\right] \tag{2}$$

transformation known as Birkhoff's method (Stiefel and Scheifele 1971; Szebehely 1967).

A good amount of literature exists which covers the KS-transformation in detail (Marchal 1990; Bruno 1994; Celletti 2002). Bettis and Szebehely (1971) were the first who applied a KS-transformation to the restricted three-body problem (R3BP) in the inertial frame and described a numerical treatment of the regularized equations of motion. Howell and Breakwell (1984) adopted the same transformation to regularize the classical circular restricted three-body problem (CR3BP) in the barycentric frame for the Earth-Moon system for finding stable periodic orbits around the collinear Lagrangian points. Prado (1996) studied minimum energy trajectories to transfer a spacecraft between the five Lagrangian points and the Earth and used Lemaître regularization to avoid singularities. Aarseth and Zare (1974) proposed a new transformation based on KS-regularization of a single binary where equations of motion numerically well behaved for close triple encounters. Numerical comparisons with standard KS-regularization showed that the new method gives improved accuracy per integration step at no extra computing time for a variety of examples. In addition, time reversal tests indicated that critical triple encounters may be studied with confidence.

In this paper, our aim is to study the KS-transformation to reduce the order of singularities arising when the spacecraft moves near the Earth in the three-body problem. In Sect. 2, we provide a mathematical formulation of the governing equations of motion in the photogravitational Sun–Earth system considering the oblateness of the Earth and radiation pressure of the Sun. Section 3 describes the regularization of the photogravitational CR3BP using the KS-transformation. Section 4 provides results and discussion, while Sect. 5 concludes our study.

In Eq. (2) G is the universal gravitational constant, θ is the angle between the body's symmetry axis and the vector to an infinitesimal body from the center at a distance ρ, $P_{2l}(\cos\theta)$ are the Legendre polynomials of first kind, and J_{2l} are the zonal harmonic coefficients. We assume that primaries have their equatorial planes coinciding with the plane of motion. Then Eq. (2) reduces to

$$V_r = -G\left[\frac{1}{\rho} + \frac{J_2 R_e^2}{2\rho^3} - \frac{3J_4 R_e^4}{8\rho^5} + \cdots\right] \qquad (3)$$

where R_e is the equatorial radius of the Earth.

Let $A_{2i} = J_{2i} R_e^{2i}$, $i = 1, 2, \ldots$ be the oblateness coefficients for the Earth, then Eq. (3) becomes

$$V_r = -G\left[\frac{1}{\rho} + \frac{A_2}{2\rho^3} - \frac{3A_4}{8\rho^5} + \cdots\right] \qquad (4)$$

Since J_2 accounts for most of the Earth's gravitational departure from a perfect sphere (Vallado 2013), hence, we account for oblateness up to the J_2 term only. Thus, in the barycentric frame, the gravitational potential per unit mass of the infinitesimal body accounting the radiation pressure of the Sun and considering oblateness of the Earth up to J_2 is given by

$$V = -G\left[\frac{m_\odot q}{d} + m_\oplus\left(\frac{1}{r} + \frac{A_2}{2r^3}\right)\right] \qquad (5)$$

where

$$\left.\begin{array}{l} d = \sqrt{(x - x_1)^2 + y^2 + z^2} \\ r = \sqrt{(x - x_2)^2 + y^2 + z^2} \end{array}\right\} \qquad (6)$$

In Eq. (6) $\|x_1\|$ and $\|x_2\|$ are the distance of the Sun and the Earth from the barycenter, respectively.

2.3 Kinetic energy

The kinetic energy per unit mass of the infinitesimal body in the barycentric frame rotating about the z-axis, with uniform angular velocity n, is expressed as

$$K = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + n(x\dot{y} - \dot{x}y) + \frac{1}{2}n^2(x^2 + y^2) \qquad (7)$$

where $\dot{x}$, $\dot{y}$, and $\dot{z}$ are the time derivatives of x, y, and z, respectively.

2.4 Equations of motion

The Lagrangian function of the system, in the barycentric reference frame, is then given by

$$L = K - V \qquad (8)$$

Equation (8) can be expressed as

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + n(x\dot{y} - \dot{x}y) - W \qquad (9)$$

where $W = V - \frac{1}{2}n^2(x^2 + y^2)$. Now using the Euler–Lagrange equation, the equations of motion of the infinitesimal body are given by

$$\left.\begin{array}{l} \ddot{x} - 2n\dot{y} = -\frac{\partial W}{\partial x} \\ \ddot{y} + 2n\dot{x} = -\frac{\partial W}{\partial y} \\ \ddot{z} = -\frac{\partial W}{\partial z} \end{array}\right\} \qquad (10)$$

It can be noted that the variables in Eq. (10) are in the dimensional form. To give them a non-dimensional form, we chose the total sum of the mass of the Sun–Earth system to be one, and the distance between the Sun and the Earth to be one. Subsequently, we get $m_\odot = 1 - \mu$, $m_\oplus = \mu$, $x_1 = -\mu$, $x_2 = 1 - \mu$ where $\mu = \frac{m_\oplus}{(m_\odot + m_\oplus)}$ is the mass ratio parameter. The unit of time is chosen so as to make the unperturbed mean motion, n_0, and gravitational constant, G, unity (Sharma and Rao 1976). Hence, the equations of motion (10), in non-dimensional form, in the barycentric coordinate system, becomes (Srivastava et al. 2016)

$$\left.\begin{array}{l} \ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x} \\ \ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y} \\ \ddot{z} = \frac{\partial U}{\partial z} \end{array}\right\} \qquad (11)$$

where

$$\left.\begin{array}{l} U = \frac{n^2}{2}(x^2 + y^2) + \frac{(1-\mu)q}{d} + \frac{\mu}{r} + \frac{A_2\mu}{2r^3} \\ d = \sqrt{(x + \mu)^2 + y^2 + z^2} \\ r = \sqrt{(x + \mu - 1)^2 + y^2 + z^2} \end{array}\right\} \qquad (12)$$

The partial derivatives of U in Eq. (11) are given by

$$\frac{\partial U}{\partial x} = n^2 x - \frac{q(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x+\mu-1)}{r^3}$$
$$- \frac{3}{2}\frac{\mu A_2(x+\mu-1)}{r^5} \qquad (13)$$

$$\frac{\partial U}{\partial y} = n^2 y - \frac{q(1-\mu)y}{d^3} - \frac{\mu y}{r^3} - \frac{3}{2}\frac{\mu A_2 y}{r^5} \qquad (14)$$

$$\frac{\partial U}{\partial z} = -\frac{q(1-\mu)z}{d^3} - \frac{\mu z}{r^3} - \frac{3}{2}\frac{\mu A_2 z}{r^5} \qquad (15)$$

It can be observed that the governing equations of motion (11) contain a pole of order five at $r = 0$ whereas the classical CR3BP contains a pole of order three at $r = 0$. Hence, the inclusion of oblateness leads to an increase of the order of the singularity from three to five.

2.5 Oblateness coefficient

The oblateness coefficient A_2 is given by

$$A_2 = J_2 R_e^2 \tag{16}$$

However, J_2 is expressed as (Fitzpatrick 2012)

$$J_2 = \frac{(I_e - I_r)}{m_\oplus R_e^2} \tag{17}$$

where $I_e \ (= \frac{1}{5} m_\oplus R_e^2)$ is the moment of inertia of the Earth about any axis in the equatorial plane, and $I_r \ (= \frac{1}{5} m_\oplus R_p^2$, R_p is the polar radius of the Earth) is the moment of inertia of the Earth about its axis of rotation. Consequently, Eq. (16) reduces to

$$A_2 = \frac{(R_e^2 - R_p^2)}{5} \tag{18}$$

In non-dimensional form, Eq. (18) is expressed by

$$A_2 = \frac{(R_e^2 - R_p^2)}{5 d_{\oplus\odot}^2} \tag{19}$$

where $d_{\oplus\odot}$ is the distance between the Sun and the Earth.

2.6 Perturbed mean motion

The perturbed mean motion, n, is expressed as (Vallado 2013)

$$n = n_0 \left[1 + \frac{3 J_2 (\sqrt{1 - e_0^2}) R_e^2}{4 p^2} \left(2 - 3 \sin^2(i_0) \right) \right] \tag{20}$$

In Eq. (20), $p = d_{\odot\oplus}(1 - e_0^2)$, e_0 and i_0 are the unperturbed eccentricity and inclination angle, respectively, of the orbit. Hence in the barycentric frame, the non-dimensional form of the perturbed mean motion for the circular motion reduces to

$$n = \left[1 + \frac{3}{2} A_2 \right] \tag{21}$$

It can be noted that for $q = 1$ and $A_2 = 0$ the governing equations of motion (11) reduce to the classical CR3BP.

3 Regularization of photogravitational CR3BP

In this section, the 3-dimensional system given in Eqs. (11) is converted into 4-dimensional system using KS-transformation. The purpose of this transformation into 4-dimensional space is to reduce the order of the singularities due to the motion of the spacecraft in a close vicinity of the Earth in the restricted three-body problem. The KS-transformation

in 3-dimensional space consists of two stages, namely, the transformation of the independent variable, t, and the transformation of the space coordinates x, y, z of the governing equations of motion (11). The time transformation transforms the independent variable t to a new independent variable τ through the relationship $dt = r d\tau$. In the rest of the work, we represent $()' = \frac{d()}{d\tau}$.

The governing equations of motion (11) in terms of the new independent variable τ become

$$x'' - 2nry' = \frac{x'}{r^2}(\vec{r}.\vec{r}') - \frac{\mu(x + \mu - 1)}{r} \\ - \frac{3}{2} \frac{\mu A_2 (x + \mu - 1)}{r^3} + r^2 F_1 \tag{22}$$

$$y'' + 2rnx' = \frac{y'}{r^2}(\vec{r}.\vec{r}') - \frac{\mu y}{r} - \frac{3}{2} \frac{\mu A_2 y}{r^3} + r^2 F_2 \tag{23}$$

$$z'' = \frac{z'}{r^2}(\vec{r}.\vec{r}') - \frac{\mu z}{r} - \frac{3}{2} \frac{\mu A_2 z}{r^3} + r^2 F_3 \tag{24}$$

where $(\vec{r}.\vec{r}')$ is the scalar product of two vectors $\vec{r}$ and $\vec{r}'$. The terms F_1, F_2, and F_3 are given by

$$F_1 = n^2 x - \frac{q(1 - \mu)(x + \mu)}{d^3} \tag{25}$$

$$F_2 = n^2 y - \frac{q(1 - \mu)y}{d^3} \tag{26}$$

$$F_3 = -\frac{q(1 - \mu)z}{d^3} \tag{27}$$

Let $\vec{R} = (x + \mu - 1, y, z, 0)^T$ and $\vec{u} = (u_1, u_2, u_3, u_4)^T$, then Eqs. (22)–(24) can be expressed in the following form:

$$\vec{R}'' = \frac{\vec{R}.\vec{R}'}{\vec{R}.\vec{R}} \vec{R}' + RB\vec{R}' - \frac{\mu \vec{R}}{R} - \frac{3}{2} \frac{\mu A_2 \vec{R}}{R^3} + R^2 \vec{F} \tag{28}$$

where $R = \|\vec{R}\|$, $\vec{F} = (F_1, F_2, F_3, 0)^T$, and $B = \begin{pmatrix} 0 & 2n & 0 & 0 \\ -2n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The coordinate transformation from 3-dimensional space to 4-dimensional space is given by

$$\vec{R} = L(\vec{u})\vec{u} \tag{29}$$

where the operator $L(\vec{u})$ is expressed by the KS-matrix,

$$L(\vec{u}) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \tag{30}$$

From Eq. (29), 3-dimensional space coordinates x, y, z are related with 4-dimensional space coordinates u_1, u_2, u_3, u_4 via the expressions

$$x = u_1^2 - u_2^2 - u_3^2 + u_4^2 + 1 - \mu \tag{31}$$

$$y = 2(u_1 u_2 - u_3 u_4) \tag{32}$$

$$z = 2(u_1 u_3 + u_2 u_4) \tag{33}$$

Equations (31)–(33) provide a transformation from 3-dimensional position to 4-dimensional counterpart and vice versa. It can be observed that, for $\mu = 1$, Eqs. (31)–(33) are the KS-transformation. It can easily be found that

$$r = \vec{u}.\vec{u} \tag{34}$$

The operator $L(\vec{u})$ has the following significant properties:

(I) $L^T(\vec{u})L(\vec{u}) = r I_4$, where I_4 is an 4×4 identity matrix.
(II) $L'(\vec{u}) = L(\vec{u}')$.
(III) $L(\vec{u})\vec{u}' = L(\vec{u}')\vec{u} = \frac{1}{2}\vec{R}'$, provided $u_4 u_1' - u_3 u_2' + u_2 u_3' - u_1 u_4' = 0$. This relation is known as the bilinear relation.
(IV) If two vectors $\vec{u}$, $\vec{u}'$ satisfy the bilinear relation, then

$$(\vec{u}.\vec{u})L(\vec{u}')\vec{u}' - 2(\vec{u}.\vec{u}')L(\vec{u})\vec{u}' + (\vec{u}'.\vec{u}')L(\vec{u})\vec{u} = 0.$$

From Eq. (29), we have

$$\vec{R}'' = 2L(\vec{u})\vec{u}'' + 2L(\vec{u}')\vec{u}' \tag{35}$$

Combining Eqs. (28), (29), (35), and property (III), we get

$$L(\vec{u})\vec{u}'' - \frac{(\vec{u}'.\vec{u}')}{(\vec{u}.\vec{u})} L(\vec{u})\vec{u}$$

$$= (\vec{u}.\vec{u})BL(\vec{u})\vec{u}' - \frac{\mu}{2}\frac{L(\vec{u})\vec{u}}{(\vec{u}.\vec{u})}$$

$$- \mu A_2 \frac{3L(\vec{u})\vec{u}}{4(\vec{u}.\vec{u})^3} + \frac{(\vec{u}.\vec{u})^2}{2}\vec{F} \tag{36}$$

Further, by property (I) of $L(\vec{u})$, we have

$$L^{-1}(\vec{u}) = \frac{L^T(\vec{u})}{(\vec{u}.\vec{u})} \tag{37}$$

Operating by $L^{-1}(\vec{u})$ on both sides of Eq. (36), we get

$$\vec{u}'' + \left[\frac{3\mu A_2 + 2\mu(\vec{u}.\vec{u})^2 - 4(\vec{u}'.\vec{u}')(\vec{u}.\vec{u})^2}{4(\vec{u}.\vec{u})^3} \right]\vec{u}$$

$$= L^T(\vec{u})BL(\vec{u})\vec{u}' + \frac{(\vec{u}.\vec{u})}{2}L^T(\vec{u})\vec{F} \tag{38}$$

Now, define

$$h = \frac{\dot{\vec{R}}.\dot{\vec{R}}}{2} - \frac{\mu}{R} - \frac{3}{2}\frac{\mu A_2}{R^3} \tag{39}$$

Also, we have

$$\frac{\dot{\vec{R}}.\dot{\vec{R}}}{2} = \frac{2L(\vec{u})\vec{u}'.L(\vec{u})\vec{u}'}{r^2} \tag{40}$$

From Eqs. (39) and (40), and using the linear algebra property, we get

$$h = -\left[\frac{3\mu A_2 + 2\mu(\vec{u}.\vec{u})^2 - 4(\vec{u}'.\vec{u}')(\vec{u}.\vec{u})^2}{2(\vec{u}.\vec{u})^3} \right] \tag{41}$$

From Eqs. (38) and (41), we get

$$\vec{u}'' - \frac{h}{2}\vec{u} = L^T(\vec{u})BL(\vec{u})\vec{u}' + \frac{(\vec{u}.\vec{u})}{2}L^T(\vec{u})\vec{F} \tag{42}$$

Further, the Jacobi constant, C, can be given by the relationship

$$\frac{\dot{\vec{R}}.\dot{\vec{R}}}{2} = n^2 \frac{(x^2 + y^2)}{2} + \frac{q(1-\mu)}{d} + \frac{\mu}{R} + \frac{\mu A_2}{2R^3} - \frac{C}{2} \tag{43}$$

It is computed at the initial time (Howell and Breakwell 1984). Therefore, from Eqs. (39) and (43), we get

$$h = \frac{n^2}{2}(x^2 + y^2) + \frac{q(1-\mu)}{d} - \frac{\mu A_2}{r^3} - \frac{C}{2} \tag{44}$$

It can be noted that after applying KS-transformation to 3-dimensional system, order of the pole at $r = 0$ reduces from three to one for the classical CR3BP and from five to three for the considered problem. For the classical CR3BP, only equation $d\tau = \frac{1}{r}dt$ contains a simple pole (i.e., a pole of order one) whereas for the CR3BP+SRP+Oblateness, Eqs. (43) and (44) contain a pole of order three. We notice that when $q = 1$ and $A_2 = 0$, Eq. (42) reduces to the KS-regularization of the classical CR3BP (Howell and Breakwell 1984). However, a KS-regularization of the classical CR3BP provides a relatively less accurate solution for any Sun-planet system due to not accounting the effects of perturbing forces such as radiation pressure of the Sun, oblateness of the planet etc. Hence, considering these perturbations in the classical CR3BP and then finding KS-regularization of the modeled equations of motion improves the accuracy for finding stable periodic orbits in the Sun-planet system.

The expressions for d, F_1, F_2, F_3, and h in terms of new four variables are given by

$$d = \left\{ \left[(1 + u_1^2 - u_2^2 - u_3^2 + u_4^2)^2 + 4(u_1 u_2 - u_3 u_4)^2 \right. \right.$$

$$\left. \left. + 4(u_1 u_3 + u_2 u_4)^2 \right] \right\}^{\frac{1}{2}} \tag{45}$$

$$F_1 = n^2(u_1^2 - u_2^2 - u_3^2 + u_4^2 + 1 - \mu)$$

$$- \frac{q(1-\mu)(1 + u_1^2 - u_2^2 - u_3^2 + u_4^2)}{d^3} \tag{46}$$

$$F_2 = 2n^2(u_1 u_2 - u_3 u_4) - \frac{2q(1-\mu)(u_1 u_2 - u_3 u_4)}{d^3} \tag{47}$$

$$F_3 = -\frac{2q(1-\mu)(u_1 u_3 + u_2 u_4)}{d^3} \tag{48}$$

$$h = \frac{n^2}{2}\big[\big(u_1^2 - u_2^2 - u_3^2 + u_4^2 + 1 - \mu\big)^2 + 4(u_1 u_2 - u_3 u_4)^2\big]$$
$$+ \frac{q(1-\mu)}{d} - \frac{\mu A_2}{r^3} - \frac{C}{2} \tag{49}$$

The scalar form of Eq. (42) can be written as

$$u_1'' = \frac{h}{2}u_1 + 2\big(u_1^2 + u_2^2\big)u_2' + 2(-u_1 u_4 + u_2 u_3)u_3'$$
$$- 2(u_1 u_3 + u_2 u_4)u_4' + \frac{(u_1^2 + u_2^2 + u_3^2 + u_4^2)}{2}$$
$$\times (u_1 F_1 + u_2 F_2 + u_3 F_3) \tag{50}$$

$$u_2'' = \frac{h}{2}u_2 - 2\big(u_1^2 + u_2^2\big)u_1' + 2(u_2 u_4 + u_1 u_3)u_3'$$
$$+ 2(u_2 u_3 - u_1 u_4)u_4' + \frac{(u_1^2 + u_2^2 + u_3^2 + u_4^2)}{2}$$
$$\times (-u_2 F_1 + u_1 F_2 + u_4 F_3) \tag{51}$$

$$u_3'' = \frac{h}{2}u_3 - 2(u_2 u_3 - u_1 u_4)u_1' - 2(u_1 u_3 + u_2 u_4)u_2'$$
$$+ 2\big(u_3^2 + u_4^2\big)u_4' + \frac{(u_1^2 + u_2^2 + u_3^2 + u_4^2)}{2}$$
$$\times (-u_3 F_1 - u_4 F_2 + u_1 F_3) \tag{52}$$

$$u_4'' = \frac{h}{2}u_4 + 2(u_2 u_4 + u_1 u_3)u_1' + 2(u_1 u_4 - u_2 u_3)u_2'$$
$$- 2\big(u_3^2 + u_4^2\big)u_3' + \frac{(u_1^2 + u_2^2 + u_3^2 + u_4^2)}{2}$$
$$\times (u_4 F_1 - u_3 F_2 + u_2 F_3) \tag{53}$$

The set of Eqs. (50)–(53) are solved by using the Runge–Kutta method for a different set of initial conditions. The conversion of initial conditions from 3-dimensional space to 4-dimensional one is explained below.

Equations (31)–(34) give

$$\frac{1}{2}(x + R + \mu - 1) = u_1^2 + u_4^2 \tag{54}$$

$$u_2 = \frac{y u_1 + z u_4}{2(u_1^2 + u_4^2)}, \qquad u_3 = \frac{z u_1 - y u_4}{2(u_1^2 + u_4^2)} \tag{55}$$

where

$$R = u_1^2 + u_2^2 + u_3^2 + u_4^2 \tag{56}$$

Keeping the arbitrariness of one of the components of $\vec{u}$, we select u_1 and u_4 in such a way that Eq. (54) is satisfied. For simplicity, choose either u_1 or u_4 identically equal to zero or u_1 equals u_4. When $x + \mu - 1 \geq 0$, Eqs. (54) and (55) are used to determine the initial values of the components of $\vec{u}$. When $x + \mu - 1 < 0$, the following relations are used (Bettis

and Szebehely 1971):

$$\left.\begin{array}{l} R - x + \mu - 1 = 2(u_2^2 + u_3^2) \\[2mm] u_1 = \frac{y u_2 + z u_3}{R - (x + \mu - 1)} \\[2mm] u_4 = \frac{z u_2 - y u_3}{R - (x + \mu - 1)} \end{array}\right\} \tag{57}$$

and choosing either u_2 or u_3 arbitrarily. Once the initial components of $\vec{u}$ are computed, their derivatives with respect to τ, $\vec{u}'$, is computed by using the property (III) of the operator $L(\vec{u})$ and is expressed as

$$\vec{u}' = \frac{1}{2}L^T(\vec{u})\dot{\vec{R}} \tag{58}$$

where $\dot{\vec{R}} = (\dot{x}, \dot{y}, \dot{z}, 0)$ is known.

4 Results and discussion

In this study, we mainly focused on modeling of the 4-dimensional regularized equations of motion using a KS-transformation on the 3-dimensional equations, which corresponds to photogravitational CR3BP accounting for radiation pressure of the Sun and oblateness of the Earth, and we addressed a comparison of the numerical solution for regularized and non-regularized motion. A numerical computation of regularized equations of motion is carried out when the infinitesimal body, rotating around L_1 and L_2 Lagrangian points, comes close to the Earth, and the obtained results are compared with the solution of non-regularized equations of motion. Initial conditions corresponding to periodic as well as non-periodic orbits are chosen and governing equations are propagated using RK-7(8) scheme with the corresponding initial conditions. In the considered model radiation pressure due to the Sun and oblateness of the Earth are included over and above the classical model. Results are computed for different values of the mass reduction factor (q) with and without considering the oblateness of the Earth. The oblateness coefficient, A_2 ($= 2.43 \times 10^{-12}$) is computed from Eq. (19). The out-of-plane amplitude, $A_z = 110\,000$ km (Srivastava et al. 2016) is considered throughout the study unless otherwise mentioned. The time period is chosen as one orbit period. In this work, two different choices of initial conditions are considered, namely initial conditions without and with differential correction applied on the third-order analytic solution of the governing equations of motion (11). The results of these two choices are discussed in Sect. 4.1. In Sect. 4.2, we have studied the variation of transformed time, τ of 4-dimensional space with respect to A_z while in Sect. 4.3 we mention the potential applications of the KS-regularized equations of motion.
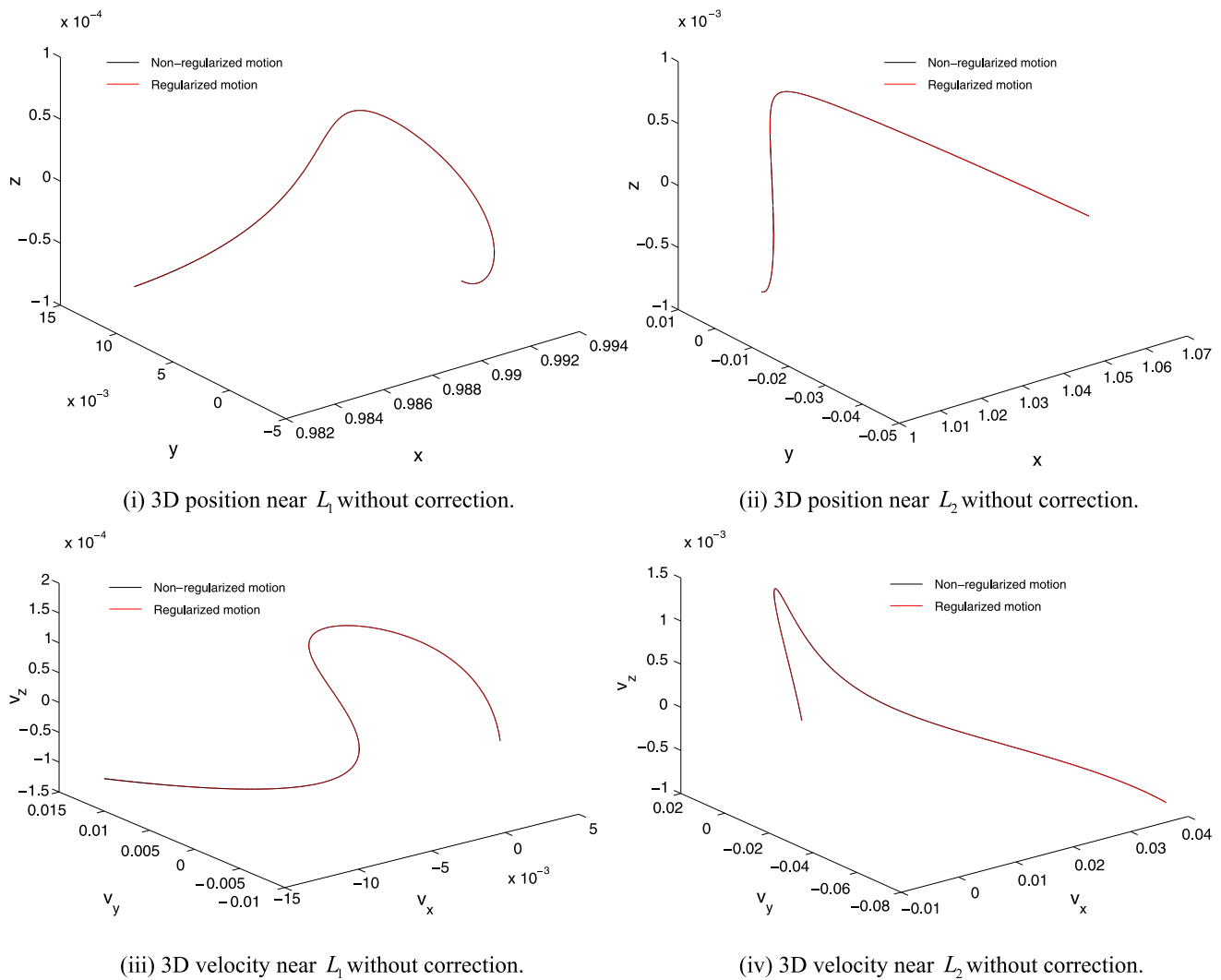
(i) 3D position near L_1 without correction.

(ii) 3D position near L_2 without correction.

(iii) 3D velocity near L_1 without correction.

(iv) 3D velocity near L_2 without correction.

Fig. 2 Three-dimensional position (non-dimensional) and velocity (non-dimensional) comparisons around L_1 and L_2 points using a third order periodic analytic approximation without and with differential correction for $q = 0.999334$, $A_2 = 2.43 \times 10^{-12}$

4.1 Initial condition obtained without and with differential correction

Figure 2 shows a comparison between the 3-dimensional (3D) position and velocity profiles (the velocity profile is shown only for the first choice of q and A_2) obtained from non-regularized and regularized equations of motion considering initial conditions without and with differential correction applied on the third-order analytic solution for $q = 0.999334$, $A_2 = 2.43 \times 10^{-12}$. Similarly, Figs. 3, 4 and 5 show a 3D position profile comparison for ($q = 1$, $A_2 = 2.43 \times 10^{-12}$), ($q = 0.999334$, $A_2 = 2.43 \times 10^{-12}$), and ($q = 1$, $A_2 = 0$), respectively. It can be seen that for the considered set of parameters the results obtained by both sets of governing equations overlap each other in 3D space, which ensures the correctness of the regularization modeling of the problem.

4.2 Variation of τ

Figure 6 depicts a typical variation of new independent transformed time, τ of 4-dimensional space with respect to time t of 3-dimensional space around L_1 and L_2 Lagrangian points and their comparisons with a classical CR3BP as well as CR3BP+SRP+Oblateness assuming both the cases with and without differential correction applied on the third-order analytic solution as the initial condition. From the figure, an almost nonlinear relation between t and τ is observed.

A quantitative variation of τ as a function of t and for different initial conditions around L_1 corresponding to q and A_2 values is given in Table 1. Table 1 contains initial conditions without and with differential correction. The variations of τ and t with respect to the out-of-plane amplitude, A_z are shown in Table 2 around the L_1 and L_2 points. It can be observed that as A_z increases, τ and t decrease for all con-

(v) 3D position near L_1 with correction.

(vi) 3D position near L_2 with correction.

(vii) 3D velocity near L_1 with correction.

(viii) 3D velocity near L_2 with correction.

Fig. 2 (*Continued*)

(i) 3D position near L_1 without correction.

(ii) 3D position near L_2 without correction.
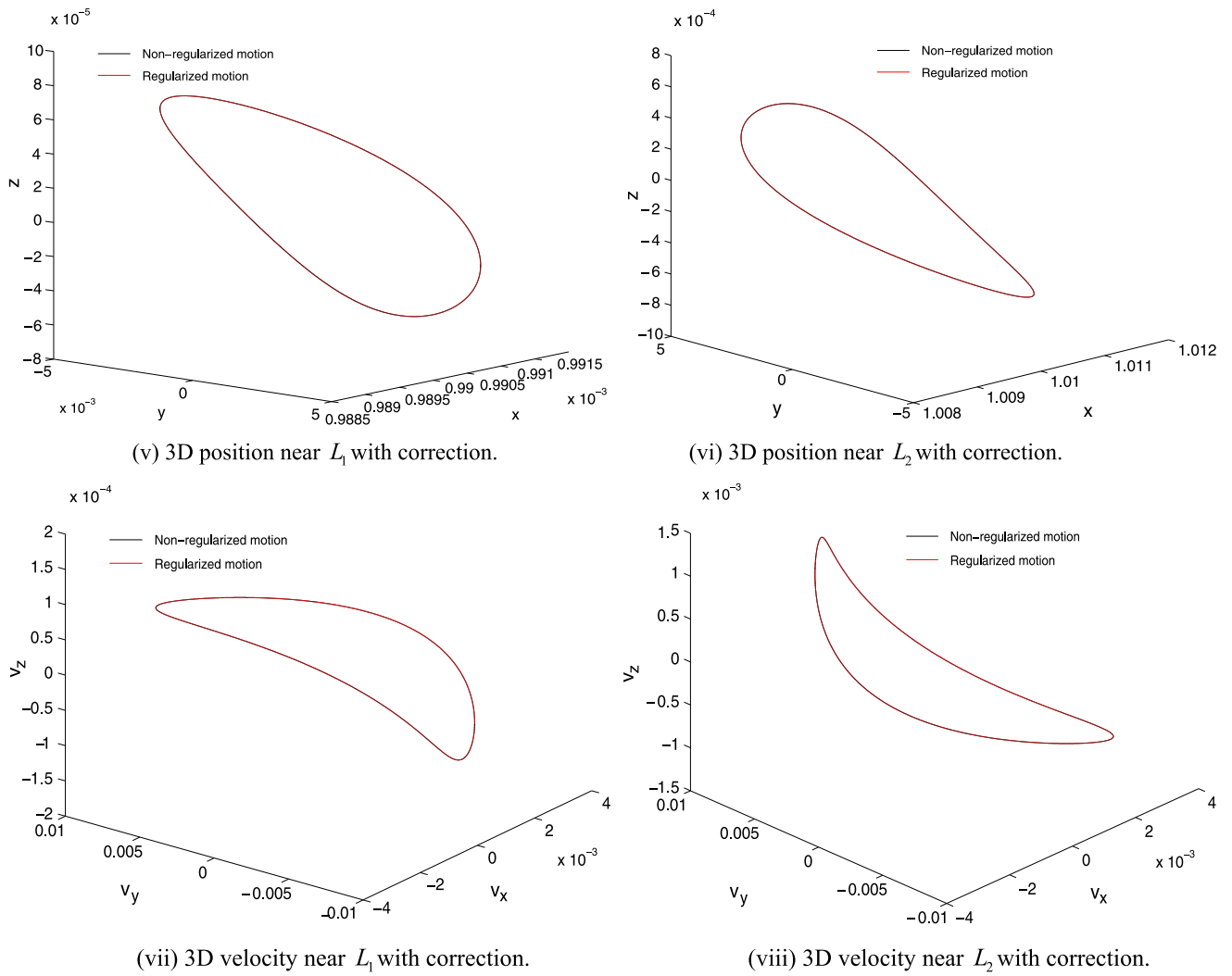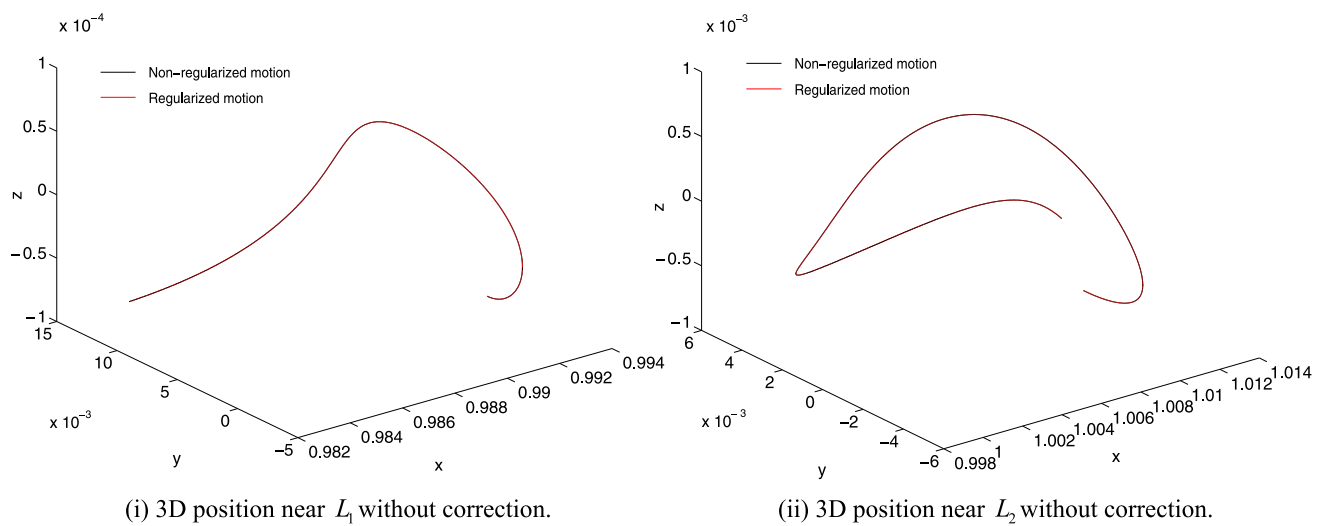
Fig. 3 Three-dimensional position (non-dimensional) comparison around L_1 and L_2 points using third order periodic analytic approximation without and with differential correction for $q = 1$ and $A_2 = 2.43 \times 10^{-12}$
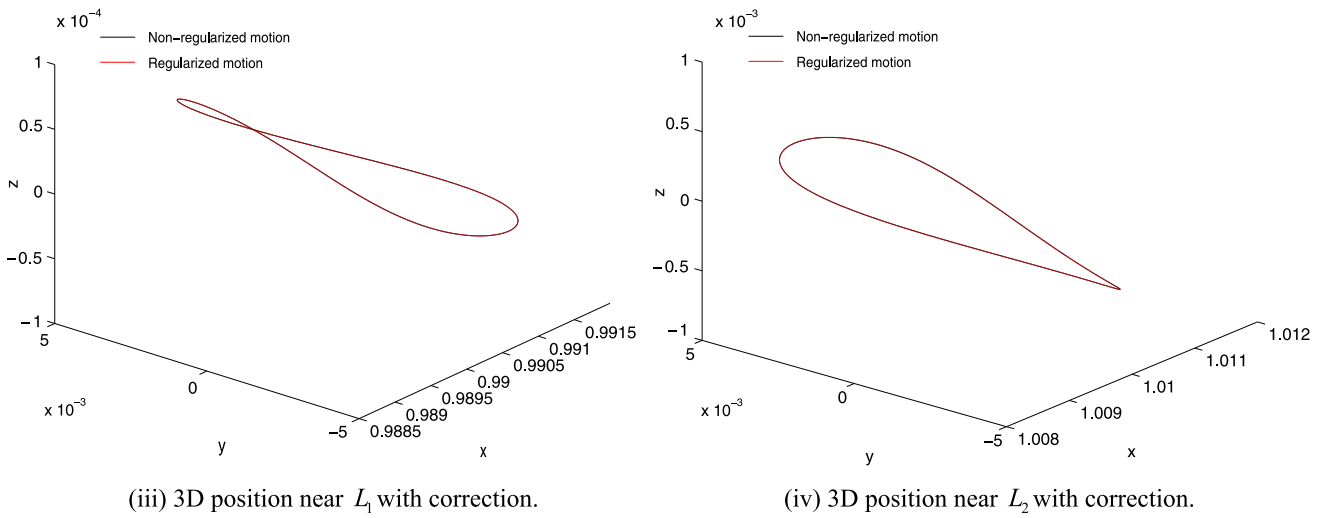
(iii) 3D position near L_1 with correction.

(iv) 3D position near L_2 with correction.

Fig. 3 (*Continued*)

(i) 3D position near L_1 without correction.

(ii) 3D position near L_2 without correction.

(iii) 3D position near L_1 with correction.
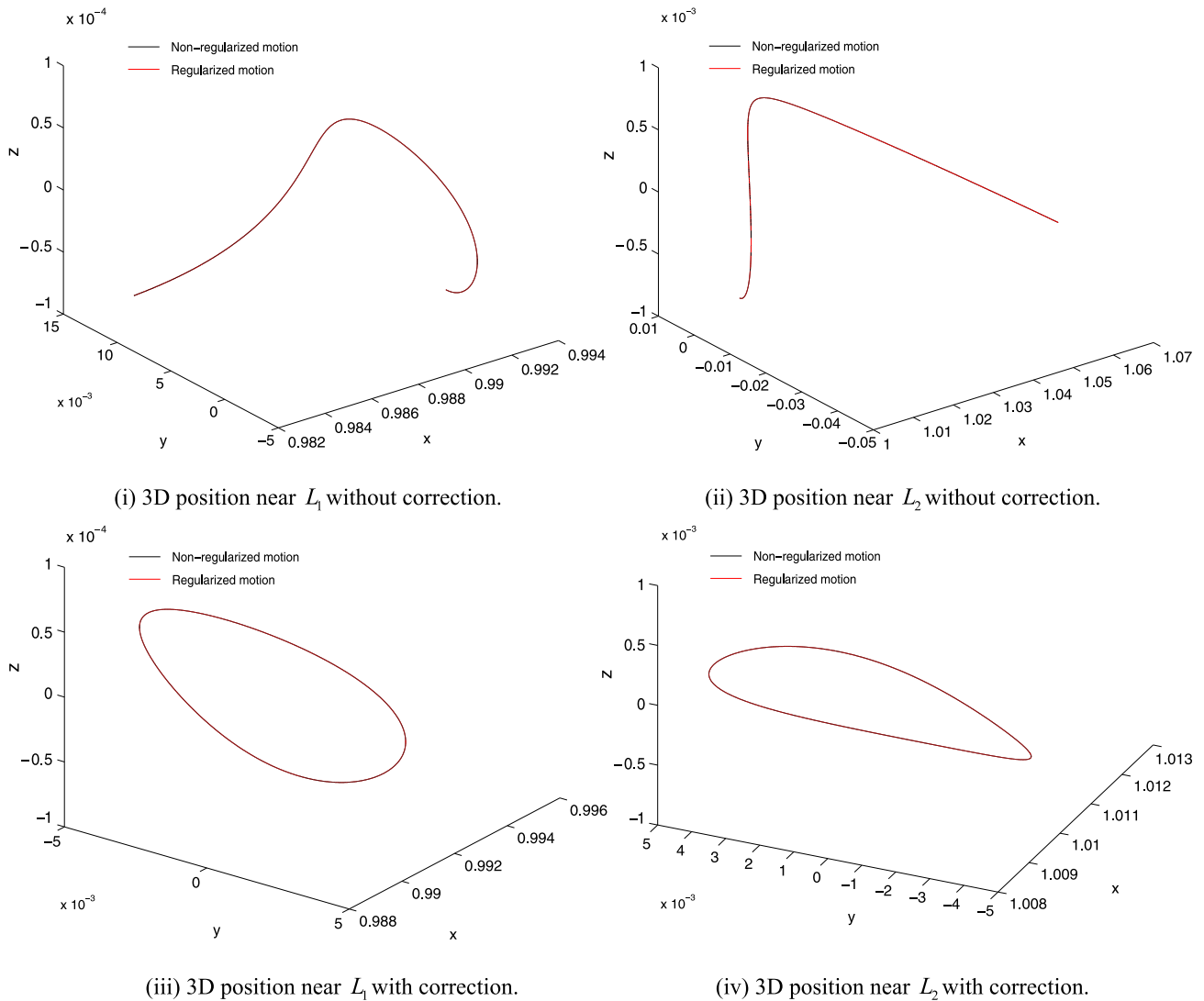

(iv) 3D position near L_2 with correction.

Fig. 4 Three-dimensional position (non-dimensional) comparison around L_1 and L_2 points using a third-order periodic analytic approximation without and with differential correction for $q = 0.999334$ and $A_2 = 0$

(i) 3D position near L_1 without correction.

(ii) 3D position near L_2 without correction.

(iii) 3D position near L_1 with correction.

(iv) 3D position near L_2 with correction.

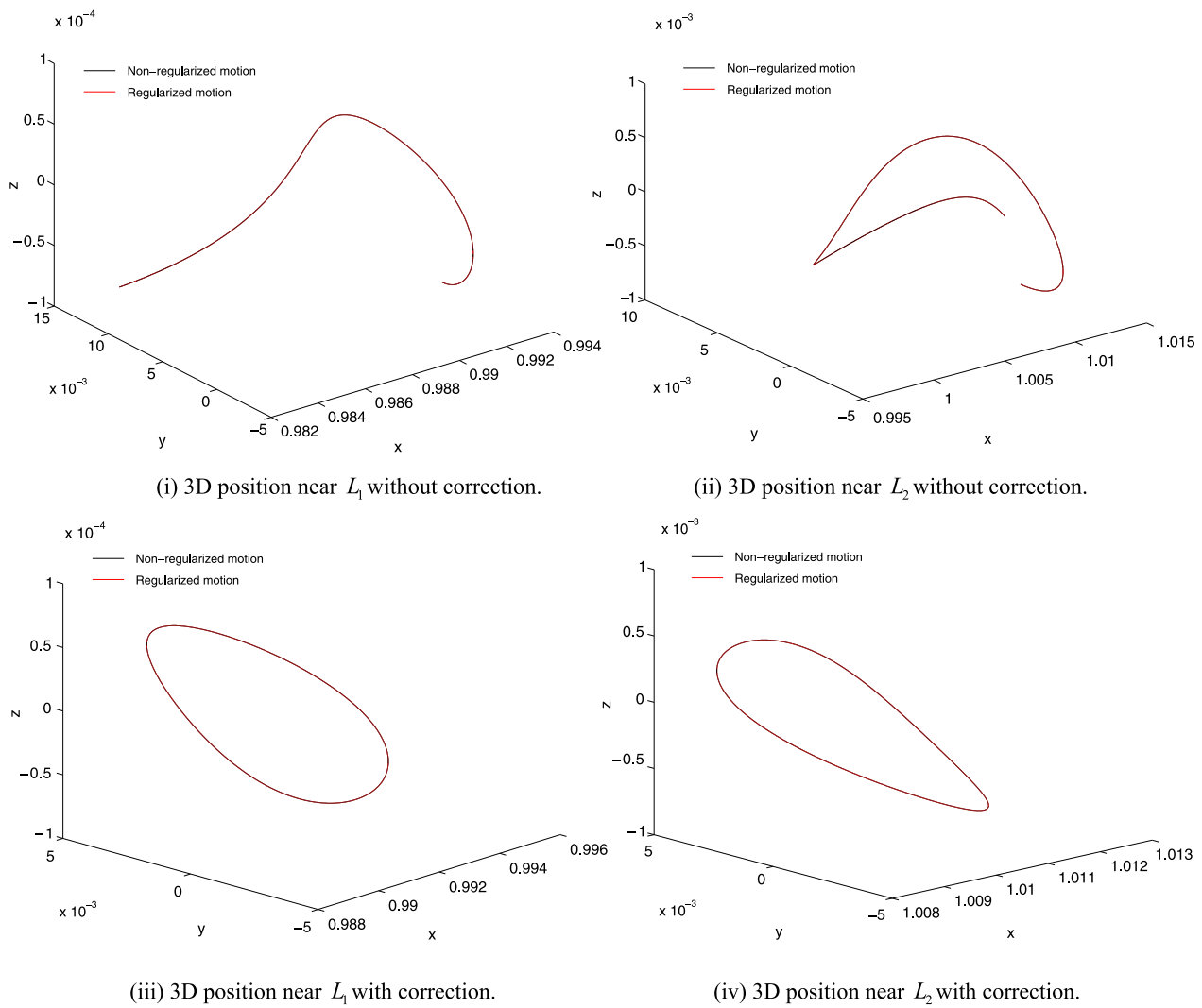**Fig. 5** Three-dimensional position (non-dimensional) comparison around L_1 and L_2 points using a third-order periodic analytic approximation without and with differential correction for $q = 1$ and $A_2 = 0$
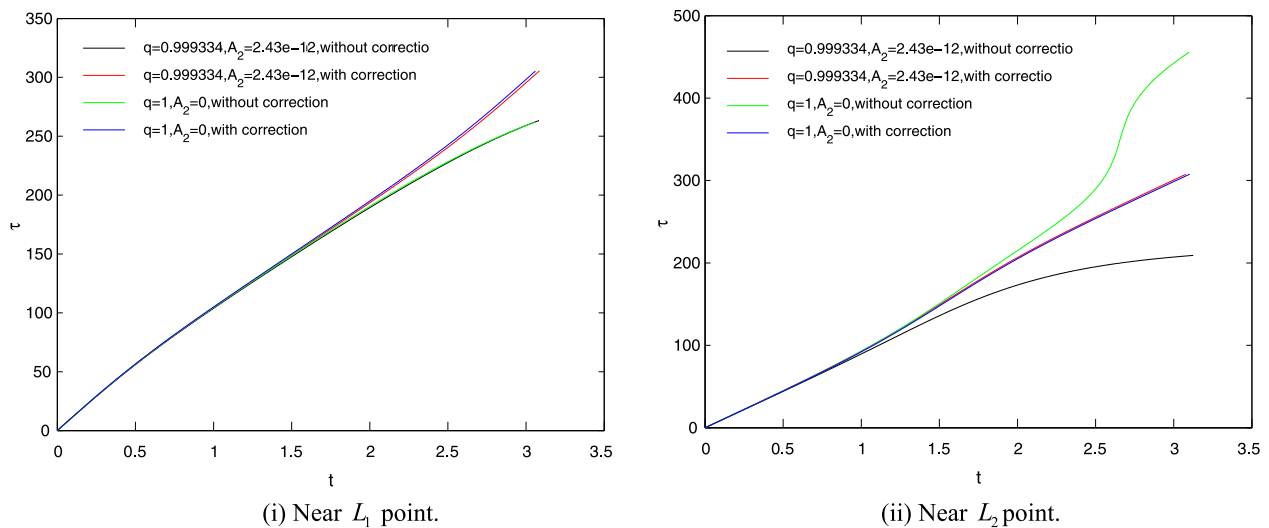
(i) Near L_1 point.

(ii) Near L_2 point.

Fig. 6 Variation of τ versus t around (i) L_1 point and (ii) L_2 point

Table 1 Third-order analytic solution without (with) differential correction taken as initial condition around L_1 point

Parameters	$q = 0.999334$, $A_2 = 2.43 \times 10^{-12}$	$q = 1$, $A_2 = 2.43 \times 10^{-12}$	$q = 0.999334$, $A_2 = 0$	$q = 1$, $A_2 = 0$
x	0.9915202 (0.9915795)	0.9915794 (0.9916252)	0.9915198 (0.9915198)	0.9915794 (0.9915794)
y	0 (0)	0 (0)	0 (0)	0 (0)
z	−0.00067028 (−0.00067028)	−0.00067063 (−0.00067063)	−0.00067028 (−0.00067028)	−0.00067063 (−0.00067063)
$\dot{x}$	0 (0)	0 (0)	0 (0)	0 (0)
$\dot{y}$	−0.00959229 (−0.00984497)	−0.00961724 (0.00979540)	−0.00959209 (−0.00959209)	−0.00961724 (−0.00961724)
$\dot{z}$	0 (0)	0 (0)	0 (0)	0 (0)
T	3.08283808 (3.08559430)	3.05704228 (3.05964337)	3.08302864 (3.08302864)	3.05704228 (3.05704228)
τ	263.371911 (305.564135)	261.948366 (305.372021)	262.661527 (262.661527)	261.948319 (261.948319)

Table 2 Variation of τ and t with respect to A_z around L_1 and L_2 Lagrangian points

Parameters		$A_z = 110\,000$ km		$A_z = 700\,000$ km	
		$\tau(t)$			
		Differential correction			
		No	Yes	No	Yes
$q = 0.999334$, $A_2 = 2.43 \times 10^{-12}$	L_1	222.58 (3.06388)	302.65 (3.05499)	264.03 (3.08332)	305.62 (3.08605)
	L_2	209.12 (3.12485)	307.50 (3.07652)	209.42 (3.09924)	304.96 (3.04096)
$q = 1$, $A_2 = 2.43 \times 10^{-12}$	L_1	217.49 (3.03640)	302.46 (3.02586)	262.93 (3.05757)	305.42 (3.06016)
	L_2	455.67 (3.09884)	307.69 (3.10197)	279.07 (3.07173)	305.16 (3.06906)
$q = 0.999334$, $A_2 = 0$	L_1	222.45 (3.06408)	302.65 (3.05499)	263.31 (3.08351)	305.61 (3.08605)
	L_2	208.96 (3.12503)	307.50 (3.07652)	209.29 (3.09944)	304.96 (3.04096)
$q = 1$, $A_2 = 0$	L_1	217.49 (3.036399)	302.46 (3.02585)	262.93 (3.05756)	305.42 (3.06015)
	L_2	455.67 (3.09884)	307.69 (3.10197)	279.07 (3.07173)	305.16 (3.06906)

sidered cases around the L_1 point. A similar behavior is observed around the L_2 point except the case ($q = 0.999334$, $A_2 = 2.43 \times 10^{-12}$) where τ increases as A_z increases.

4.3 Applications of KS-regularized motion

Stable or nearly stable periodic orbits are very important from mission design prospectives. The fuel expenditure for maintaining the stable or nearly stable halo orbits is very nominal. The stable orbit is found to be near the smaller primary. After a threshold value of the out-of-plane amplitude without regularization of the 3-dimensional governing equations, the stable periodic orbits cannot be obtained due to non-convergence of the numerical continuation method (Srivastava et al. 2016). Hence, the obtained regularized system of motion can be used for finding stable periodic orbits and also for the transfer trajectory from the Earth parking orbit to a periodic orbit near the collinear Lagrangian points in the photogravitational CR3BP accounting radiation pressure of the Sun and oblateness of the Earth.

5 Conclusions

In this study, the well-known KS-transformation is described to reduce the order of singularities arising when the spacecraft moves closer toward the smaller primary in the Sun–Earth system accounting radiation pressure of the Sun and oblateness of the Earth. The obtained regularized motion of the modeled equations are solved numerically and compared with numerical solution of the original governing equation of motion in 3-dimensional space to establish agreement between the two systems of equations and very satisfactory results are achieved by computing both components of motion i.e., position and velocity. The new independent transformed time τ is also computed as a function of time t of 3-dimensional space and the initial conditions. A nonlinear relationship is found between τ and t. It is noticed that the KS-transformation reduces the order of the pole from five to three at the point of singularity of the governing equations of motion, whereas it reduces a pole of order three to a simple pole for the classical CR3BP.

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