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ORIGINAL ARTICLE

Hawking radiation as tunneling of vector particles from Kerr-Newman black hole

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Abstract In this paper, by applying the WKB approximation and Hamilton-Jacobi ansatz to the Proca equation, we investigate the tunneling of vector bosons across the event horizon of Kerr-Newman black hole and also the resulting vector particles' Hawking radiation. Universality of the properties of the emitted spectra of different types of particles is established for Kerr-Newman black hole. The coordinate problem for Hawking radiation of the vector particles is also investigated using three coordinate systems. The thermal spectrum of the radiated vector bosons determined using a direct computation corresponds to a temperature which is twice the Hawking temperature of Kerr-Newman black hole for scalar particles. If the well behaved Eddington coordinate system and Painleve coordinate system are used, the correct result of Hawking temperature is obtained. The reason for the discrepancy in the results of naive coordinate and well behaved coordinates is also discussed.

Keywords Kerr-Newman black hole · Hawking radiation · Proca equation

1 Introduction

In the early 1970s, on the basis of quantum theory Stephen Hawking discovered that black hole emits particles and the energy spectrum of the emitted particles is thermal (Hawking 1974, 1975). Page (1976) also showed that black holes emit all sorts of particles e.g. neutrinos, photons, gravitons, electrons, positrons etc. and calculated the emission rate for the particles having zero or negligible rest mass emitted from the Kerr black hole. The emission of vector particles e.g. $W^{\pm}$, Z^{o} which constitute a fundamental part of the standard model for electroweak interactions should also be an important aspect of the study of Hawking radiation.

Various methods have been proposed for the study of Hawking radiation as tunneling across the event horizon of the black hole (Hartle and Hawking 1976; Kraus and Wilczek 1995; Parikh and Wilczek 2000; Srinivasan and Padmanabhan 1999; Sankaranarayanan et al. 2001, 2002; Padmanabhan 2004; Angheben et al. 2005; Kerner and Mann 2008; Kruglov 2014a,b; Ge-Rui et al. 2015). The tunneling method provides a conceptual means of understanding the underlying physical process of black hole radiation. Creation of the positive and negative energy virtual pair of particles is similar to the particle-antiparticle pair creation. Just like the antiparticle counterpart the negative energy virtual particle might be considered as a positive energy virtual particle moving forward in time and tunneling out of the black hole event horizon. At the moment when the negative energy virtual particle is just created the positive energy virtual particle vanishes and another positive energy virtual particle is created and it is emitted as a part of the Hawking radiation. The principle of conservation of energy is obeyed in the process. Kruglov (2014a,b) investigated the radiation of vector particles from black holes using the WKB approximation to the Proca equation. For the Schwarzschild background geometry the emission temperature is in agreement with the Hawking temperature corresponding to scalar particle emission. Ge-Rui et al. (2014, 2015) investigated vector particles tunneling from the BTZ black holes and four dimensional Schwarzschild black hole respectively. They could recover the expected Hawking temperature. Sakalli

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and Ovgun (2015a,b,c) also studied the Hawking radiation of vector particles from a 3-dimensional rotating black hole with scalar hair using the Hamilton-Jacobi ansatz. The tunneling spectrum of the vector particles was determined and the standard Hawking temperature could be recovered. They investigated quantum tunneling of massive vector particles from Schwarzschild black hole in the Kruskal coordinates and the Lemaitre coordinates and also from Lorentzian wormholes in $3+1$ dimensions. Gursel and Sakalli (2015) studied Hawking radiation of massive vector particles from Warped anti de Sitter black hole in $2+1$ dimensions. Xiang-Qian and Ge-Rui (2015) investigated massive vector particles tunneling from Kerr and Kerr-Newman black holes. Angheben et al. (2005) found that naive coordinate leads to an incorrect result for Hawking radiation of scalar particles, however, when isotropic coordinate or invariant radial distance is used the correct result is obtained. Wang et al. (2010) and Ibungochouba et al. (2013) also observed that naive coordinate leads to incorrect result whereas well behaved Painleve and Eddington coordinate systems give the correct results in the case of tunneling of scalar particles across the event horizons of Kerr-Newman and Kerr-de Sitter black hole respectively. The Painleve coordinate system has many attractive features. Firstly, the metric components are analytic, hence non-diverging at the black hole event horizon. Secondly, the constant time slice are flat Euclidean space. The second one is very important because the WKB may be used to calculate the tunneling rate and WKB approximation is obtained from the quantum mechanics of flat space time (Zhang and Zhao 2005a,b; Ren and Zhao 2006, 2007). Thirdly, there exists a time like Killing vector field which is important to the energy conservation. Lastly, particle tunneling of a barrier is a spontaneous process in quantum mechanic. This new coordinate system will eliminate the singularity of the metric components at the black hole event horizon and the components of metric may satisfy Landau's coordinate clock synchronization condition (Landau and Lifshitz 1975).

In this paper by applying the WKB approximation and Hamilton-Jacobi ansatz to the Proca equation we observe the spectrum of the vector bosons emitted via tunneling across the event horizon of Kerr-Newman black hole. It is observed that similar to the tunneling of scalar particles the coordinate problem exists in the case of vector particles' tunneling across the event horizon.

The layout of the paper is as follows. In Sect. 2, we give a brief review of the Kerr-Newman black hole and using the naive coordinate the imaginary part of the action is determined and the probability of tunneling across the event horizon is calculated. In Sects. 3 and 4 the tunneling probabilities are calculated using the Eddington and Painleve coordinate systems respectively. A brief discussion of our findings are given in Sect. 5.

2 Kerr-Newman black hole

The line element of Kerr-Newman black hole in Boyer-Lindguist coordinates can be written in the form

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{R^2}dt^2 + \frac{R^2}{\Delta}dr^2 + R^2 d\theta^2 \\ &+ \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{R^2}\sin^2\theta d\phi^2 \\ &- 2a\sin^2\theta \frac{r^2+a^2-\Delta}{R^2} dt\, d\phi, \end{aligned} \tag{1}$$

where $R^2 = r^2 + a^2\cos^2\theta$, and $\Delta = r^2 + a^2 + Q^2 - 2Mr$, and the parameters M, Q and $J = Ma$ denote the mass, charge and angular momentum of the black hole respectively. The event horizon is given by

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}. \tag{2}$$

For the Kerr-Newman black hole horizon the semi-classical Hawking temperature (Ibungochouba et al. 2014) is given by

$$T_H = \frac{1}{4\pi}\cdot\frac{r_+ - r_-}{(r_+^2 + a^2)}. \tag{3}$$

The Kerr-Newman black hole has a frame dragging effect in the coordinate system and matter field in the ergosphere near the event horizon $r = r_+$ could be dragged. It is not convenient to study the Hawking thermal radiation effect and a suitable approach to study Hawking radiation would be in the dragging coordinate system. For this we used the dragging coordinate system, let $\frac{d\phi}{dt} = -\frac{g_{03}}{g_{33}}$, the line element of Kerr-Newman black hole (1) becomes

$$\begin{aligned} ds^2 &= -\frac{R^2\Delta}{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}dt_k^2 + \frac{R^2}{\Delta}dr^2 + R^2 d\theta^2 \\ &= g_{00}dt_k^2 + g_{11}dr^2 + g_{22}d\theta^2, \end{aligned} \tag{4}$$

where $B = (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$ and angular velocity at the black hole event horizon is given by

$$\Omega_+ = \frac{a}{r_+^2 + a^2}. \tag{5}$$

From Eq. (4), the contravariant components are given by

$$g^{00} = -\frac{B}{\Delta R^2}, \qquad g^{11} = \frac{\Delta}{R^2}, \qquad g^{22} = \frac{1}{R^2}. \tag{6}$$

The line element (4) must satisfy the Landau's condition of the coordinate clock synchronization. We know that the black hole event horizon and the infinite red-shift surface are coincident with each other, which means the geometrical optics limit may be used. Applying WKB approximation, the relationship between the tunneling probability and the imaginary part of the action may be derived as $\Gamma \sim e^{-2\,\mathrm{Im}\, S}$ (Keski-Vakkuri and Kraus 1997). We will study massive vector particles in this space-time. Within the semi-classical approximation, the wave function Ψ satisfies the Proca equation as

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\Psi^{ab}\right)+\frac{m^2}{\hbar^2}\Psi^b=0, \tag{7}$$

where $\Psi_{ab}=\partial_a\Psi_b-\partial_b\Psi_a$ and Ψ^{ab} is an anti-symmetric tensor. From Eq. (2), we obtain the components of wave function Ψ are as follows

$$\begin{aligned}
&\Psi^0=-\frac{B}{\Delta R^2}\Psi_0, \qquad \Psi^1=\frac{\Delta}{R^2}\Psi_1, \qquad \Psi^2=\frac{1}{R^2}\Psi_2\\
&\Psi^{01}=-\frac{B}{R^2R^2}\Psi_{01}, \qquad \Psi^{02}=-\frac{B}{\Delta R^2R^2}\Psi_{02}\\
&\Psi^{12}=\frac{\Delta}{R^2R^2}\Psi_{12}.
\end{aligned} \tag{8}$$

Then the Proca equations are reduced to

$$\begin{aligned}
&B\left(\frac{\partial^2\Psi_2}{\partial\theta\partial t_k}-\frac{\partial^2\Psi_0}{\partial\theta^2}\right)+\Delta B\left(\frac{\partial^2\Psi_1}{\partial r\partial t_k}-\frac{\partial^2\Psi_0}{\partial r^2}\right)\\
&\quad+\Psi_{01}\left[\Delta B\frac{\partial}{\partial r}\log\sqrt{-g}+\Delta R^2R^2\frac{\partial}{\partial r}\left(\frac{B}{R^2R^2}\right)\right]\\
&\quad+\Psi_{02}\left[B\frac{\partial}{\partial\theta}\log\sqrt{-g}+\Delta R^2R^2\frac{\partial}{\partial\theta}\left(\frac{B}{\Delta R^2R^2}\right)\right]\\
&\quad+\frac{m^2BR^2}{\hbar^2}\Psi_0=0,\\
&B\left(\frac{\partial^2\Psi_0}{\partial t_k\partial r}-\frac{\partial^2\Psi_1}{\partial t_k^2}\right)-\Delta\left(\frac{\partial^2\Psi_2}{\partial\theta\partial r}-\frac{\partial^2\Psi_1}{\partial\theta^2}\right)\\
&\quad-\Psi_{12}\left[\Delta\frac{\partial}{\partial\theta}\log\sqrt{-g}+R^2R^2\frac{\partial}{\partial\theta}\left(\frac{\Delta}{R^2R^2}\right)\right]\\
&\quad+\Psi_{10}\left[\Delta\frac{\partial}{\partial t}\log\sqrt{-g}+R^2R^2\frac{\partial}{\partial t}\left(\frac{\Delta}{R^2R^2}\right)\right]\\
&\quad-\frac{\Delta m^2R^2}{\hbar^2}\Psi_1=0,\\
&B\left(\frac{\partial^2\Psi_0}{\partial t_k\partial\theta}-\frac{\partial^2\Psi_2}{\partial t_k^2}\right)-\Delta^2\left(\frac{\partial^2\Psi_1}{\partial r\partial\theta}-\frac{\partial^2\Psi_2}{\partial r^2}\right)\\
&\quad-\Psi_{20}\left[\Delta R^2R^2\frac{\partial}{\partial t}\left(\frac{B}{\Delta R^2R^2}\right)+B\frac{\partial}{\partial t}\log\sqrt{-g}\right]\\
&\quad-\Psi_{21}\left[\Delta^2\frac{\partial}{\partial r}\log\sqrt{-g}+\Delta R^2R^2\frac{\partial}{\partial r}\left(\frac{\Delta}{R^2R^2}\right)\right]\\
&\quad-\frac{\Delta m^2R^2}{\hbar^2}\Psi_2=0.
\end{aligned} \tag{9}$$

The vector function can be defined as

$$\Psi_a=(c_0,c_1,c_2)\exp\left[\frac{i}{\hbar}S(t_k,r,\theta)\right]. \tag{10}$$

Using WKB approximation, the action can be written as

$$\begin{aligned}
S(t_k,r,\theta)&=S_0(t_k,r,\theta)+\hbar S_1(t_k,r,\theta)\\
&\quad+\hbar^2S_2(t_k,r,\theta)\ldots.
\end{aligned} \tag{11}$$

Using Eqs. (10) and (11) in (9), the higher order terms of $o(\hbar)$ are neglected. Then resulting equations can be obtained as follows

$$\begin{aligned}
&c_2\frac{\partial S_0}{\partial\theta}\frac{\partial S_0}{\partial t_k}-c_0\left(\frac{\partial S_0}{\partial\theta}\right)^2+\Delta c_1\frac{\partial S_0}{\partial r}\frac{\partial S_0}{\partial t_k}\\
&\quad-\Delta c_0\left(\frac{\partial S_0}{\partial r}\right)^2-m^2R^2c_0=0,\\
&c_0B\frac{\partial S_0}{\partial t_k}\frac{\partial S_0}{\partial r}-Bc_1\left(\frac{\partial S_0}{\partial t_k}\right)^2-\Delta c_2\frac{\partial S_0}{\partial\theta}\frac{\partial S_0}{\partial r}\\
&\quad+\Delta c_1\left(\frac{\partial S_0}{\partial\theta}\right)^2+\Delta m^2R^2c_1=0,\\
&c_0B\left(\frac{\partial S_0}{\partial t_k}\right)\left(\frac{\partial S_0}{\partial\theta}\right)-Bc_2\left(\frac{\partial S_0}{\partial t_k}\right)^2\\
&\quad-\Delta^2c_1\left(\frac{\partial S_0}{\partial r}\right)\left(\frac{\partial S_0}{\partial\theta}\right)+\Delta^2c_2\left(\frac{\partial S_0}{\partial r}\right)^2\\
&\quad+\Delta m^2R^2c_2=0.
\end{aligned} \tag{12}$$

It is very difficult to find the action S_0 directly from Eqs. (12) because it is a functions of t_k, r and θ. So we assume the solution as

$$S_0=-\omega t_k+W(r)+K(\theta)+\zeta, \tag{13}$$

where ω is the energy of the vector particles and ζ is a complex constant. Putting Eq. (13) into Eq. (12), we obtain the matrix equation

$$\Lambda(c_0,c_1,c_2)^T=0, \tag{14}$$

where Λ is a 3×3 matrix and superscript T means the transition to the transposed vector. Then the components of Λ matrix are given below

$$\begin{aligned}
&\Lambda_{00}=K_\theta^2+\Delta W_r^2+m^2R^2, \qquad \Lambda_{01}=\Delta\omega W_r,\\
&\Lambda_{02}=K_\theta\omega,\ \Lambda_{10}=-\omega BW_r, \qquad \Lambda_{12}=-\Delta K_\theta W_r,\\
&\Lambda_{11}=-\omega^2B+\Delta K_\theta^2+\Delta m^2R^2,\\
&\Lambda_{20}=-B\omega K_\theta, \qquad \Lambda_{21}=-\Delta^2W_rK_\theta,\\
&\Lambda_{22}=-B\omega^2+\Delta^2W_r^2+\Delta R^2m^2,
\end{aligned} \tag{15}$$

where $W_r = \frac{\partial W}{\partial r}$ and $K_\theta = \frac{\partial K}{\partial \theta}$. Equation (14) is a homogeneous system linear equations and admits non-trivial solution if and only if det $\Lambda(c_0, c_1, c_2) = 0$. Then we get

$$\begin{aligned}&\left(\Delta m^2 R^2 - B\omega^2 + \Delta^2 W_r^2 + \Delta K_\theta^2\right)\\&\quad\times\left[\left(\Delta m^2 R^2 - B\omega^2\right)\left(K_\theta^2 + m^2 R^2 + \Delta W_r^2\right)\right.\\&\quad\left.+ B K_\theta^2 \omega^2 + \Delta B \omega^2 W_r^2\right] = 0. \end{aligned} \tag{16}$$

And integrating, we obtain

$$W_\pm = \int \sqrt{\frac{L - B K_\theta^2 \omega^2}{\Delta^2 m^2 R^2}} dr, \tag{17}$$

where W_+ stands for outgoing spin-1 particle (moving away from the black hole) and W_- corresponds to the ingoing (moving towards the black hole) spin-1 particle. Using Eq. (17) in Eq. (13), we get

$$S_0 = -\omega t_k + K(\theta) + \zeta + \int \sqrt{\frac{L - B K_\theta^2 \omega^2}{\Delta^2 m^2 R^2}} dr, \tag{18}$$

where $L = (B\omega^2 - \Delta m^2 R^2)(K_\theta^2 + m^2 R^2)$. From Eq. (18), we see that there is a pole at the event horizon $r = r_+$ and imaginary part of the action can be derived from the pole. Using Feyman prescription and completing the integral, the imaginary part of the action can be derived as

$$\operatorname{Im} S_0 = \frac{\pi \omega (r_+^2 + a^2)}{r_+ - r_-}. \tag{19}$$

Applying WKB approximation, the tunneling probability will be

$$\Gamma \sim e^{-2 \operatorname{Im} S_0} = e^{\frac{-2\pi\omega(r_+^2 + a^2)}{r_+ - r_-}} = e^{-\beta\omega}, \tag{20}$$

where $T = \frac{1}{\beta} = \frac{1}{2\pi} \cdot \frac{r_+ - r_-}{(r_+^2 + a^2)}$, which is equal to twice the Hawking temperature for a Kerr-Newman black hole given in Eq. (3).

3 Eddington coordinate

For describing tunneling probability near the event horizon, we will consider a well behaved coordinate system named as Eddington coordinate. Let

$$dt_k = du - \frac{\sqrt{B}}{\Delta} dr, \tag{21}$$

the line element (4) can be expressed as

$$ds^2 = -\frac{R^2 \Delta}{B} du^2 + \frac{2R^2}{\sqrt{B}} du dr + R^2 d\theta^2, \tag{22}$$

then the contravariant components are given by

$$g^{01} = g^{10} = \frac{\sqrt{B}}{R^2}, \qquad g^{11} = \frac{\Delta}{R^2}, \qquad g^{22} = \frac{1}{R^2}. \tag{23}$$

For the vector particle, the wave function Ψ satisfies the Proca equation as

$$\frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} \Psi^{ab}\right) + \frac{m^2}{\hbar^2} \Psi^b = 0. \tag{24}$$

The components of wave function are given by

$$\begin{aligned}&\Psi^0 = \frac{\sqrt{B}}{R^2}\Psi_1, \qquad \Psi^1 = \frac{\sqrt{B}}{R^2}\Psi_0 + \frac{\Delta}{R^2}\Psi_1,\\&\Psi^2 = \frac{1}{R^2}\Psi_2, \qquad \Psi^{01} = \frac{B}{R^2 R^2} = \Psi_{01}\\&\Psi^{02} = \frac{\sqrt{B}}{R^2 R^2}\Psi_{12},\\&\Psi^{12} = \frac{\sqrt{B}}{R^2 R^2}\Psi_{02} + \frac{\Delta}{R^2 R^2}\Psi_{12}.\end{aligned} \tag{25}$$

Then Proca equation can be written as

$$\begin{aligned}&B\left(\frac{\partial^2 \Psi_0}{\partial r^2} - \frac{\partial^2 \Psi_1}{\partial r \partial u}\right) + B\left(\frac{\partial^2 \Psi_2}{\partial \theta \partial r} - \frac{\partial^2 \Psi_1}{\partial \theta^2}\right)\\&\quad + \Psi_{10}\left[B\frac{\partial}{\partial r}\log\sqrt{-g} + R^2 R^2 \frac{\partial}{\partial r}\left(\frac{B}{R^2 R^2}\right)\right]\\&\quad + \Psi_{12}\left[R^2 R^2 \frac{\partial}{\partial \theta}\left(\frac{\sqrt{B}}{R^2 R^2}\right) + \sqrt{B}\frac{\partial}{\partial \theta}\log\sqrt{-g}\right]\\&\quad + \frac{m^2 \sqrt{B} R^2 \Psi_1}{\hbar^2} = 0,\\&B\left(\frac{\partial^2 \Psi_1}{\partial u^2} - \frac{\partial^2 \Psi_0}{\partial u \partial r}\right) + \sqrt{B}\left(\frac{\partial^2 \Psi_2}{\partial \theta \partial u} - \frac{\partial^2 \Psi_0}{\partial \theta^2}\right)\\&\quad + \Delta\left(\frac{\partial^2 \Psi_2}{\partial \theta \partial r} - \frac{\partial^2 \Psi_1}{\partial \theta^2}\right) + \Psi_{10} R^2 R^2 \frac{\partial}{\partial u}\left(\frac{B}{R^2 R^2}\right)\\&\quad + \Psi_{02}\left[\sqrt{B}\frac{\partial}{\partial \theta}\log\sqrt{-g} + R^2 R^2 \frac{\partial}{\partial \theta}\left(\frac{B}{R^2 R^2}\right)\right]\\&\quad + \Psi_{12}\left[\Delta\frac{\partial}{\partial \theta}(\log\sqrt{-g}) + R^2 R^2 \frac{\partial}{\partial \theta}\left(\frac{\Delta}{R^2 R^2}\right)\right]\\&\quad + \frac{m^2 R^2 \sqrt{B}\Psi_0}{\hbar^2} + \frac{\Delta m^2 R^2 \Psi_1}{\hbar^2} = 0,\\&\sqrt{B}\left(\frac{\partial^2 \Psi_1}{\partial u \partial \theta} - \frac{\partial^2 \Psi_2}{\partial u \partial r}\right) + \Delta\left(\frac{\partial^2 \Psi_1}{\partial r \partial \theta} - \frac{\partial^2 \Psi_2}{\partial r^2}\right)\\&\quad + \sqrt{B}\left(\frac{\partial^2 \Psi_2}{\partial r \partial u} - \frac{\partial^2 \Psi_0}{\partial r \partial \theta}\right) + \Psi_{21}\left[\Delta\frac{\partial}{\partial r}(\log\sqrt{-g})\right.\end{aligned} \tag{26}$$

$$+R^2R^2\frac{\partial}{\partial r}\left(\frac{\Delta}{R^2R^2}\right)\bigg]+\Psi_{02}\bigg[\sqrt{B}\frac{\partial}{\partial r}(\log\sqrt{-g})$$
$$+R^2R^2\frac{\partial}{\partial r}\left(\frac{B}{R^2R^2}\right)\bigg]+\frac{m^2R^2\Psi_2}{\hbar^2}=0.$$

Let us define vector function as

$$\Psi_a=(c_0,c_1,c_2)\exp\left[\frac{i}{\hbar}S(u,r,\theta)\right]. \tag{27}$$

Using WKB approximation, the action can be written as

$$S(u,r,\theta)=S_0(u,r,\theta)+\hbar S_1(u,r,\theta)+\hbar^2S_2(u,r,\theta)\dots. \tag{28}$$

Using Eqs. (27) and (28) in Eq. (26), the leading order terms of $o(\hbar)$ become

$$\begin{aligned}
&Bc_0\left(\frac{\partial S_0}{\partial r}\right)^2-Bc_1\left(\frac{\partial S_0}{\partial r}\right)\left(\frac{\partial S_0}{\partial u}\right)\\
&\quad+\sqrt{B}c_2\left(\frac{\partial S_0}{\partial \theta}\right)\left(\frac{\partial S_0}{\partial r}\right)-\sqrt{B}c_1\left(\frac{\partial S_0}{\partial \theta}\right)^2\\
&\quad-m^2\sqrt{B}c_1R^2=0,\\
&Bc_1\left(\frac{\partial S_0}{\partial u}\right)^2-Bc_0\left(\frac{\partial S_0}{\partial u}\right)\left(\frac{\partial S_0}{\partial r}\right)\\
&\quad+\Delta c_2\left(\frac{\partial S_0}{\partial \theta}\right)\left(\frac{\partial S_0}{\partial r}\right)-\Delta c_1\left(\frac{\partial S_0}{\partial \theta}\right)^2\\
&\quad+\sqrt{B}c_2\left(\frac{\partial S_0}{\partial \theta}\right)\left(\frac{\partial S_0}{\partial u}\right)-c_0\sqrt{B}\left(\frac{\partial S_0}{\partial \theta}\right)^2\\
&\quad-m^2\sqrt{B}R^2c_0-m^2\Delta R^2c_1=0,\\
&\sqrt{B}c_1\left(\frac{\partial S_0}{\partial u}\right)\left(\frac{\partial S_0}{\partial \theta}\right)-\sqrt{B}c_2\left(\frac{\partial S_0}{\partial u}\right)\left(\frac{\partial S_0}{\partial r}\right)\\
&\quad+\Delta c_1\left(\frac{\partial S_0}{\partial r}\right)\left(\frac{\partial S_0}{\partial \theta}\right)-\Delta c_2\left(\frac{\partial S_0}{\partial r}\right)^2\\
&\quad+\sqrt{B}c_2\left(\frac{\partial S_0}{\partial r}\right)\left(\frac{\partial S_0}{\partial u}\right)-\sqrt{B}c_0\left(\frac{\partial S_0}{\partial r}\right)\left(\frac{\partial S_0}{\partial \theta}\right)\\
&\quad-m^2R^2c_2=0.
\end{aligned} \tag{29}$$

Due to symmetry of the space-time, we assume a solution of the form

$$S_0=-\omega u+\tilde{W}(r)+\tilde{K}(\theta)+\Xi, \tag{30}$$

where Ξ is an complex constant. Using Eq. (30) into Eqs. (29), we obtain the matrix equation as

$$\tilde{\Lambda}(c_0,c_1,c_2)^T=0, \tag{31}$$

where $\tilde{\Lambda}$ is a 3×3 matrix. Its components are given below

$$\begin{aligned}
&\tilde{\Lambda}_{00}=B\tilde{W}_r^2,\qquad \tilde{\Lambda}_{20}=\tilde{\Lambda}_{02}=\sqrt{B}\tilde{K}_\theta\tilde{W}_r,\\
&\tilde{\Lambda}_{01}=\tilde{\Lambda}_{10}=B\tilde{W}_r\omega-\sqrt{B}\tilde{K}_\theta^2-m^2\sqrt{B}R^2,\\
&\tilde{\Lambda}_{11}=B\omega^2-\Delta\tilde{K}_\theta^2-m^2\Delta R^2,\\
&\tilde{\Lambda}_{12}=\tilde{\Lambda}_{21}=\Delta\tilde{K}_\theta\tilde{W}_r-\sqrt{B}\tilde{K}_\theta\omega,\\
&\tilde{\Lambda}_{22}=-\Delta\tilde{W}_r^2-m^2R^2,
\end{aligned} \tag{32}$$

where $\tilde{W}_r=\frac{\partial\tilde{W}}{\partial r}$ and $\tilde{K}_\theta=\frac{\partial\tilde{K}}{\partial\theta}$. Equation (31) is a homogeneous system of linear equations and there exists non-trivial solution only when $\det\tilde{\Lambda}=0$. Integrating the resulting equation and using WKB approximation we obtain tunneling probability as

$$\Gamma\sim e^{-2\,\mathrm{Im}\,S_0}=e^{\frac{-4\pi\omega(r_+^2+a^2)}{r_+-r_-}}=e^{-\beta\omega}, \tag{33}$$

where $T=\frac{1}{\beta}=\frac{1}{4\pi}\cdot\frac{r_+-r_-}{(r_+^2+a^2)}$, which is exactly equal to the Hawking radiation temperature given in Eq. (3).

4 Painleve coordinate

Next we consider Painleve coordinate system. For this we consider the following coordinate transformation

$$dt_k=dt+F(r,\theta)dr+G(r,\theta)d\theta, \tag{34}$$

where $F(r,\theta)$ and $G(r,\theta)$ are the two functions to be determined and they must satisfy the integrability condition

$$\frac{\partial F(r,\theta)}{\partial\theta}=\frac{\partial G(r,\theta)}{\partial r}. \tag{35}$$

We know that the constant-time slices are flat Euclidean space, so we get the condition

$$g_{22}+F^2(r,\theta)=1, \tag{36}$$

we obtain another form of the Kerr-Newman metric as

$$\begin{aligned}
ds^2&=g_{00}dt^2+2\sqrt{g_{00}(1-g_{11})}dtdr+dr^2\\
&\quad+\left[g_{00}G^2(r,\theta)+g_{22}\right]d\theta^2+2g_{00}G(r,\theta)dtd\theta\\
&\quad+2\sqrt{g_{00}(1-g_{11})}G(r,\theta)drd\theta,
\end{aligned} \tag{37}$$

so the contravariant components of Eq. (37) are

$$\begin{aligned}
&g^{00}=\frac{R^2G^2-B}{R^2R^2},\qquad g^{11}=\frac{\Delta}{R^2},\qquad g^{22}=\frac{1}{R^2}\\
&g^{12}=g^{21}=0,\qquad g^{02}=g^{20}=-\frac{G}{R^2},\\
&g^{01}=g^{10}=\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2}.
\end{aligned} \tag{38}$$

We consider a vector particle moving in this space-time using the Proca equation

$$\frac{1}{\sqrt{-g}}\partial_a\left(\sqrt{-g}\Psi^{ab}\right)+\frac{m^2}{\hbar^2}\Psi^b=0. \tag{39}$$

The components of the wave function are given by

$$\begin{aligned}
\Psi^0 &= \frac{R^2G^2-B}{R^2R^2}\Psi_0+\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2}\Psi_1-\frac{G}{R^2}\Psi_2,\\
\Psi^1 &= \frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2}\Psi_0+\frac{\Delta}{R^2}\Psi_1,\\
\Psi_2 &= -\frac{G}{R^2}\Psi_0+\frac{1}{R^2}\Psi_1,\\
\Psi^{01} &= \frac{B(R^2-\Delta)}{R^2R^2R^2}\Psi_{10}-\frac{G\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\Psi_{20}\\
&\quad+\frac{R^2G^2-B}{R^2R^2}\frac{\Delta}{R^2}\Psi_{01}-\frac{G\Delta}{R^2R^2}\Psi_{21},\\
\Psi^{02} &= -\frac{G}{R^2R^2}\sqrt{BR^2\left(R^2-\Delta\right)}\Psi_{10}\\
&\quad+\frac{1}{R^2R^2R^2}\left(R^2G^2-B\right)\Psi_{02}+\frac{G^2}{R^2R^2}\Psi_{20}\\
&\quad+\frac{1}{R^2R^2R^2}\sqrt{BR^2\left(R^2-\Delta\right)}\Psi_{12},\\
\Psi^{12} &= -\frac{G\Delta}{R^2R^2}\Psi_{10}+\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\Psi_{02}+\frac{\Delta}{R^2R^2}\Psi_{12}.
\end{aligned} \tag{40}$$

Then the Proca equation can be expressed as

$$\begin{aligned}
&\frac{\partial}{\partial r}\Psi^{01}+\frac{\partial}{\partial\theta}\Psi^{02}+\Psi^{01}\frac{\partial}{\partial r}(\log\sqrt{-g})\\
&\quad+\Psi^{02}\frac{\partial}{\partial\theta}(\log\sqrt{-g})+\frac{m^2\Psi^0}{\hbar^2}=0,\\
&\frac{\partial}{\partial t}\Psi^{10}+\frac{\partial}{\partial\theta}\Psi^{12}+\Psi^{12}\frac{\partial}{\partial\theta}(\log\sqrt{-g})+\frac{m^2\Psi^1}{\hbar^2}=0,\\
&\frac{\partial}{\partial t}\Psi^{20}+\frac{\partial}{\partial r}\Psi^{12}+\Psi^{21}\frac{\partial}{\partial r}(\log\sqrt{-g})+\frac{m^2\Psi^2}{\hbar^2}=0.
\end{aligned} \tag{41}$$

Assuming the vector function by

$$\Psi_a=(c_0,c_1,c_2)\exp\left[\frac{i}{\hbar}S(t,r,\theta)\right] \tag{42}$$

and utilizing WKB approximation, the action may be written as

$$\begin{aligned}
S(t,r,\theta) &= S_0(t,r,\theta)+\hbar S_1(t,r,\theta)\\
&\quad+\hbar^2S_2(t,r,\theta)\ldots.
\end{aligned} \tag{43}$$

For the separation of variables in Eqs. (41), we assume

$$S_0=-\omega t+H(r)+\Theta(\theta)+\Upsilon, \tag{44}$$

where Υ is a complex constant. Substituting Eqs. (42)–(44) into Eqs. (41), we derive the matrix equation

$$\Lambda^*(c_0,c_1,c_2)^T=0, \tag{45}$$

where the matrix components are given by

$$\begin{aligned}
\Lambda^*_{00} &= -\frac{B(R^2-\Delta)}{R^2R^2R^2}H_r^2+\frac{\Delta(R^2G^2-B)}{R^2R^2R^2}H_r^2\\
&\quad+\frac{H_r\Theta_\theta G\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\\
&\quad-\frac{G}{R^2R^2}\sqrt{BR^2\left(R^2-\Delta\right)}H_r\Theta_\theta-\frac{G^2\Theta_\theta^2}{R^2R^2}\\
&\quad+\frac{G}{R^2R^2}\sqrt{BR^2\left(R^2-\Delta\right)}H_r\Theta_\theta\\
&\quad+\frac{m^2}{R^2R^2}\left(R^2G^2-B\right)+\frac{(R^2G^2-B)}{R^2R^2R^2}\Theta_\theta^2,\\
\Lambda^*_{01} &= \frac{-\omega B(R^2-\Delta)}{R^2R^2R^2}H_r+\frac{G\Delta\Theta_\theta H_r}{R^2R^2}\\
&\quad+\frac{\Delta\omega}{R^2R^2R^2}\left(R^2G^2-B\right)H_r\\
&\quad+\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2}\left(G\omega\Theta_\theta+m^2\right)\\
&\quad+\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\Theta_\theta^2,\\
\Lambda^*_{02} &= -\frac{G\Delta H_r^2}{R^2R^2}-\frac{G^2\Theta_\theta}{R^2R^2}-\frac{m^2G}{R^2}\\
&\quad+\frac{\omega\Theta_\theta(R^2G^2-B)}{R^2R^2R^2}\\
&\quad-\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\Theta_\theta H_r,\\
\Lambda^*_{10} &= \frac{\Delta(R^2G^2-B)}{R^2R^2R^2}\omega H_r+\frac{G\Delta\Theta_\theta H_r}{R^2R^2}\\
&\quad+\frac{B(R^2+\Delta)}{R^2R^2R^2}\Theta_\theta H_r\\
&\quad+\frac{G\omega\Theta_\theta}{R^2R^2R^2}\sqrt{BR^2\left(R^2-\Delta\right)}\\
&\quad+\frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\left(R^2m^2-\Theta_\theta^2\right),\\
\Lambda^*_{11} &= \frac{\Delta(R^2G^2-B)}{R^2R^2R^2}\omega^2-\frac{B(R^2-\Delta)}{R^2R^2R^2}\omega^2\\
&\quad+\frac{2\Delta G\omega\Theta_\theta}{R^2R^2}+\frac{\Delta\Theta_\theta^2}{R^2R^2}+\frac{m^2\Delta}{R^2},
\end{aligned} \tag{46}$$

$$\Lambda_{12}^{*} = \frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\left(G\omega^2+\omega\Theta_\theta\right) - \frac{\Delta G\omega H_r}{R^2R^2} - \frac{\Delta\Theta_\theta H_r}{R^2R^2},$$

$$\Lambda_{20}^{*} = \frac{H_r\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\left(G\omega R^2-\Theta_\theta\right) - \frac{G}{R^2R^2}\left(\omega\Theta_\theta+\Delta H_r^2\right) + \frac{(R^2G^2-B)\omega\Theta_\theta}{R^2R^2R^2} + \frac{m^2G}{R^2},$$

$$\Lambda_{21}^{*} = \frac{\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2}\left(G\omega^2R^2+\omega\Theta_\theta\right) + \frac{\Delta H_r}{R^2R^2}(\Theta_\theta-G\omega),$$

$$\Lambda_{22}^{*} = \frac{2\omega H_r\sqrt{BR^2(R^2-\Delta)}}{R^2R^2R^2} + \frac{m^2}{R^2} - \frac{(G\omega^2+\Delta H_r^2)}{R^2R^2} - \frac{(R^2G^2-B)\omega^2}{R^2R^2R^2},$$

where $H_r = \frac{\partial H}{\partial r}$ and $\Theta_\theta = \frac{\partial \Theta}{\partial \theta}$. The homogeneous Eq. (45) has non-trivial solutions for $\det \Lambda^* = 0$ using Feyman prescription and the WKB approximation, the tunneling probability can be obtained as

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S_0} = e^{\frac{-4\pi\omega(r_+^2+a^2)}{r_+-r_-}} = e^{-\beta\omega}, \tag{47}$$

where $T = \frac{1}{\beta} = \frac{1}{4\pi}\cdot\frac{r_+-r_-}{(r_+^2+a^2)}$, which is equal to the Hawking temperature given in Eq. (3). From these results given in Eqs. (20), (33) and (44), we observe that well behaved coordinate system is required to study Hawking radiation temperature using the Proca equation.

5 Discussion and conclusion

By applying the WKB approximation and Hamilton-Jacobi ansatz to the Proca equation the tunneling of vector bosons (e.g. $W^\pm$, Z^o) across the event horizon of Kerr-Newman black hole and the resulting Hawking radiation are studied. The tunneling probability of the vector particles is determined using the imaginary part of the action of the emitted particles. The energy spectrum of the emitted vector particles is observed to be thermal. However, as in the case of scalar particles there exists coordinate problem in vector particles' Hawking radiation from Kerr-Newman black hole. Three coordinate systems, namely naive, Eddington and Painleve are used to study the tunneling of the vector particles. A direct calculation gives twice the correct value of the Hawking temperature. However, if Painleve coordinate system and Eddington coordinate system are used, the correct value of Hawking temperature can be obtained. It is due to the fact that the well behaved coordinate system has some special features. For example, in the well behaved coordinate system, the constant time slices are flat Euclidean space and there exist time like Killing vector fields in the Eddington and Painleve coordinate systems. Also for Eddington and Painleve coordinate systems the metric components are analytic, hence non-diverging at the event horizon.

The discrepancy between well behaved coordinates and naive coordinate can be seen in a curve manifold, the non-locally integrable function $1/r$ does not lead to a covariant distribution of the form $1/r + i0$. It is necessary to make use of invariant distance or change the line element into some well behaved coordinate system. The metric components in such coordinate systems are regular at the black hole event horizon surface and it has time like Killing vector, which are the enough conditions to be utilized in the Hawking radiation calculation via Proca equation. Similar conclusions are also seen in (Wang et al. 2010; Ibungochouba et al. 2013; Ren and Zhao 2006, 2007).

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