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ORIGINAL ARTICLE

Dynamical instability of cylindrical symmetric collapsing star in generalized teleparallel gravity

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Abstract This paper is devoted to an analysis of the dynamical instability of a self-gravitating object that undergoes a collapse process. We take the framework of generalized teleparallel gravity with a cylindrically symmetric gravitating object. The matter distribution is represented by a locally anisotropic energy-momentum tensor. We develop basic equations such as the dynamical equations along with the matching conditions and the Harrison–Wheeler equation of state. By applying a linear perturbation strategy, we construct a collapse equation, which is used to obtain the instability ranges in the Newtonian and post-Newtonian regimes. We find these ranges for isotropic pressure and reduce to the results in general relativity. The unstable behavior depends on matter-, metric-, mass-, and torsion-based terms.

Keywords $f(T)$ gravity · Instability · Cylindrical symmetry · Newtonian and post-Newtonian regimes

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1 Introduction

The generalized teleparallel theory of gravity ($f(T)$ where T is the torsion scalar) is one of the modified theories which occupies a vast area of research in modern cosmology and astrophysics. General relativity is identified as the cornerstone of modern cosmology. Einstein introduced the cosmological constant, a simplest of models, which is backed by cosmological observations on the basis of general relativity (GR). This theory is based on curvature via the Levi–Civita connection. This constant happened to inherit some flaws later like fine-tuning and cosmic coincidence problems. Later on, Einstein proposed a theory which is equivalent to GR established through a parallel transported vierbein field, named teleparallel gravity. In this theory, the torsion scalar determines the gravitational field taking into account the Weitzenböck connection. The modified form of GR leads to $f(R)$ gravity, while $f(T)$ is the modification of teleparallel gravity by modifying the curvature and torsion scalars up to higher order terms.

First of all, the $f(T)$ theory of gravity is proposed under the Born–Infeld strategy which helped to solve the particle horizon problem and one found a singularity-free solution (Ferraro and Fiorini 2008). Later, many phenomena included the expanding universe with acceleration through different cosmological parameters, planes, perturbations, cosmographic techniques, energy conditions, spherically symmetric solutions via solar system constraints, Birkhoff's theorem, connection between different regions of the universe through static and wormhole solutions, viability of thermodynamical laws, unstable behavior of spherically symmetric collapsing stars applying different dynamical conditions, reconstruction scenario via dynamical dark energy models, etc. (Jamil et al. 2012a, 2012b, 2013; Houndjo et al. 2012; Wang 2011a, 2011b; Daouda et al. 2012a, 2012b; Gonzlez et al. 2012; Bamba et al. 2014; Sharif and Rani 2013, 2014a, 2014b; Jawad and Rani 2015a, 2015b). The $f(T)$ theory of gravity is a non-local Lorentz invariant theory. To resolve this problem, a lot of work has been done in this direction Li et al. 2011a, 2011b. Nashed (2015) proposed a general tetrad field by regularization of the $f(T)$ field equations which has two tetrad matrices. This regularized process with a general tetrad field removed the effect of the local Lorentz invariance.

The dynamics of a self-gravitating object during the collapse process has been analyzed in GR (Skripkin 1960; Chandrasekhar 1964; Herrera et al. 1989; Chan et al. 1993; Herrera and Santos 1995; Herrera et al. 2012) and modified theories of gravity such as $f(R)$, $f(R,T)$, Brans–Dicke, Einstein, and Gauss–Bonnet gravity in spherical as well as cylindrical symmetry (Sharif and Manzoor 2014, 2016; Kausar et al. 2015; Kausar 2013), while in $f(T)$ gravity one used spherical symmetry (Sharif and Rani 2014c, 2015; Jawad and Rani 2015a, 2015b). A disturbance in the hydrostatic equilibrium of a stellar object leads to the collapse process. The instability occurs when the weight of the outer region surrounding the object overcomes very quickly the pressure inside the object and the gravitational force consequently pushes the matter toward the center of the object, initiating collapse. In order to study the unstable behavior of self-gravitating collapsing objects, the adiabatic index which is a stiffness parameter is used. For a spherically symmetric matter configuration of the collapsing star, this index gives the numerical range of the instability as less than $4/3$ in GR. However, in modified theories of gravity, there appear effective terms resulting from a modification of the theory. Nowadays, a lot of work is done on the Jeans instability too (Roshan and Abbass 2014, 2015).

In $f(R)$ gravity, Sharif and Kausar (2011) and Kausar and Noureen (2014) analyzed the dynamical instability ranges for the Newtonian and post-Newtonian regimes of a spherically symmetric collapsing star with and without charge. These ranges depend on matter-, metric-, mass-, and curvature-based terms. For cylindrical symmetry, Kausar (2013) studied the effects of the CDTT model which has an inverse curvature term on the unstable behavior. The asymptomatic behavior is also obtained for both regimes. Sharif and Manzoor (2014) examined the instability of a cylindrically symmetric collapsing object in the frame work of Brans–Dicke gravity. They concluded that the adiabatic index remains less than 1 for the unstable behavior, while it remained greater than 1 in a special case. In $f(T)$ gravity, Sharif and Rani (2014c, 2015) analyzed the dynamics of a self-gravitating object with spherical symmetry via expansion and expansion-free cases. Recently, Jawad and Rani (2015a, 2015b) examined the instability ranges taking into account the shear-free condition for the Newtonian and post-Newtonian regimes.

The collapse process happens when the stability of matter is disturbed and at long last experiences collapse which leads to different structures. We have taken the self-gravitating object as a cylindrically symmetric collapsing star. The dynamical instability analysis is mostly done for spherically symmetric objects, which includes galactic halos, globular clusters, etc. However, the non-spherical objects were such that cylindrical symmetry and plates came into being by post-shock clouds on the verge of gravitational collapse at stellar scales as well as galaxy formations. The cylindrical symmetry is associated with the problem of fragmentation of pre-stellar clouds. Specifically, the final fate of the collapse of a non-spherical cloud coming from numerical relativity and certain analytical solutions in cylindrical symmetry provide some new examples of gravitational collapse.

In this connection, we extend our work on the dynamical instability to the cylindrically symmetric collapsing stars in $f(T)$ gravity. We analyze the dynamical instability ranges in the Newtonian and post-Newtonian regimes. The paper is organized as follows: in the next section, we provide the basics of generalized teleparallel gravity. Section 3 is devoted to the construction of some basic equations such as the field, dynamical, and matching equations. In Sect. 4, we develop a collapse equation for a cylindrically symmetric collapsing star. The instability ranges for anisotropic as well as isotropic fluids for the Newtonian and post-Newtonian regimes are examined in Sect. 5. The last section provides the results of the paper.

2 Generalized teleparallel gravity

In this section, we provide the basics and formulation of generalized teleparallel gravity. This gravity is defined through the action (Ferraro and Fiorini 2007, 2008; Linder 2010; Bamba et al. 2010; Wu and Yu 2010a, 2010b, 2011; Bamba et al. 2011; de la Cruz-Dombriz et al. 2014)

$$\mathcal{I}=\frac{1}{\kappa^2}\int d^4x\big(\mathcal{L}_m+f(T)\big)h, \tag{1}$$

where $f(T)$ is an arbitrary differentiable function, T denotes the torsion scalar, $\mathcal{L}_m$ represents the matter density, and κ^2 denotes the coupling constant. The term $h=\det(h^a{}_\beta)$ is the determinant of the vierbein (or tetrad) field, $h^a{}_\beta$. This field has a basic and central part in the construction of this torsion-based gravity. This is an orthonormal set of vector fields related with the metric tensor by the relation $g_{\beta\alpha}=\eta_{ab}h^a{}_\beta h^b{}_\alpha$; $\eta_{ab}=\text{diag}(1,-1,-1,-1)$ is the Minkowski space. It is noted here that the indices $(a,b,\dots)$ are the coordinates of the tangent space, while $(\alpha,\beta,\dots)$ expresses the coordinate indices on the manifold and all these indices run from 0, 1, 2, 3. The parallel transport of vierbein field by the significant component Weitzenböck connection in Weitzenböck spacetime leads to the basic construction of teleparallel as well as generalized teleparallel gravity where $\widetilde{\Gamma}^{\beta}_{\alpha\gamma} = h_a{}^{\beta}\partial_\gamma h^a{}_\alpha$ is the Weitzenböck connection. We obtain the torsion tensor by the antisymmetric part of this connection as follows:

$$T^{\beta}{}_{\alpha\gamma} = \widetilde{\Gamma}^{\beta}_{\gamma\alpha} - \widetilde{\Gamma}^{\beta}_{\alpha\gamma} = h_a{}^{\beta}\left(\partial_\gamma h^a{}_\alpha - \partial_\alpha h^a{}_\gamma\right), \quad (2)$$

which is antisymmetric in its lower indices, i.e., $T^{\beta}{}_{\alpha\gamma} = -T^{\beta}{}_{\gamma\alpha}$. It is noted that on parallel transport of the tetrad field the curvature of the Weitzenböck connection rapidly vanishes.

The torsion scalar takes the form

$$T = S_\beta{}^{\alpha\gamma} T^{\beta}{}_{\alpha\gamma}, \quad (3)$$

where

$$S_\beta{}^{\alpha\gamma} = \frac{1}{2}\left[\delta^\alpha_\beta T^{\mu\gamma}{}_\mu - \delta^\gamma_\beta T^{\mu\alpha}{}_\mu - \frac{1}{2}\left(T^{\alpha\gamma}_\beta - T^{\gamma\alpha}_\beta - T^{\alpha\gamma}_\beta\right)\right]. \quad (4)$$

Varying of the action of $f(T)$ gravity w.r.t. the vierbein field, we obtain the field equations:

$$h_a{}^\beta S_\beta{}^{\alpha\gamma}\partial_\alpha T f_{TT} + \left[\frac{1}{h}\partial_\alpha\left(h h_a{}^\beta S_\beta{}^{\alpha\gamma}\right) + h_a{}^\beta T^\lambda{}_{\alpha\beta} S_\lambda{}^{\gamma\alpha}\right] f_T + \frac{1}{4} h_a{}^\gamma f = \frac{1}{2}\kappa^2 h_a{}^\beta \Theta^\gamma_\beta, \quad (5)$$

where the subscripts T and TT represent first and second order derivatives of f with respect to T. In terms of the covariant derivative instead of the partial derivatives, the $f(T)$ field equations can be reconstructed. In the covariant formalism, we obtain an important condition, $R = -T - 2\nabla^\beta T^\gamma{}_{\beta\gamma}$. This implies that the covariant derivative of the torsion tensor shows the only difference between the Ricci and torsion scalars.

By implying the strategy of the covariant derivative (widely discussed in (Sharif and Rani 2013, 2014a, 2014b; Jawad and Rani 2015a, 2015b), we get the field equations in $f(T)$ gravity:

$$f_T G_{\alpha\gamma} + \frac{1}{2} g_{\alpha\gamma}(f - T f_T) + \mathcal{D}_{\alpha\gamma} f_{TT} = \kappa^2 \Theta_{\alpha\gamma}, \quad (6)$$

where $\mathcal{D}_{\alpha\gamma} = S_{\gamma\alpha}{}^\beta \nabla_\beta T$. The trace of the above equation is

$$\mathcal{D} f_{TT} - (R + 2T) f_T + 2f = \kappa^2 \Theta, \quad (7)$$

with $\mathcal{D} = \mathcal{D}^\gamma{}_\gamma$ and $\Theta = \Theta^\gamma{}_\gamma$. Equation (6) can be rewritten as

$$G_{\alpha\gamma} = \frac{\kappa^2}{f_T}\left(\Theta^m_{\alpha\gamma} + \Theta^T_{\alpha\gamma}\right). \quad (8)$$

The $\Theta^m_{\alpha\gamma}$ constitutes the matter contribution, while the torsion contribution is represented by

$$\Theta^T_{\alpha\gamma} = \frac{1}{\kappa^2}\left[-\mathcal{D}_{\alpha\gamma} f_{TT} - \frac{1}{4} g_{\alpha\gamma}\left(\Theta - \mathcal{D} f_{TT} + R f_T\right)\right]. \quad (9)$$

It can be observed that (6) has an equivalent structure to $f(R)$ gravity and reduces to GR for $f = T$.

We consider the collapsing star as a cylindrically symmetric surface which is characterized by a hypersurface Σ. This timelike $3D$ hypersurface isolates for the manifold the $4D$ interior and exterior portions. The interior region is taken as a cylindrically symmetric collapsing star having a line element as follows:

$$ds^{2(-)} = X^2(t,r)dt^2 - Y^2(t,r)dr^2 - Z^2(t,r)d\phi^2 - dz^2, \quad (10)$$

where the coordinates t, r, ϕ, and z are constrained,

$$-\infty \le t \le \infty, \qquad 0 \le r < \infty,$$
$$0 \le \phi \le 2\pi, \qquad -\infty < z < \infty,$$

in order to preserve cylindrical symmetry. The line element for the exterior portion in terms of the retarded time coordinate τ and the gravitational mass M is given by Chao-Guang (1995)

$$ds^{2(+)} = \left(-\frac{2M}{r}\right)d\tau^2 + 2dr\,d\tau - r^2\left(d\phi^2 + \gamma^2 dz^2\right), \quad (11)$$

where γ is an arbitrary constant and $M = M(\tau)$ represents the total mass inside the cylindrical surface. Also, we take the interior region to be distributed by locally anisotropic matter for which the energy-momentum description is

$$\Theta^m_{\alpha\beta} = (\rho + P_r)U_\alpha U_\beta - P_r g_{\alpha\beta} + (P_z - P_r)V_\alpha V_\beta + (P_\phi - P_r)L_\alpha L_\beta, \quad (12)$$

where $\rho = \rho(t,r)$ denotes the energy density, and $P_r = P_r(t,r)$, $P_z = P_z(t,r)$, $P_\phi = P_\phi(t,r)$ represent the corresponding principal pressure components. The four-velocity $U_\alpha = X\delta^0_\alpha$ and the four-vectors $L_\alpha = Z\delta^2_\alpha$ and $V_\alpha = \delta^3_\alpha$ satisfy the relations

$$U_\alpha U^\alpha = 1, \qquad L_\alpha L^\alpha = 1 = V_\alpha V^\alpha,$$
$$U^\alpha L_\alpha = L^\alpha V_\alpha = V^\alpha U_\alpha = 0.$$

3 Basic equations

The choice of the tetrad field in $f(T)$ gravity is the key to set up the framework of $f(T)$ gravity. Bad choices of the tetrad are those which constrain the torsion scalar to be constant or vanishing of the modification of theory. These tetrads have the form of a diagonal representation except for the cartesian symmetry. For spherical and cylindrical symmetry, good tetrads are the non-diagonal tetrads having no restriction for the torsion scalar and keeping the modification of the theory. We consider the following tetrad in non-diagonal form for the interior spacetime:

$$h^a{}_\alpha=\begin{pmatrix} X & 0 & 0 & 0\\ 0 & Y\cos\phi & -Z\sin\phi & 0\\ 0 & Y\sin\phi & Z\cos\phi & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

with its inverse

$$h_a{}^\alpha=\begin{pmatrix} \frac{1}{X} & 0 & 0 & 0\\ 0 & \frac{\cos\phi}{Y} & \frac{\sin\phi}{Y} & 0\\ 0 & -\frac{\sin\phi}{Z} & \frac{\cos\phi}{Z} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

The torsion scalar takes the form

$$T=-2\left[\frac{X'}{XY^2}\left(\frac{Y}{Z}-\frac{Z'}{Z}\right)+\frac{\dot{Y}\dot{Z}}{X^2YZ}\right],$$

where a dot and a prime denote derivatives with respect to time and radial coordinate. The corresponding field equations are

$$\frac{X^2}{Y^2}\left(\frac{Y'Z'}{YZ}-\frac{Z''}{Z}\right)+\frac{\dot{Y}\dot{Z}}{YZ}=\frac{X^2\kappa^2}{f_T}\left[\rho+\frac{1}{\kappa^2}\left\{\frac{Tf_T-f}{2}+\frac{1}{2Y^2}\left(\frac{Y}{Z}-\frac{Z'}{Z}\right)f'_T\right\}\right], \tag{13}$$

$$\frac{\dot{Z}'}{Z}-\frac{X'\dot{Z}}{XZ}-\frac{\dot{Y}Z'}{YZ}=\frac{\dot{Z}}{2Z}\frac{f'_T}{f_T}, \tag{14}$$

$$\frac{\dot{Z}'}{Z}-\frac{X'\dot{Z}}{XZ}-\frac{\dot{Y}Z'}{YZ}=\frac{1}{2}\left(\frac{Z'}{Z}-\frac{Y}{Z}\right)\frac{\dot{T}}{T'}\frac{f'_T}{f_T}, \tag{15}$$

$$\frac{Y^2}{X^2}\left(\frac{\dot{X}\dot{Z}}{XZ}-\frac{\ddot{Z}}{Z}\right)+\frac{X'Z'}{XZ}=\frac{Y^2\kappa^2}{f_T}\left[P_r+\frac{1}{\kappa^2}\left\{\frac{f-Tf_T}{2}-\frac{\dot{Z}\dot{T}}{2X^2ZT'}f'_T\right\}\right], \tag{16}$$

$$\frac{Z^2}{XY}\left[\frac{X''}{Y}-\frac{\ddot{Y}}{X}+\frac{\dot{X}\dot{Y}}{X^2}-\frac{X'Y'}{Y^2}\right]=\frac{Z^2\kappa^2}{f_T}\left[P_\phi+\frac{1}{\kappa^2}\left\{\frac{f-Tf_T}{2}-\left(\frac{\dot{Y}\dot{T}}{2X^2YT'}-\frac{X'}{2XY^2}\right)f'_T\right\}\right], \tag{17}$$

$$\frac{1}{Y^2}\left[\frac{X''}{X}-\frac{X'Y'}{XY}+\frac{X'Z'}{XZ}+\frac{Z''}{Z}-\frac{Y'Z'}{YZ}\right]+\frac{1}{X^2}\left[-\frac{\ddot{Y}}{Y}+\frac{\dot{X}\dot{Y}}{XY}-\frac{\ddot{Z}}{Z}+\frac{\dot{X}\dot{Z}}{XZ}-\frac{\dot{Y}\dot{Z}}{YZ}\right]$$
$$=\frac{\kappa^2}{f_T}\left[P_z+\frac{1}{\kappa^2}\left\{\frac{f-Tf_T}{2}-\left(\frac{\dot{T}}{2X^2T'}\left(\frac{\dot{Y}}{Y}+\frac{\dot{Z}}{Z}\right)-\frac{1}{2Y^2}\left(\frac{X'}{X}-\frac{Y}{Z}+\frac{Z'}{Z}\right)\right)f'_T\right\}\right]. \tag{18}$$

We find a relationship using Eqs. (14) and (15) as follows:

$$\frac{\dot{Z}}{Z}=\frac{\dot{T}}{T'}\left(\frac{Z'}{Z}-\frac{Y}{Z}\right). \tag{19}$$

In order to match the interior and exterior regions of the cylindrically symmetric collapsing star, we use junction conditions defined by Darmois. For this purpose, we consider the C-energy, i.e., the mass function representing matter inside the cylinder is given by Throne (1965)

$$m(t,r)=\frac{l}{8}\left(1-\frac{1}{l^2}\nabla^\alpha\hat{r}\nabla_\alpha\hat{r}\right)=E(t,r), \tag{20}$$

where E is the gravitational energy per unit specific length l of the cylinder. The areal radius $\hat{r}$ is defined as $\hat{r}=\mu l$ where the circumference radius has the relation $\mu^2=\xi_{(1)i}\xi^i_{(1)}$ while $l^2=\xi_{(2)i}\xi^i_{(2)}$. The terms $\xi_{(1)}=\frac{\partial}{\partial\phi}$ and $\xi_{(2)}=\frac{\partial}{\partial z}$ are the Killing vectors corresponding to cylindrical systems. For Eq. (10), the C-energy turns out to be

$$m(t,r)=\frac{l}{8}\left[1+\left(\frac{\dot{Z}}{X}\right)^2-\left(\frac{Z'}{Y}\right)^2\right]. \tag{21}$$

The continuity of the Darmois conditions (junction conditions) establishes the following constraints:

$$\frac{l}{8}\overset{\Sigma^{(e)}}{=}m(t,r)-M,\qquad l\overset{\Sigma^{(e)}}{=}4C,\qquad p_r\overset{\Sigma^{(e)}}{=}\frac{\overset{T}{\Theta}_{11}}{Y^2}-\frac{\overset{T}{\Theta}_{01}}{XY}, \tag{22}$$

where $\Sigma^{(e)}$ indicates the outer region for measurements with $r=r_{\Sigma^{(e)}}=$ constant. The conservation of the total energy of a system is obtained from the Bianchi identities through the dynamical equations. The dynamical equations in the framework of $f(T)$ gravity are

$$\left(\overset{m}{\Theta^{\alpha\beta}}+\overset{T}{\Theta^{\alpha\beta}}\right)_{;\beta}U_\alpha=0,\qquad \left(\overset{m}{\Theta^{\alpha\beta}}+\overset{T}{\Theta^{\alpha\beta}}\right)_{;\beta}\chi_\alpha=0. \tag{23}$$

Using these equations, the dynamical equations for the cylindrically symmetric collapsing star become

$$\dot{\rho}+\frac{\dot{Y}}{Y}(\rho+P_r)+\frac{\dot{Z}}{Z}(\rho+P_\phi)+\frac{J_0}{\kappa^2}=0, \tag{24}$$

$$P_r{}'+\frac{X'}{X}(\rho+P_r)+\frac{Z'}{Z}(P_r-P_\phi)+\frac{J_1}{\kappa^2}=0, \tag{25}$$

where

$$J_0 = X^2\left\{\frac{1}{X^2}\left(\frac{Tf_T - f}{2} + \frac{1}{2Y^2}\left(\frac{Y}{Z} - \frac{Z'}{Z}\right)f'_T\right)\right\}_{,0}$$
$$+ X^2\left\{\frac{\dot{Z}}{2X^2Y^2Z}f'_T\right\}_{,1} + \frac{\dot{X}}{X}(Tf_T - f)$$
$$+ \left\{\frac{1}{YZ}\left(\frac{\dot{X}}{X} - \frac{\dot{X}Z'}{XY} + \frac{3X'\dot{Z}}{2XY} + \frac{\dot{Y}}{2Y} + \frac{\dot{Z}}{2Z} - \frac{\dot{Y}Z'}{2Y^2}\right.\right.$$
$$\left.\left. + \frac{Y'\dot{Z}}{2Y^2}\right) + \left(\frac{X'Z'}{2XY^2Z} - \frac{X'Y}{2XY^2Z}\right)\frac{\dot{T}}{T'}\right\}f'_T,$$

$$J_1 = Y^2\left\{\frac{1}{2X^2Y^2}\left(\frac{Z'}{Z} - \frac{Y}{Z}\right)\frac{\dot{T}}{T'}f'_T\right\}_{,0}$$
$$+ Y^2\left\{\frac{1}{Y^2}\left(\frac{f - Tf_T}{2} - \frac{\dot{Z}\dot{T}}{2X^2ZT'}f'_T\right)\right\}_{,1}$$
$$+ \frac{Y'}{Y}(f - Tf_T) + \left\{\left(\frac{3\dot{Y}Z'}{2X^2YZ} - \frac{\dot{Y}}{X^2Z} - \frac{Y'\dot{Z}}{X^2YZ}\right.\right.$$
$$+ \frac{X'}{2XYZ} - \frac{X'Z'}{2XY^2Z} + \frac{\dot{X}Z'}{2X^3Z} - \frac{Y\dot{X}}{2X^3Z} - \frac{Y\dot{Z}}{2X^2Z^2}$$
$$\left.\left. - \frac{X'Z'}{2X^2Z}\right)\frac{\dot{T}}{T'} - \frac{X'Z'}{XY^2Z} + \frac{\dot{Y}\dot{Z}}{2X^2YZ}\right\}f'_T.$$

In order to develop the collapse equation representing the dynamics of the cylindrically symmetric collapsing star in the framework of $f(T)$ gravity, the choice of $f(T)$ model keeps central importance. Here we assume a model in the form of a power-law up to a quadratic torsion scalar term to discuss the instability ranges of the cylindrically symmetric collapsing star. This is given by (Sharif and Rani 2014c, 2015; Jawad and Rani 2015a, 2015b)

$$f(T) = T + \omega T^2, \tag{26}$$

where ω is an arbitrary constant. This particular model is analogous to a viable model from $f(R)$ gravity such as $f(R) = R + \lambda R^2$, which reduces to GR by taking $\lambda \to 0$. In $f(R)$ gravity, this model presents the dynamics of the collapsing star and gives instability ranges for different regimes under many scenarios (Sharif and Kausar 2011; Kausar and Noureen 2014). The power-law $f(T)$ model is very simple, providing a direct comparison with GR by choosing ω as zero. The finite time singularities are also discussed for a power-law model of the type T^m, which results in the vanishing of these singularities for $m > 1$ (Bamba et al. 2012). Also, this model leads to the accelerated expansion of the universe in the phantom phase, the possibility of realistic wormhole solutions, solar system tests, and instability conditions for a collapsing star. In order to discuss the dynamical instability ranges in the underlying scenario, we impose the condition of static equilibrium (dependent on the radial coordinate only) on metric, matter as well as effective parts of the system initially. After some time t, these parts also become time dependent along with the r dependency. We represent this strategy by linear perturbation strategy to construct the dynamical equations in order to explore the instability ranges for the cylindrically symmetric collapsing star. These perturbations are described as follows:

$$X(t,r) = x_0(r) + \varepsilon\Lambda(t)\hat{x}(r), \tag{27}$$
$$Y(t,r) = y_0(r) + \varepsilon\Lambda(t)\hat{y}(r), \tag{28}$$
$$Z(t,r) = z_0(r) + \varepsilon\Lambda(t)\hat{z}(r), \tag{29}$$
$$\rho(t,r) = \rho_0(r) + \varepsilon\hat{\rho}(t,r), \tag{30}$$
$$P_r(t,r) = p_{r0}(r) + \varepsilon\hat{p}_r(t,r), \tag{31}$$
$$P_\phi(t,r) = p_{\phi0}(r) + \varepsilon\hat{p}_\phi(t,r), \tag{32}$$
$$P_z(t,r) = p_{z0}(r) + \varepsilon\hat{p}_z(t,r), \tag{33}$$
$$m(t,r) = m_0(r) + \varepsilon\hat{m}(t,r), \tag{34}$$
$$T(t,r) = T_0(r) + \varepsilon\Lambda(t)e(r). \tag{35}$$

That is, the quantities with zero subscript refer to a zero order perturbation of the corresponding functions, while $0 < \varepsilon \ll 1$. The $f(T)$ model under perturbation of the torsion scalar becomes

$$f(T) = T_0(1 + \omega T_0) + \varepsilon\Lambda e(1 + 2\omega T_0),$$
$$f_T(T) = 1 + 2\omega T_0 + 2\varepsilon\delta\Lambda e. \tag{36}$$

4 Collapse equation

Here, we construct the collapse equation using the $f(T)$ model along with a perturbation scheme for the underlying scenario. The static configuration of the torsion scalar and mass function with $Z_0 = r$ are given as follows:

$$T_0 = -\frac{2x'_0(y_o - 1)}{rx_0y_0}, \qquad m_0 = \frac{l}{8}\left(1 - \frac{1}{y_0^2}\right),$$

while the perturbed configuration is given by

$$e = -\frac{2}{rx_0y_0}\left[x'_0\left(\hat{y} - \hat{z}' + \frac{\hat{z}}{r}(1 - y_0)\right)\right.$$
$$\left. + (y_0 - 1)\left(\hat{x}' - \frac{x'_0}{x_0y_0}(\hat{x}y_0 + \hat{y}x_0)\right)\right],$$
$$\hat{m} = -\frac{\Lambda l}{4y_0^2}\left(\hat{z}' - \frac{\hat{y}}{y_0}\right),$$

respectively. The dynamical equations play their part in order to construct the collapse equation for the dynamical instability ranges of the cylindrical collapsing star. For this purpose, the non-static configuration of the second dynamical equation (25) takes the form

$$\hat{p}_r' + \frac{x_0'}{x_0}(\hat{\rho} + \hat{p}_r) + \left(\frac{\hat{x}'}{x_0} - \frac{\hat{x}x_0'}{x_0}\right)(\rho_0 + p_{r0})\Lambda + \frac{1}{r}(\hat{p}_r - \hat{p}_\phi) + \frac{\hat{z}'}{r}(p_{r0} - p_{\phi 0})\Lambda + \frac{J_{1p}}{\kappa^2} = 0, \tag{37}$$

where

$$J_{1p} = \frac{e\omega T_0^{2'}}{r x_0^2}(1 - y_0)\ddot{\Lambda} + 2y_0\hat{y}\left(\frac{\omega T_0^2}{2y_0^2}\right)_{,1}\Lambda + y_0^2\left\{\frac{1}{y_0^2}\left(\frac{\hat{y}\omega T_0^2}{y_0} - \omega e T_0\right)\right\}_{,1}\Lambda - \frac{2\omega e T_0 y_0'}{y_0}\Lambda - \frac{1}{y_0}\left(\hat{y}' - \frac{y_0'\hat{y}}{y_0}\right)\omega T_0^2\Lambda + \frac{e\omega T_0^{2'}}{r x_0}\left(\frac{x_0'}{y_0} - \frac{\hat{x}'}{y_0^2} - \frac{\hat{x}'}{x_0}\right) - \frac{2\hat{x}'\omega e'}{r x_0 y_0^2}\Lambda - 2\omega T_0'\left(\frac{\hat{x}}{x_0} + \frac{\hat{y}}{y_0} + \frac{\hat{z}}{r}\right)\Lambda.$$

Equation (37) is the general collapse equation, which depicts the instability of hydrostatic equilibrium of a cylindrical gravitating fluid in $f(T)$ gravity. To analyze the instability of the fluid using this equation, we need the expressions for $\hat{\rho}$, $\hat{p}_r$, $\hat{p}_\phi$, and Λ through the basic equations of the underlying system.

Applying a perturbation strategy on the first dynamical equation, the perturbed part is given as follows:

$$\dot{\hat{\rho}} + \left[\frac{\hat{y}}{y_0}(\rho_0 + p_{r0}) + \frac{\hat{z}}{r}(\rho_0 + p_{\phi 0}) + \frac{J_{0p}}{\kappa^2}\right]\dot{\Lambda} = 0, \tag{38}$$

where

$$J_{0p} = e\omega T_0 + \frac{\omega e'(y_0 - 1)}{r y_0^2} + \frac{\omega T_0'}{r y_0^2}\left(\hat{y} - \hat{z}' + \frac{\hat{z}}{r}(1 - y_0) - \frac{2\hat{y}(y_0 - 1)}{y_0}\right) - \frac{\hat{x}}{x_0}\left(\omega T_0^2 + \frac{2\omega T_0'(y_0 - 1)}{r y_0^2}\right) + x_0^2\left(\frac{z\omega T_0'}{r x_0^2 y_0^2}\right) + \frac{\hat{x}\omega T_0^2}{x_0} + \frac{2\omega T_0'}{r y_0}\left(\frac{\hat{x}}{x_0} - \frac{\hat{x}}{x_0 y_0} + \frac{\hat{z}x_0'}{2x_0 y_0} + \frac{\hat{y}}{2y_0} + \frac{\hat{z}}{2r} - \frac{\hat{y}}{2y_0^2} + \frac{\hat{z}x_0'}{x_0 y_0} + \frac{\hat{z}y_0'}{2y_0^2} + \frac{e x_0' T_0'(1 - y_0)}{2x_0 y_0}\right).$$

Integrating Eq. (38) with respect to time, we obtain the non-static energy density, which is given by

$$\hat{\rho} = -\left[\frac{\hat{y}}{y_0}(\rho_0 + p_{r0}) + \frac{\hat{z}}{r}(\rho_0 + p_{\phi 0}) + \frac{J_{0p}}{\kappa^2}\right]\Lambda. \tag{39}$$

The Harrison–Wheeler equation of state establishes a relationship between the energy density and pressure (Harrison et al. 1965):

$$\hat{p}_i = \Gamma\frac{p_{i0}}{\rho_0 + p_{i0}}\hat{\rho}, \tag{40}$$

where Γ is called the adiabatic index. We use this index in order to examine the instability ranges in the context of $f(T)$ gravity. The adiabatic index Γ determines the rigidity of the fluid and evaluates the change of pressure to the corresponding density. Substituting the value of $\hat{\rho}$ from Eq. (39) in (40) for $\hat{p}_r$, and $\hat{p}_\phi$, we have

$$\hat{p}_r = -\Lambda\left[\frac{\hat{y}}{y_0}p_{r0} + \frac{c}{r}\frac{\rho_0 + p_{\phi 0}}{\rho_0 + p_{r0}}p_{r0} + \frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0 + p_{r0}}J_{0p}\right]\Gamma, \tag{41}$$

$$\hat{p}_\phi = -\Lambda\left[\frac{\hat{y}}{y_0}\frac{\rho_0 + p_{r0}}{\rho_0 + p_{\phi 0}}p_{\phi 0} + \frac{c}{r}p_{\phi 0} + \frac{1}{\kappa^2}\frac{p_{\phi 0}}{\rho_0 + p_{\phi 0}}J_{0p}\right]\Gamma. \tag{42}$$

To find the value of $\Lambda(t)$, we perturb the field equation (16) and its non-static part is given by

$$\frac{\hat{z}}{r x_0^2}\ddot{\Lambda} + \frac{1}{r x_0 y_0^2}\left(-\frac{2\hat{y}x_0'}{y_0} - \frac{\hat{x}x_0'}{x_0} - \frac{\hat{z}x_0'}{r} + \hat{x}' + \hat{z}'x_0'\right)\Lambda = -\frac{2\omega e\kappa^2\Lambda}{(1 + 2\omega T_0)^2}p_{r0} + \frac{\kappa^2\hat{p}_r}{1 + 2\omega T_0} - \frac{e\omega T_0\Lambda}{1 + 2\omega T_0} - \frac{e\omega^2 T_0^2\Lambda}{(1 + 2\omega T_0)^2}. \tag{43}$$

We use the last junction condition given in Eq. (22) with $r = r_{\Sigma^{(e)}} =$ constant, which under a perturbation strategy yields

$$p_{r0} \overset{\Sigma^{(e)}}{=} \frac{\omega T_0^2}{2\kappa^2}, \qquad \hat{p}_r \overset{\Sigma^{(e)}}{=} \frac{\omega e T_0}{\kappa^2}\Lambda. \tag{44}$$

Using this equation along with $r = r_{\Sigma^{(e)}} =$ constant in Eq. (43), we obtain

$$\ddot{\Lambda} - \sigma_{\Sigma^{(e)}}\Lambda \overset{\Sigma^{(e)}}{=} 0, \quad \text{where } \sigma_{\Sigma^{(e)}} = \frac{2r_{\Sigma^{(e)}}\omega^2 e T_0^2 x_0^2}{\hat{z}(1 + 2\omega T_0)^2}.$$

Its solution is

$$\Lambda(t) = c_1 e^{\sqrt{\sigma_{\Sigma^{(e)}}}\, t} + c_2 e^{-\sqrt{\sigma_{\Sigma^{(e)}}}\, t},$$

representing the stability and instability phases of the cylindrical gravitating fluid through static, and non-static parts and c_1, c_2 are constants. We assume the hydrostatic equilibrium for which Λ is zero at large past time, $t = -\infty$. After this scenario with the evolution of time, the system grows into the present phase, reducing its areal radius and commencing to collapse. As regards the instability analysis of the cylindrical collapsing star, we take only the static solution ($\Lambda(-\infty)=0$), which gives $c_2=0$. Choosing $c_1=-1$, we get

$$\Lambda(t)=-e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t},\quad \text{where } \sigma_{\Sigma^{(e)}}>0. \tag{45}$$

Inserting all the corresponding values in the general collapse equation (37), we obtain

$$\begin{aligned}\Gamma\Bigg[\Bigg\{&\frac{\hat{y}}{y_0}p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi 0}}{\rho_0+p_{r0}}p_{r0}+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}\Bigg\}_{,1}\\&+\frac{x_0'}{x_0}\Bigg\{\frac{\hat{y}}{y_0}p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi 0}}{\rho_0+p_{r0}}p_{r0}+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}\Bigg\}\\&+\frac{1}{r}\Bigg\{\frac{\hat{y}}{y_0}p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi 0}}{\rho_0+p_{r0}}p_{r0}+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}\\&-\left(\frac{\hat{y}}{y_0}\frac{\rho_0+p_{r0}}{\rho_0+p_{\phi 0}}p_{\phi 0}+\frac{c}{r}p_{\phi 0}+\frac{1}{\kappa^2}\frac{p_{\phi 0}}{\rho_0+p_{\phi 0}}J_{0p}\right)\Bigg\}\Bigg]\\&+\frac{x_0'}{x_0}\Bigg\{\frac{\hat{y}}{y_0}(\rho_0+p_{r0})+\frac{\hat{z}}{r}(\rho_0+p_{\phi 0})+\frac{J_{0p}}{\lambda^2}\Bigg\}\\&-\left(\frac{\hat{x}'}{x_0}-\frac{\hat{x}x_0'}{x_0}\right)(\rho_0+p_{r0})-\frac{\hat{z}'}{r}(p_{r0}-p_{\phi 0})\\&+\frac{J_{1p}}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}=0.\end{aligned} \tag{46}$$

This equation represents the collapse equation for cylindrically gravitating object in the framework of $f(T)$ gravity.

5 Dynamical instability analysis

In this section, we analyze the instability ranges of cylindrically symmetric self-gravitating fluid in $f(T)$ gravity using the Newtonian and post-Newtonian conditions in the collapse equation for the corresponding regimes with the help of the adiabatic index.

5.1 Newtonian regime

For the Newtonian regime, we have the constraints as follows:

$$x_0=1=y_0,\qquad \rho_0\geq p_{r0},\qquad \rho_0\geq p_{\phi 0}.$$

These constraints imply that $x_0'=0=y_0'$, $\frac{p_{r0}}{\rho_0+p_{r0}}\to 0$ and $\frac{p_{\phi 0}}{\rho_0+p_{\phi 0}}\to 0$. We use these expressions in Eq. (46) and consequently the collapse equation reduces to

$$\begin{aligned}\Gamma\left[\left(\hat{y}p_{r0}+\frac{z}{r}p_{r0}\right)_{,1}+\frac{1}{r}\left(\hat{y}+\frac{\hat{z}}{r}\right)(p_{r0}-p_{\phi 0})\right]\\-\hat{x}'\rho_0-\frac{\hat{z}'}{r}(p_{r0}-p_{\phi 0})+\frac{J_{1p}^N}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}=0,\end{aligned} \tag{47}$$

which represents the cylindrically symmetric self-gravitating fluid in hydrostatic equilibrium. Here J_{1p}^N expresses the Newtonian approximation terms in J_{1p}, i.e., those terms remaining after applying the above constraints. The system turns to collapse or instability if

$$\Gamma<\frac{\hat{x}'\rho_0+\frac{\hat{z}'}{r}(p_{r0}-p_{\phi 0})-\frac{J_{1p}^N}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}}{(\hat{y}p_{r0}+\frac{z}{r}p_{r0})_{,1}+\frac{1}{r}(\hat{y}+\frac{\hat{z}}{r})(p_{r0}-p_{\phi 0})}=\frac{A_N}{B_N}. \tag{48}$$

It is noted that we have assigned to the numerator and the denominator A_n and B_n for the sake for simplicity. We assume the adiabatic index to be positive throughout the scenario to maintain a difference between the gradients of the principal pressure components and the gravitational forces. Under this condition, the left hand side of the above inequality, depending on dynamical properties such as density, pressure components, and torsion terms, remains positive. The system remains unstable as long as this inequality holds. Thus it leads to the following possibilities:

- I: $\hat{x}'\rho_0+\frac{\hat{z}'}{r}(p_{r0}-p_{\phi 0})-\frac{J_{1p}^N}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}=(\hat{y}p_{r0}+\frac{z}{r}p_{r0})_{,1}$ $+\frac{1}{r}(\hat{y}+\frac{\hat{z}}{r})(p_{r0}-p_{\phi 0})$ or equivalently $A_N=B_N$.
- II: $\hat{x}'\rho_0+\frac{\hat{z}'}{r}(p_{r0}-p_{\phi 0})-\frac{J_{1p}^N}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}<(\hat{y}p_{r0}+\frac{z}{r}p_{r0})_{,1}$ $+\frac{1}{r}(\hat{y}+\frac{\hat{z}}{r})(p_{r0}-p_{\phi 0})$ or equivalently $A_N<B_N$.
- III: $\hat{x}'\rho_0+\frac{\hat{z}'}{r}(p_{r0}-p_{\phi 0})-\frac{J_{1p}^N}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}>(\hat{y}p_{r0}+$ $\frac{z}{r}p_{r0})_{,1}+\frac{1}{r}(\hat{y}+\frac{\hat{z}}{r})(p_{r0}-p_{\phi 0})$ or equivalently $A_N>B_N$.

In case I, we obtain $\Gamma<\frac{A_N}{B_N}=1$, which implies that $0<\Gamma<1$, the instability range for the cylindrical system in the Newtonian range. For case II, the condition $A_N<B_N$ yields $\Gamma<\frac{A_N}{B_N}<1$, which leads to the instability range as $0<\Gamma<\frac{A_N}{B_N}$ where $\frac{A_N}{B_N}<1$. If $A_N>B_N$ holds, we obtain the instability range by the inequality as $0<\Gamma<\frac{A_N}{B_N}$ where $\frac{A_N}{B_N}>1$. It is noted that we may recover GR in the Newtonian limit for the instability range as $\Gamma<\frac{4}{3}$ in this case. This limit also includes the first two instability ranges.

Isotropic pressure Here we discuss the instability ranges for the case when all pressure components become equal, i.e., we have an isotropic pressure fluid ($p_r=p_\phi=p_z=p$). For the isotropic cylindrical fluid system, Eq. (48) reduces to

$$\Gamma<\frac{\hat{x}'\rho_0-\frac{J_{1p}^N}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}}{(\hat{y}p+\frac{\hat{z}}{r}p)_{,1}}. \tag{49}$$

This inequality indicates the unstable behavior and expresses the correspondingly instability ranges.

Asymptotic behavior Substituting $\omega=0$ in Eq. (48) implies $J_{1p}^{N}=0$, yielding the inequality for the anisotropic pressure fluid:

$$\Gamma<\frac{\hat{x}'\rho_0+\frac{\hat{z}'}{r}(p_{r0}-p_{\phi0})}{(\hat{y}p_{r0}+\frac{z}{r}p_{r0})_{,1}+\frac{1}{r}(\hat{y}+\frac{\hat{z}}{r})(p_{r0}-p_{\phi0})}. \tag{50}$$

The adiabatic index indicates the results for GR as long as the above inequality is satisfied. For a cylindrically symmetric isotropic fluid, we obtain

$$\Gamma<\frac{\hat{x}'\rho_0}{(\hat{y}p+\frac{\hat{z}}{r}p)_{,1}}. \tag{51}$$

In these cases, the instability ranges are retrieved accordingly as obtained for Eq. (48).

5.2 Post-Newtonian regime

Here we study the dynamical instability of a cylindrical self-gravitating object with post-Newtonian limits. These limits are $x_0=1-\frac{m_0}{r}$, $y_0=1+\frac{m_0}{r}$ up to order $O(\frac{m_0}{r})$. Consequently the fluid for hydrostatic equilibrium by inserting x_0 and y_0 in the collapse equation takes the form

$$\begin{aligned}
&\Gamma\Bigg[\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}\\
&\quad+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}^{pN}\Bigg\}_{,1}+\frac{m_0}{r^2}\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p_{r0}\\
&\quad+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}^{pN}\Bigg\}\\
&\quad+\frac{1}{r}\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}\\
&\quad+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}^{pN}-\left(\hat{y}\left(1-\frac{m_0}{r}\right)\frac{\rho_0+p_{r0}}{\rho_0+p_{\phi0}}p_{\phi0}\right.\\
&\quad\left.+\frac{c}{r}p_{\phi0}+\frac{1}{\kappa^2}\frac{p_{\phi0}}{\rho_0+p_{\phi0}}J_{0p}^{pN}\right)\Bigg\}\Bigg]\\
&\quad+\frac{m_0}{r^2}\Bigg\{\hat{y}(\rho_0+p_{r0})+\frac{\hat{z}}{r}(\rho_0+p_{\phi0})+\frac{J_{0p}^{pN}}{\lambda^2}\Bigg\}\\
&\quad-\left(\hat{x}'\left(1+\frac{m_0}{r}\right)-\frac{\hat{x}m_0}{r^2}\right)(\rho_0+p_{r0})\\
&\quad-\frac{\hat{z}'}{r}(p_{r0}-p_{\phi0})+\frac{J_{1p}^{pN}}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}}=0.
\end{aligned} \tag{52}$$

The terms with superscript pN point to the terms with post-Newtonian approximations in the corresponding expressions. For the dynamical instability, we find

$$\Gamma<\frac{A_{pN}}{B_{pN}}, \tag{53}$$

where

$$\begin{aligned}
A_{pN}&=\left(\hat{x}'\left(1+\frac{m_0}{r}\right)-\frac{\hat{x}m_0}{r^2}\right)(\rho_0+p_{r0})\\
&\quad-\frac{m_0}{r^2}\Bigg\{\hat{y}(\rho_0+p_{r0})-\frac{J_{0p}^{pN}}{\kappa^2}+\frac{\hat{z}}{r}(\rho_0+p_{\phi0})\Bigg\}\\
&\quad+\frac{\hat{z}'}{r}(p_{r0}-p_{\phi0})-\frac{J_{1p}^{pN}}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}},\\
B_{pN}&=\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}\\
&\quad+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}^{pN}\Bigg\}_{,1}+\frac{m_0}{r^2}\Bigg\{\hat{y}p_{r0}\\
&\quad+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}^{pN}\Bigg\}\\
&\quad+\frac{1}{r}\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}\\
&\quad+\frac{1}{\kappa^2}\frac{p_{r0}}{\rho_0+p_{r0}}J_{0p}^{pN}-\left(\hat{y}\left(1-\frac{m_0}{r}\right)\frac{\rho_0+p_{r0}}{\rho_0+p_{\phi0}}p_{\phi0}\right.\\
&\quad\left.+\frac{c}{r}p_{\phi0}+\frac{1}{\kappa^2}\frac{p_{\phi0}}{\rho_0+p_{\phi0}}J_{0p}^{pN}\right)\Bigg\}.
\end{aligned}$$

Similar to the case of the Newtonian regime, we can develop three possibilities on A_{pN} and B_{pN}, i.e., $A_{pN}=B_{pN}$, $A_{pN}<B_{pN}$, $A_{pN}>B_{pN}$. These possibilities yield the instability ranges: $0<\Gamma<\frac{A_{pN}}{B_{pN}}$, which contains $0<\Gamma<1$.

Isotropic pressure Taking isotropy of the pressure in Eq. (52), we get the unstable behavior:

$$\Gamma<\frac{A_{ipN}}{B_{ipN}}, \tag{54}$$

where

$$\begin{aligned}
A_{ipN}&=\left(\hat{x}'\left(1+\frac{m_0}{r}\right)-\frac{\hat{x}m_0}{r^2}\right)(\rho_0+p)\\
&\quad-\frac{m_0}{r^2}\Bigg\{\hat{y}(\rho_0+p)-\frac{J_{0p}^{pN}}{\kappa^2}+\frac{\hat{z}}{r}(\rho_0+p)\Bigg\}\\
&\quad-\frac{J_{1p}^{pN}}{\kappa^2}\frac{1}{e^{\sqrt{\sigma_{\Sigma^{(e)}}}\,t}},\\
B_{ipN}&=\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p_{r0}+\frac{c}{r}p+\frac{1}{\kappa^2}\frac{p}{\rho_0+p}J_{0p}^{pN}\Bigg\}_{,1}\\
&\quad+\frac{m_0}{r^2}\Bigg\{\hat{y}p+\frac{c}{r}p+\frac{1}{\kappa^2}\frac{p}{\rho_0+p}J_{0p}^{pN}\Bigg\}\\
&\quad+\frac{1}{r}\Bigg\{\hat{y}\left(1-\frac{m_0}{r}\right)p+\frac{c}{r}p+\frac{1}{\kappa^2}\frac{p}{\rho_0+p}J_{0p}^{pN}
\end{aligned}$$

$$-\left(\hat{y}\left(1-\frac{m_0}{r}\right)p+\frac{c}{r}p+\frac{1}{\kappa^2}\frac{p}{\rho_0+p}J^{pN}_{0p}\right)\Bigg\}.$$

For the stability ranges for the cylindrically symmetric isotropic fluid we have the criteria as for inequality (53).

Asymptotic behavior In order to recover the results in GR, we insert $\omega=0$, which leads to $J^{pN}_{0p}=0=J^{pN}_{1p}$ for isotropic as well as anisotropic fluids.

6 Conclusion

The dynamical instability of a self-gravitating general relativistic object undergoing a gravitational collapse process has become a widely considered phenomenon in GR as well as in modified theories of gravity. This process stays significantly at the center of structure formation and holds the evolutionary development of these objects. We have taken a cylindrically symmetric line element as interior spacetime, while we took the exterior spacetime in retarded time coordinate in the framework of $f(T)$ gravity. The locally anisotropic matter distribution is considered for which we have obtained an important result at the boundary of matching both interior and exterior regions. We have developed $f(T)$ field equations along with some basic equations such as the dynamical equations through Bianchi identities. In order to get insight in a more realistic way, we have assumed a specific power-law $f(T)$ model with linear and quadratic torsion scalar terms. This model is used to discuss many cosmological scenarios as well as general symmetric solutions, such as the stability of a spherical collapsing star and of wormhole solutions.

We kept the system in hydrostatic equilibrium initially and then perturbed with the evolution of time by a linear perturbation strategy. This strategy is applied on all matter, metric, mass, and torsion components. In order to construct the collapse equation, the dynamical equations are used in an appropriate way along with the adiabatic index Γ. With the help of the second law of thermodynamics, this index gives the ratio of the specific heat using the energy density and the pressure components. We have obtained the solution for non-static perturbed quantity using the matching conditions which satisfied the initial state of equilibrium. We have applied the constraints of the Newtonian and post-Newtonian regimes in the collapse equation to find the instability ranges for the cylindrically symmetric collapsing object in the framework of $f(T)$ gravity.

We have found the instability ranges for both regimes represented by the adiabatic index. We have also found these ranges taking into account the isotropic pressure as well as the asymptotic behavior, i.e., GR solutions. We have found the instability range: $0<\Gamma<\frac{A}{B}$ for $\frac{A}{B}=1$, $\frac{A}{B}<1$, $\frac{A}{B}>1$ where A and B represent the corresponding expressions in each case. These expressions depend on matter, metric, and torsion terms. This range also admits the GR condition of unstable behavior through the adiabatic index, which is $\Gamma<\frac{4}{3}$. It is noted that for other forms of $f(T)$ models instead of the power-law form, the quantitative consequences are changed, while the qualitative consequences remain the same. For instance, we consider the exponential model, $f(T)=T-\alpha_1 T(1-e^{\frac{pT_c}{T}})$, for which we may obtain static and perturbed parts using a perturbation scheme adopting some more steps. The collapse equation as well as the instability ranges through the adiabatic index will depend on exponential terms throughout. However, the qualitative consequences remain unchanged due to the fact that all the instability ranges via the adiabatic index depend on matter, metric, and torsion dependent terms.

The dynamical instability in $f(R)$ gravity has been discussed taking the CDTT model for a cylindrically symmetric collapsing star (Kausar 2013). It has been found that the adiabatic index depends on immense perturbed terms of this model along with some positivity constraints for the dynamical unstable behavior. In Brans–Dicke theory of gravity (Sharif and Manzoor 2016), the instability ranges of a collapsing stellar object having cylindrically symmetry have been investigated. We found that the instability ranges through the collapse equation depend on the dynamical variables of the collapsing fluid. The ranges for unstable behavior are obtained: $0<\Gamma<1$, while for a special case, $\Gamma>1$. In $f(T)$ gravity, we have found the dynamical instability ranges for a spherically symmetric collapsing star with and without expansion as well as with shear-free conditions taking an anisotropic fluid (Sharif and Rani 2014c, 2015; Jawad and Rani 2015a, 2015b). However, in the present paper, we have analyzed the dynamical unstable behavior, taking a cylindrically symmetric object, which gives less complexity in the expressions A and B. Also, we have reduced the results in the limit of an isotropic fluid distribution and to GR limit.

Chandrasekhar (1964) was the first who explored dynamical instability ranges of a spherically symmetric isotropic fluid in GR. He established these ranges through the adiabatic index which depends on its numerical value. That is, the weight of the outer layer increases rapidly as compared to the pressure in a star for $\Gamma<\frac{4}{3}$, which leads to the unstable behavior of the star. For $\Gamma>\frac{4}{3}$, the pressure overcomes the weight of the outer layers and this yields the stability of the star. In $f(T)$ gravity, Sharif and Rani (2014c, 2015) analyzed the dynamics of a self-gravitating object with spherical symmetry via expansion and expansion-free cases. Jawad and Rani (2015a, 2015b) examined the instability ranges taking into account the shear-free condition for the

Newtonian and post-Newtonian regimes via the adiabatic index. These works are only discussed for an anisotropic fluid. In order to compare the results of the present paper, we analyze that the results depend on the physical quantities, like energy density, pressure, curvature terms, and mass of the cylinder. However, to make a correspondence with the results of an isotropic sphere, we have established possibilities (after Eq. (48) as I, II, III) on these physical quantities in each case (isotropic as well as anisotropic fluids) to have numerical results like Chandrasekhar (1964).

So far we know that the cosmographic features of $f(T)$ gravity mimic the LCDM model as well as phantom dark energy models. Although there are some ambiguities as regards the validity of solar system tests for $f(T)$ due to the absence of the Schwarzschild solution as the vacuum (Rodrigues et al. 2013), the dynamics of stellar objects and what we studied here as the (in)stability of cylindrical objects provide a good sample to check the validity of $f(T)$ gravity in astrophysics. Briefly the stability conditions were obtained in our paper having a significant physical meaning in comparison to the classical results. Furthermore, the model which we studied here, the torsion-based version of the Starobinsky model, was tested among several types of models with cosmological data, so we believe that our paper will be useful for astrophysical tests of compact objects in $f(T)$ gravity.

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