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ORIGINAL ARTICLE

Tunneling radiation as new perspective of understanding the thermodynamics in $f(R)$ gravity

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Abstract We use the Parikh-Wilczek method to study the tunneling radiation from the event horizon of a charged AdS black hole in $f(R)$ gravity. The emission rate of a particle is calculated. The emission spectrum deviates from the pure thermal spectrum but consists with an underlying unitary theory. The emission rate of massive particles takes the same functional form as that of massless particles. Contradictory but interesting phenomenon is discovered. The expression of the emission rate for a black hole in $f(R)$ gravity differs from that for a black hole in Einstein gravity, if the area-law of the black hole entropy is insisted on. Conversely, based on abandoning the area-law and admitting the conventional tunneling rate, we obtain the expressions of the entropy and the first law of thermodynamics for the $f(R)$ gravity black hole, which is in accordance with the early results. So the research of tunneling radiation in this paper may serve as a new perspective of understanding the thermodynamics of black holes in $f(R)$ gravity.

Keywords Tunneling radiation · $f(R)$ gravity · Charged AdS black hole · Self-gravitation

Hawking (1975) proved that the black hole could radiate particles quantum-mechanically. Since Gibbons and Hawking (1997) demonstrated that the radiation is exactly thermal, much work to prove that the energy spectrum is precisely thermal spectrum has been done (Zhang and Zhao 2002; Liu and Zhao 2001; Yang and Lin 2002; Jing 2003). However, there are two puzzles: One is where the barrier appears during the radiation. The other is the purely thermal spectrum, from which we can not obtain any other information but one parameter, i.e. temperature, this means that if things are absorbed into the black hole, then their important information such as unitarity will be lost during the emission and there will be no marks left once the black hole is evaporated out. Later, Parikh and Wilczek (Kraus and Wilczek 1995; Parikh and Wilczek 2000; Parikh 2004a) proposed a method to calculate the emission rate at which particles tunnel across the event horizon. They treated Hawking radiation as a tunneling process. They found that the barrier is created by the outgoing particle itself, and their key insight is to find a coordinate system which is well behaved at the event horizon to calculate the emission rate. In this way they have calculated the corrected emission spectrum of the spherically symmetric black holes, such as Schwarzschild black holes and Reissner-Nordström black holes. This method was used to calculate the emission rate of particles from other spherically symmetric black holes (Zhang and Zhao 2005a, 2005b; Jiang et al. 2005; Han and Yang 2005; Ren et al. 2005) and also extended to investigate the tunneling radiation from axisymmetric black holes (Zhang and Zhao 2005c, 2005d; Yang 2005; Li 2006a; Zhang and Fan 2007). We also made use of this technique to calculate the emission rate at which a particle tunnel from the black plane (Li 2006b), black string (Li 2006c) and black toroidal (Li 2007). An not purely thermal emission spectrum is obtained in these work and this result is due to particle's self-gravitation and energy conservation, which are also used to calculate the corrections to entropy and Cardy–Verlinde formula (Setare 2005,

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2006, 2007; Setare and Vagenas 2004, 2005). In this paper, we wish to extend the Parikh-Wilczek method to a charged AdS black hole in $f(R)$ gravity and calculate the corrected emission spectrum of particles from its event horizon. This background is chosen for the following reasons. On the one hand, $f(R)$ gravity is one kind of modified gravity theories which successfully mimic the cosmological history especially the cosmic acceleration. It has various applications in both gravitation and cosmology. For nice reviews, see Refs. (De Felice and Tsujikawa 2010; Capozziello and De Laurentis 2011). On the other hand, it is believed that the thermodynamics of $f(R)$ black holes distinguishes from that of black holes in Einstein gravity. So the black hole solutions in $f(R)$ gravity and their thermodynamics attract extensive attention. Recently, we investigated the coexistence line (Mo and Li 2015), Ehrenfest scheme (Mo 2014), the critical phenomena and thermodynamic geometry (Mo and Li 2016) of black holes in $f(R)$ gravity. To the best of our knowledge, the tunneling radiation from a black hole in $f(R)$ gravity has not been covered in the literature yet. And some novel and interesting phenomena may be disclosed.

Charged AdS black hole solution in the $R+f(R)$ gravity with constant curvature scalar $R=R_0$ was obtained in Moon et al. (2011). The metric reads

$$\mathrm{d}s^2 = -N(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{N(r)} + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\big), \tag{1}$$

where

$$N(r) = 1 - \frac{2m}{r} + \frac{q^2}{br^2} - \frac{R_0}{12}r^2, \qquad b = 1 + f'(R_0) \tag{2}$$

and the curvature scalar is identified as $R_0 = -12/l^2 = 4\Lambda$. m and q are parameters related to the black hole ADM mass M and the electric charge Q respectively as follows (Moon et al. 2011)

$$M = mb, \qquad Q = \frac{q}{\sqrt{b}}. \tag{3}$$

When $b=1$, the $f(R)$ black hole becomes the Reissner-Nordström AdS one exactly.

The horizons are determined by

$$N(r) = 1 - \frac{2m}{r} + \frac{q^2}{br^2} - \frac{R_0}{12}r^2 = 0. \tag{4}$$

We use r_{H} to denote the position of the outer event horizon and factorize $N(r)$ into the form as

$$N(r) = (r - r_{\mathrm{H}})\eta(r). \tag{5}$$

According to the Parikh-Wilczek method, we do a transformation $\mathrm{d}T = \mathrm{d}t - g(r)\mathrm{d}r$. When $N(r) + N^2(r)g^2(r) = 1$ is satisfied, we obtain the Painlevé line element

$$\begin{aligned}\mathrm{d}s^2 &= -N(r)\mathrm{d}T^2 + 2\sqrt{1-N(r)}\mathrm{d}T\mathrm{d}r \\ &\quad + \mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\big).\end{aligned} \tag{6}$$

Obviously, the line element (6) displays the stationary, nonstatic, and nonsingular nature of the spacetime. Meanwhile, it is flat Euclidean space in radial to constant-time slices. Moreover, it is easy to show that the metric in this new coordinate system satisfies Landau's condition of coordinate clock synchronization which is given by Landau and Lifshitz (1975)

$$\frac{\partial}{\partial x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\partial}{\partial x^i}\left(-\frac{g_{0j}}{g_{00}}\right), \quad (i,j=1,2,3). \tag{7}$$

That is, the coordinate clock synchronization in the Painlevé coordinates can be transmitted from one place to another though the line element is not diagonal. In quantum mechanics, it is an instantaneous process that particle tunnels across a barrier. This feature is necessary for us to discuss the tunneling process.

The radial outgoing null geodesic is given by

$$\dot{r} = 1 - \sqrt{1 - N(r)}, \tag{8}$$

where the dot denotes differentiation with respect to T.

Equation (8) is the motion equation of a massless particle when it tunnels across the horizon. The world-line of a massive quanta is timelike, so it does not follow radial-lightlike geodesic (8). Similar to Zhang and Zhao (2005a, 2005b), we treat the outgoing massive particle as a de Broglie wave and can easily obtain its motion equation

$$\dot{r} = -\frac{g_{00}}{g_{01}} = \frac{N(r)}{2\sqrt{1-N(r)}}. \tag{9}$$

The total energy of a stationary space-time should be conservational during the emission. When particle's self-gravitation is taken into account, Eqs. (1–9) should be modified. If we fix the total energy of the space-time and allow the black hole mass to fluctuate, when a particle of energy ωb is emitted, the black hole ADM mass will become $b(m-\omega)$ and Eqs. (1–9) should be used with $m \to m - \omega$. Since the metric is of spherical symmetry, so regarding the outgoing particle as an s-wave, i.e. a shell of energy is reasonable. Assuming that the outgoing wave is traced back toward the horizon, its wave-length, as measured by local fiducial observers, will be blue-shifted. Near the horizon, the radial wave number approaches infinity, so that the Wentzel-Kramers-Brillouin (WKB) approximation is appropriate (Parikh and Wilczek 2000).

The s-wave function of the outgoing positive energy particle can be expressed as $\varphi(r) = e^{iZ(r)}$, where Z is the action. Upon WKB approximation, the relationship between the tunneling probability Γ and the imaginary part of Z is described by (Keski-Vakkuri and Kraus 1997)

$$\Gamma \sim \exp(-2\,\mathrm{Im}\,Z). \tag{10}$$

The imaginary part of the action for a particle crossing the horizon outwards from the initial radius $r_{\rm i}$ to the final radius $r_{\rm f}$ can be expressed as

$$\operatorname{Im} Z = \operatorname{Im}\int_{T_{\rm i}}^{T_{\rm f}} L\,\mathrm{d}T = \operatorname{Im}\int_{r_{\rm i}}^{r_{\rm f}} p_r\,\mathrm{d}r = \operatorname{Im}\int_{r_{\rm i}}^{r_{\rm f}}\int_0^{p_r}\mathrm{d}p_r\,\mathrm{d}r, \tag{11}$$

where p_r is canonical momentum conjugate to r. Taking the Hamilton equation into account, then we can obtain

$$\dot{r} = \left.\frac{\mathrm{d}H}{\mathrm{d}p_r}\right|_r, \tag{12}$$

where $(\mathrm{d}H)_r = \mathrm{d}M$. Changing the variable from the momentum to the energy and switching the order of integration, we have

$$\operatorname{Im} Z = \operatorname{Im}\int_{M_{\rm i}}^{M_{\rm f}}\int_{r_{\rm i}}^{r_{\rm f}}\frac{\mathrm{d}r}{\dot{r}}\mathrm{d}M$$
$$= \begin{cases} \operatorname{Im}\int_{M_{\rm i}}^{M_{\rm f}}\int_{r_{\rm i}}^{r_{\rm f}}\frac{1+\sqrt{1-N(r)}}{N(r)}\mathrm{d}r\,\mathrm{d}M \\ \quad\text{(massless particle)}, \\ \operatorname{Im}\int_{M_{\rm i}}^{M_{\rm f}}\int_{r_{\rm i}}^{r_{\rm f}}\frac{2\sqrt{1-N(r)}}{N(r)}\mathrm{d}r\,\mathrm{d}M \\ \quad\text{(massive particle)}, \end{cases} \tag{13}$$

where $M_{\rm i} = M = bm$; $M_{\rm f} = b(m-\omega)$.

It is easy to find that the integrand is singular at the point $r = r_{\rm H}$. The integral can be evaluated by deforming the contour around the pole, so as to ensure that positive energy solution decay in time. Note that all real parts, divergent or not, can be discarded since they only contribute a phase. Doing the r integral, we obtain

$$\operatorname{Im} Z = -2\pi\int_{M_{\rm i}}^{M_{\rm f}}\frac{1}{\eta(r_{\rm H})}\mathrm{d}M. \tag{14}$$

From Eqs. (3) and (4), we obtain

$$M = \frac{br_{\rm H}}{2} + \frac{q^2}{2r_{\rm H}} - \frac{bR_0}{24}r_{\rm H}^3. \tag{15}$$

So

$$\mathrm{d}M = \frac{1}{2}\left(b - \frac{q^2}{r_{\rm H}^2} - \frac{bR_0}{4}r_{\rm H}^2\right)\mathrm{d}r_{\rm H}. \tag{16}$$

On the other hand,

$$\eta(r_{\rm H}) = N'(r_{\rm H}) = \frac{1}{r_{\rm H}}\left(1 - \frac{q^2}{br_{\rm H}^2} - \frac{R_0}{4}r_{\rm H}^2\right). \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (14), we have

$$\operatorname{Im} Z = -\pi\int_{r_{\rm i}}^{r_{\rm f}} br_{\rm H}\,\mathrm{d}r_{\rm H} = \frac{b\pi(r_{\rm i}^2 - r_{\rm f}^2)}{2}, \tag{18}$$

upon which Eq. (10) reads

$$\begin{aligned}\Gamma \sim \exp\left[b\pi\left(r_{\rm f}^2 - r_{\rm i}^2\right)\right] &= \exp b\left(\pi r_{\rm f}^2 - \pi r_{\rm i}^2\right) \\ &= \exp\left(b\pi r_{\rm f}^2 - b\pi r_{\rm i}^2\right).\end{aligned} \tag{19}$$

The emission spectrum obviously deviates from the pure thermal spectrum but consists with an underlying unitary theory. It should be noted that the emission spectrums of massless particles and massive ones have the same functional forms.

According to the area-law governing the Bekenstein-Hawking (BH) entropy, namely,

$$S_{\rm BH} = \frac{A_{\rm H}}{4} = \pi r_{\rm H}^2, \tag{20}$$

where $A_{\rm H} = \iint\{\sqrt{g_{22}g_{33}}\}_{r_{\rm H}}\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi r_{\rm H}^2$ is the outer event horizon area, the tunneling rate (19) can be written as

$$\Gamma \sim e^{b\Delta S_{\rm BH}}. \tag{21}$$

where $\Delta S_{\rm BH} = S_{\rm BH}(M - b\omega) - S_{\rm BH}(M)$ is the difference of the BH entropy of the black hole before and after the emission.

Compared with the conventional tunneling rate which is shown in all of the early reference about tunneling radiation, an apparent difference in Eq. (21) is that there exists an efficient b. From Eq. (2), we think the presence of b reflects the effect of $f(R)$ gravity on black hole radiation.

Conversely, we may give up the area-law and admit the conventional tunneling rate, namely, $\Gamma \sim e^{\Delta S^f_{\rm BH}}$, then the BH entropy of the black hole in $f(R)$ gravity reads

$$S^f_{\rm BH} = \frac{bA_{\rm H}}{4} = b\pi r_{\rm H}^2. \tag{22}$$

The result is in accord with that derived from the Wald method (Moon et al. 2011) and the Euclidean action approach (De Felice and Tsujikawa 2010). Meanwhile, we find from Eqs. (16–17) and (22) that the first law of thermodynamics is satisfied for both cases

$$dM = T_{\rm H}dS^f_{\rm BH}, \qquad dm = T_{\rm H}dS_{\rm BH}, \tag{23}$$

where $T_{\rm H} = \eta(r_{\rm H})/4\pi$.

For further discussion, we expand $r_{\rm f}^2$ in ω and only consider the linear term and the quadratic one, the tunneling rate (19) then reduces to

$$\Gamma \sim \exp\left(b\pi r_{\rm f}^2 - b\pi r_{\rm i}^2\right) \approx \exp\left(-\beta b\omega + a_2 b^2\omega^2\right), \tag{24}$$

with $\beta = 1/T_{\rm H}$ is the inverse of the Hawking temperature and

$$a_2 = \frac{\beta(4br_{\rm H}^2 - 12q^2 + bR_0 r_{\rm H}^4)}{4b^2 r_{\rm H}^5\eta^2(r_{\rm H})}. \tag{25}$$

Since the energy of the tunneling particle is given by $e = b\omega$, Eq. (24) can be written as

$$\Gamma \sim \exp\left(b\pi r_{\rm f}^2 - b\pi r_{\rm i}^2\right) \approx \exp\left(-\beta e + a_2 e^2\right). \tag{26}$$

Obviously, the first term in Eq. (26) gives the familiar thermal Boltzmann factor for the emanating radiation, while the second one can be interpreted as correction from the response of the background geometry to the emission of a quantum, which is caused by energy conservation and indicatived of a "greybody" factor in the emission spectrum; that is, a deviation from pure thermality (Zhang and Zhao 2005d). The result that the black hole radiation is not purely thermal radiation means information is preserved in black hole formation and evaporation, as argued by Hawking (2005).

Especially, when $b = 1$ and $q = \Lambda = 0$, the $f(R)$ gravity black hole reduces to the Schwarzschild black hole. We can obtain $\beta = 8\pi M, a_2 = 4\pi$ from Eqs. (17) and (25). Then the tunneling rate (24) becomes

$$\Gamma \sim \exp\left[-8\pi\omega\left(M - \frac{\omega}{2}\right)\right]. \tag{27}$$

Obviously, the result is in accord with Parikh (2004b).

In summary, we have extended the Parikh-Wilczek method to study the tunneling radiation of particles across the event horizon of a black hole in $f(R)$ gravity and to obtain the emission spectrum which deviates from the pure thermal spectrum but consists with an underlying unitary theory. Some interesting results are demonstrated. The expression of the emission rate in $f(R)$ gravity differs from that in Einstein gravity if the black hole entropy has to obey area-law. Conversely, if the conventional tunneling rate is accepted and adopted, the area formula of black hole entropy breaks down and the entropy expression and two different forms of the thermodynamics first law for the $f(R)$ gravity black hole are obtained, which are in accord with the early results. The research of tunneling radiation in this paper is important for further study about correlative subjects such as investigating the entropy corrections, understanding the thermodynamics of black holes in modified gravity and the second law of thermodynamics, and studying the conformal (or trace) anomaly in curved spacetime.

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