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LETTER

Power-law solution for anisotropic universe in $f(G)$ gravity

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Abstract We try to study the theory of modified Gauss-Bonnet gravity in non-isotope universe. It is considered the exact power-law solution in modified gravity models. A $f(G)$ function corresponding with power law solutions for given scale factor are calculated. We show that BI-like power-law solutions only exist for a very special class of $f(G)$ theories. It is shown that transition to phantom phase is happened by applied some bound on free parameters. We also explore the stability issue of modified gravitational models.

Keywords Anisotropic Universe · Phantom phase · $f(G)$ gravity

1 Introductions

It is well known that recently it has been found strong evidence for an accelerated expansion of the universe, apparently due to the presence of an effective positive cosmological constant and associated with this acceleration there exists the so called dark energy issue (see for example Padmanabhan 2003). According to the observations of type Ia supernovae, cosmic microwave background (CMB) and large scale structures, the Universe is expanding at an accelerating rate which is linked with dark energy (DE) (having strong negative pressure). Dark energy dynamical models and modified theories of gravity are two candidates to explain the mysterious nature of DE. In physical cosmology and astrophysics, the simplest candidate for the dark energy (DE) is the cosmological constant (Λ) (Carroll 2001; Peebles and Ratra 2003). However, it needs to be extremely fine-tuned to satisfy the current value of the DE density, which is a serious problem (Overduin and Cooperstock 1998). The simplest candidate of dark energy is a cosmological constant with the equation of state parameter $\omega = -1$. However, this scenario suffers from serious problems like a huge fine tuning and the coincidence problem (Shani and Starobinsky 2000, 2006). Alternative models of dark energy suggest a dynamical form of dark energy, which is often realized by one or two scalar fields. In this respect, dark energy has many dynamical components such as quintessence (Ratra and Peebles 1988; Wetterich 1988; Caldwell et al. 1998), K-essence (Armendariz-Picon et al. 2000), tachyon (Sen 2002; Padmanabhan 2002; Setare 2007), phantom (Caldwell 2002), ghost condensate and quintom (Feng et al. 2005; Guo et al. 2005; Setare and Saridakis 2008), and so forth. Alternative to dark energy, modified theories of gravity is extremely attractive, such as $f(R)$ gravity (see, for instance, Felice and Tsujikawa (2010) for reviews), here $f(R)$ is an arbitrary function of the Ricci scalar R. Cosmic acceleration can be explained by $f(R)$ gravity (Nojiri and Odintsov 2003), and the conditions of viable cosmological models have been derived in Capozziello et al. (2006). A general model of $f(R)$ gravity has been proposed in Allemandi et al. (2005), which contains a non-minimal coupling between geometry and matter. Viable cosmological models have been found in Capozziello et al. (2006), Nojiri and Odintsov (2006, 2007a) under some conditions, and weak field constraints obtained from the classical tests of general relativity for the solar system regime seem to

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rule out most of the models so far (Erickcek et al. 2006; Chiba et al. 2007). For this kind of modification, one assumes that the gravitational action may contain some additional terms which starts to grow with decreasing curvature and obtain a late time acceleration epoch. One of the other of modification of Einstein's gravity is Gauss-Bonnet (GB) modification. As a possibility the Einstein-Gauss-Bonnet gravity is low energy limit of the string theory is of particular interest because of its special features. The reconstruction scenario for $f(G)$ gravity is proposed and developed in Nojiri and Odintsov (2007b, 2011). The GB generalization adds quadratic terms, involving second order curvature invariants (specifically Gauss-Bonnet term is a topological invariant in four dimensions) to the Einstein-Hilbert Lagrangian (Carter and Neupane 2006, see also Carroll et al. 2005). Finally, such class of modified gravities may successfully describe the universe expansion history from the early-time inflation till late-time acceleration (Cognola et al. 2006; Nojiri et al. 2008, 2010) with the unification of the inflation with DE.

Under the above circumstances, it is observed that in recent years Bianchi universes have been gaining an increasing interest and tremendous impetus of observational cosmology. According to this, the Universe should achieve the following features: (i) a slightly anisotropic special geometry in spite of the inflation, and (ii) a nontrivial isotropization history of Universe due to the presence of an anisotropic energy source. The anomalies found in the cosmic microwave background (CMB) and large scale structure observations stimulated a growing interest in anisotropic cosmological model of Universe. Here we confine ourselves to model LRS Bianchi type I whose spatial sections are flat but the expansion or contraction rate are direction dependent. For studying the possible effects of anisotropy in the early Universe based on the present day observations many researchers (Saha 2006a, 2006b; Pradhan and Singh 2004) have investigated Bianchi type I models from different point of view. We would like to further mention that unlike the FRW model this Bianchi type I model describes a different kind of Universe in which the scale factor is not restricted to be the same in each direction. Recently Aluri et al. (2013) have shown the importance of BI model to discuss the effects of anisotropy on the basis of recent evidences. Fayaz et al. (2014) discussed holographic and new agegraphic dark energy for anisotropic cosmological models in $f(R,T)$. Hossienkhani et al. (2014) investigated accelerating of the universe for Bianchi models in $f(R,T)$.

In this paper, we investigate the cosmological viability of $f(G)$ models by investigating the conditions under which one can find power law solutions that mimic the standard BI expansion history of the Universe. We discover that such solutions only exist for a very special class of $f(G)$ theories. Furthermore, we extend these results to show the existence of phantom phase power law solutions for an special form of $f(G)$ gravity. The investigation is organized as follows: The metric and field equations are presented in Sect. 2. In Sects. 3 and 4, we deals with the exact matter dominant and phantom phase power law solutions and physical behavior of the model. de Sitter solutions and stability of the model is investigated in Sects. 5 and 6 respectively. The results of the paper are summarized in the last section.

2 Field equations and $f(G)$ gravity

In this section, we consider the following gravitational action (Nojiri and Odintsov 2005).

$$S=\int d^4xe\left[\frac{1}{2\kappa^2}R+f(G)+\mathcal{L}_m\right], \tag{1}$$

where $\mathcal{L}_m$ corresponds to the matter Lagrangian and $e=\det(e^i_\mu)=\sqrt{-g}$, with g being the determinant of the metric tensor, R is the Ricci scalar curvature and $f(G)$ is an arbitrary differentiable function of G which is generally defined as: $G=R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$, where $R_{\mu\nu}$ and $R_{\mu\nu\rho\lambda}$ are Ricci curvature tensor and Riemann curvature tensor respectively. By varying the action S given in Eq. (1) with respect to the metric tensor $g_{\mu\nu}$, the corresponding field equations are obtained as follow (Nojiri and Odintsov 2005):

$$\begin{aligned}0={}&\frac{1}{2\kappa^2}\left(\frac{1}{2}g^{\mu\nu}R-R^{\mu\nu}\right)+T^{\mu\nu}+\frac{1}{2}g^{\mu\nu}f(G)\\&-2f_GRR^{\mu\nu}+4f_GR^\mu_\rho R^{\nu\rho}-2f_GR^{\mu\rho\lambda\tau}R^\nu_{\rho\lambda\tau}\\&-4f_GR^{\mu\rho\lambda\nu}R_{\rho\lambda}+2\left(\nabla^\mu\nabla^\nu f_G\right)R-2g^{\mu\nu}\left(\nabla^2f_G\right)R\\&-4\left(\nabla_\rho\nabla^\mu f_G\right)R^{\nu\rho}-4\left(\nabla_\rho\nabla^\nu f_G\right)R^{\mu\rho}+4\left(\nabla^2f_G\right)R^{\mu\nu}\\&+4g^{\mu\nu}(\nabla_\rho\nabla_\lambda f_G)R^{\rho\lambda}-4(\nabla_\rho\nabla_\lambda f_G)R^{\mu\rho\nu\lambda},\end{aligned} \tag{2}$$

where $f_G=df/dG$ is the first derivative with respect to G of the function f and $T^{\mu\nu}$ represents the energy-momentum tensor of the perfect fluid.

We consider the homogeneous and anisotropic space-time described by Bianchi type I metric in the form

$$ds^2=-dt^2+A(t)^2dx^2+B(t)^2dy^2+C(t)^2dz^2, \tag{3}$$

where $A(t)$, $B(t)$ and $C(t)$ are the scale factors (metric tensors) and functions of the cosmic time t. This metric does not cover Robertson-Walker metric, but gets its closest form to RW metric when $A(t)=B(t)=C(t)$, thus we may talk about its approaching to isotropy, but not a total isotropization of this metric. The anisotropy of the expansion can be parameterized after defining the directional Hubble parameters and the mean Hubble parameter of the expansion. The directional Hubble parameters in the directions of x, y and

z for the Bianchi type I metric defined in (3) may be defined as follows,

$$H_x = \frac{\dot{A}}{A}, \qquad H_y = \frac{\dot{B}}{B}, \qquad H_z = \frac{\dot{C}}{C}, \tag{4}$$

and the mean Hubble parameter is given as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right), \tag{5}$$

with $a = (ABC)^{1/3}$ being the average scale factor of Bianchi type I model. In this background, the above field equations can be represented as (Hossienkhani and Pasqua 2014; Saaidi and Hossienkhani 2011; Fayaz et al. 2012, 2013; Fayaz and Hossienkhani 2013)

$$3H^2 - \sigma^2 = \kappa^2 \rho_{\rm eff}, \tag{6}$$

$$3H^2 + 2\dot{H} + \sigma^2 = -\kappa^2 p_{\rm eff}, \tag{7}$$

where $\rho_{\rm eff}$ and $p_{\rm eff}$ are the effective energy density and effective pressure densities respectively, and σ^2 is the shear scalar. The equation of state parameter, $\omega_{\rm eff} = p_{\rm eff}/\rho_{\rm eff}$, can be expressed in terms of the Hubble parameter and shear tensor as

$$\omega_{\rm eff} = -1 - \frac{2(\dot{H} + \sigma^2)}{3H^2 - \sigma^2}. \tag{8}$$

The corresponding shear scalar, Ricci scalar and Gauss-Bonnet invariant become

$$\sigma^2 = \frac{1}{3}\left(\left(\frac{\dot{A}}{A}\right)^2 + \left(\frac{\dot{B}}{B}\right)^2 + \left(\frac{\dot{C}}{C}\right)^2\right) - \frac{1}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right), \tag{9}$$

$$R = 2\left(6H^2 + 3\dot{H} + \sigma^2\right) = 2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right), \tag{10}$$

$$G = \left(\frac{\ddot{A}}{A}\right)^2 + \left(\frac{\ddot{B}}{B}\right)^2 + \left(\frac{\ddot{C}}{C}\right)^2 + \left(\frac{\dot{A}\dot{B}}{AB}\right)^2 + \left(\frac{\dot{B}\dot{C}}{BC}\right)^2 + \left(\frac{\dot{A}\dot{C}}{AC}\right)^2. \tag{11}$$

Using Eqs. (2) and (3), the field equations are given by

$$-\frac{1}{\kappa^2}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right) - 24\frac{\dot{A}}{A}\frac{\dot{B}}{B}\frac{\dot{C}}{C}\dot{G} f_{GG} + G f_G - f + \rho_m = 0, \tag{12}$$

$$+\frac{1}{3\kappa^2}\left[2\left(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\right) + \left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right)\right] + \frac{8}{3}\dot{f}_G\left(\frac{\dot{A}\ddot{B}}{AB} + \frac{\dot{B}\ddot{A}}{AB} + \frac{\dot{A}\ddot{C}}{AC} + \frac{\dot{C}\ddot{A}}{AC} + \frac{\dot{C}\ddot{B}}{C}B + \frac{\dot{B}\ddot{C}}{BC}\right) + \frac{8}{3}\left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}\right)\ddot{f}_G + f - G f_G + p_m = 0, \tag{13}$$

where dot denotes the first order derivative with respect to t. We take physical assumption for the scale factor A as $A = B^m = C^m$, where $m \neq 0, 1$ (Collins et al. 1980). Note that the Kantowski-Sachs (KS) is recovered by taking $B = C$. We propose to start with the Bianchi type-I case, from which KS can be recovered. Hubble parameter H, scale factor a and shear scalar σ, for this model are

$$H = \frac{m+2}{3}\frac{\dot{B}}{B}, \qquad a = a_0 B^{\frac{m+2}{3}}, \qquad \sigma^2 = \frac{(1-m)^2}{3}\left(\frac{\dot{B}}{B}\right)^2 = \frac{3(1-m)^2}{(m+2)^2}H^2. \tag{14}$$

For the metric (3), the Gauss-Bonnet invariant G and the Ricci scalar R may be defined as functions of the Hubble parameter

$$R = 2\left((m+2)\frac{\ddot{B}}{B} + \left(1 + m + m^2\right)\left(\frac{\dot{B}}{B}\right)^2\right) = 2\left(3\dot{H} + \frac{m^2 + 2m + 3}{(m+2)^2}H^2\right), \tag{15}$$

and

$$G = \frac{648m}{(m+2)^3}H^2\left(H^2 + \dot{H}\right). \tag{16}$$

Thus Eqs. (12) and (13) take the form

$$-\frac{1}{\kappa^2}\left(3H^2 - \sigma^2\right) - \frac{648m}{(m+2)^3}H^3\dot{f}_G + G f_G - f + \rho_m = 0, \tag{17}$$

$$-\frac{96H}{(m+2)^2}\left[2(1+4m)\dot{H} + \frac{3(m^2+7m+2)}{m+2}H^2\right]\dot{f}_G + \frac{1}{\kappa^2}\left(2\dot{H} + 3H^2 + \sigma^2\right) + \frac{8}{3}\left(3H^2 - \sigma^2\right)\ddot{f}_G - f + G f_G + p_m = 0. \tag{18}$$

The conservation equation law can be expressed as the standard continuity equation

$$\dot{\rho_m} + \theta(\rho_m + p_m) = 0, \tag{19}$$

where θ is the volume expansion which defines a scale factor $a(t)$ along the fluid flow lines via the standard relation $\theta = 3H$. We often consider the case that ρ_m and p_m satisfy the simple EoS, $p_m = \omega\rho_m$. Then if ω is a constant, Eq. (19) can be easily integrated as

$$\rho_m = \rho_0 a^{-3(1+\omega)}. \tag{20}$$

3 Exact matter dominant and accelerating power law solutions

Let us now assume there exists an exact power-law solution to the field equations, i.e., the scale factor behaves as: $B(t) = B_0 t^h$, where h is a fixed real number. If $0 < h < 1$, then the requisite power-law solution is decelerating, while for $h > 1$ it is accelerating. Since we know that within the standard paradigm, the expansion history of the Universe underwent a power-law decelerating phase, it is important to study these kinds of exact solutions in our modified gravity models. Hubble parameter H, time-derivative Hubble parameter $\dot{H}$ and shear scalar σ, for this model are (Hossienkhani and Pasqua 2014; Saaidi and Hossienkhani 2011; Fayaz et al. 2012, 2013; Fayaz and Hossienkhani 2013)

$$H = \frac{h(m+2)}{3t}, \qquad \dot{H} = -\frac{h(m+2)}{3t^2}, \qquad \sigma^2 = \frac{h^2(1-m)^2}{3t^2}, \tag{21}$$

we see that at the beginning of the universe e.g., about Planck time $t_p = 10^{-43}$ s σ^2 is very larger of one in the late time e.g., the proceeding inflation, the particle horizon is smaller than the Hubble length, this indicates the Universe is very anisotropic at the early time and when time is growing the shear tensor tends to zero. Hence, it is expected, the anisotropic parameter is disappeared and the universe tends to be isotropic at late time. From the energy conservation equation, we obtain

$$\rho_m = \rho_0 t^{-h(m+2)(1+\omega)}, \tag{22}$$

and the Ricci scalar and the Gauss-Bonnet term becomes

$$R = \frac{2h}{t^2}\left[h\left(m^2+2m+3\right) - (m+2)\right], \tag{23}$$

$$G = \frac{8mh^3(h(m+2)-3)}{t^4} = \chi_{hm} t^{-4}. \tag{24}$$

The negative sign of G for all decelerating models is reflected by $\chi_{hm} < 0$ for the power-law models with $0 < h < 1$. By substituting (22) and (24) into (17) we obtain the BI equation

$$\alpha G^2 f_{GG} + G f_G - f - \gamma G^{\frac{1}{2}} + \rho_0\left(\frac{G}{\chi_{hm}}\right)^{\epsilon} = 0, \tag{25}$$

where

$$\alpha = \frac{96mh^3}{\chi_{hm}}, \qquad \gamma = \frac{(1+2m)h^2}{\kappa^2\sqrt{\chi_{hm}}}, \qquad \epsilon = \frac{h(m+2)(1+\omega)}{4}. \tag{26}$$

Note that for the power-law solution $a(t) = a_0 t^{\frac{h(m+2)}{3}}$, G/χ_{hm} is positive all the time by definition (24), and therefore Eq. (25) is real-value over the range of G. Solving Eq. (25) get the $f(G)$-gravity as

$$f(G) = -\frac{1+2m}{3+h(m+2)}\sqrt{\frac{h(-3+h(m+2))}{2m\kappa^4}}G^{\frac{1}{2}} + C_1 G + C_2 G^{-\frac{1}{\alpha}} - A_{hm\omega}G^{\epsilon}, \tag{27}$$

where

$$A_{hm\omega} = \frac{4\rho_0(8mh^3)^{-\epsilon}((h(m+2)-3))^{1-\epsilon}}{[-4+h(m+2)(1+\omega)][-3+h(m+2)(4+3\omega)]}, \tag{28}$$

and C_1 and C_2 are arbitrary constants of integration. This solution is in agreement with the one obtained in Goheer et al. (2009) and, as is explained there, we can without any lost of generality assume the constants $C_1 = C_1 = 0$. Hence, the required form of the function $f(G)$ becomes

$$f(G) = -\frac{1+2m}{3+h(m+2)}\sqrt{\frac{h}{2m\kappa^4}\left(-3+h(m+2)\right)}G^{\frac{1}{2}} - A_{hm\omega}G^{\epsilon}. \tag{29}$$

We note that the above form of f identically satisfies the other field equations, if we similarly transform them as differential equations in G space. First, we note that a real-value solution for $f(G)$ requires the values h and m. It has been seen that isotropic universe, $\sigma = 0$ or $m = 1$, Eq. (29) reduces to

$$f(G) = -\frac{1}{1+h}\sqrt{\frac{3h}{2\kappa^4}(-1+h)}G^{\frac{1}{2}} - A_{h\omega}G^{\frac{3}{4}h(1+\omega)}, \tag{30}$$

$$A_{h\omega} = \frac{\rho_0(h-1)(24h^3(h-1))^{-\frac{3}{4}h(1+\omega)}}{1+h[\frac{13}{4}+\frac{9}{4}\omega+3h(1+\omega)(1+\frac{3}{4}\omega)]}. \tag{31}$$

The case $h = 1$ and $m = 1$ leads to $G = 0$ and $R = 6/t^2$ which is the general relativity limit. The coefficient $A_{hm\omega}$ are real-values and non-zero unless $h = 1$ and $m = 1$, in which case $a(t) \propto t$. In general, the function $f(G)$ is real-value only if $G/\chi_{hm} > 0$, which is satisfied based on the exact power law solution. According to (22), it is readily seen by taking $h = 2/[(m+2)(1+\omega)]$ the $f(G)$ reduce to general relativity, and taking $h = 1$ and $m = 1$ the equation of state parameter is fixed by $\omega = -1/3$ which accounts for a negative pressure but not still an accelerating Universe. The

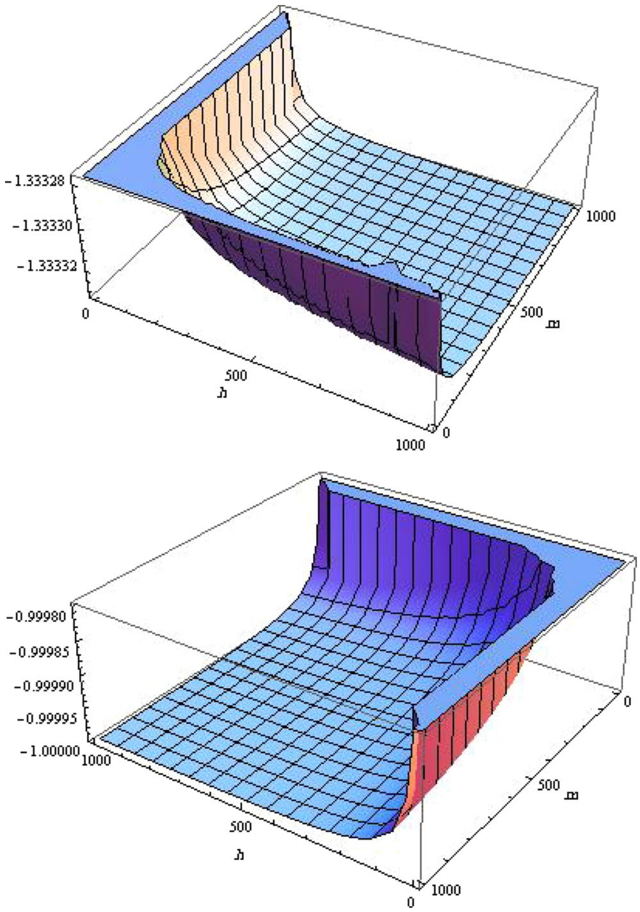

Fig. 1 The evolution of the EoS parameter in $f(G)$-gravity model, Eq. (32), versus h and m

case $h > 1$, leads to a nonzero real Gauss-Bonnet term G and a positive Ricci scalar R. However, in order to avoid divergence in the Gauss-Bonnet term we have to keep h, m and ω far away from the values in which $A_{hm\omega}$ diverges according to the following equation

$$\left[-4+h(m+2)(1+\omega)\right]\left[-3+h(m+2)(4+3\omega)\right]=0 \quad (32)$$

In this case, taking $h > 1$ and $m > 1$ predicts an accelerating universe. The evolution of the EoS parameter obtained in (32), is plotted in Fig. 1. Thus, power-law solutions time dependent type $a(t)=a_0 t^{h(m+2)/3}$ or $H=h(m+2)/3t$ held for the actions, that are as $[R+f(G)+L_m]$ with $f(G)$ given by (27) except for its values of h and m which satisfy (32). It is interesting to note that an exact GR-like solution $h=2/[(m+2)(1+\omega)]$ is possible with non-zero C_2 for $f(G)=C_2 G^{\frac{12(1+\omega)}{1+3\omega}}$ for values of ω for which $f(G)$ is real-valued. Using (22) the effective EoS for this model is obtained as

$$\omega_{\rm eff}=-1-\frac{2[-2-m+h(1-m)^2]}{3h(1+2m)}, \quad (33)$$

which is less than -1 if $h > 1$ and $m > 1$, which corresponds to phantom era. In the phantom phase, the Big Rip type singularity at $t=t_s$ might occur. Near the Big Rip singularity, however, the curvature becomes dominant and $f(G)$-term may be neglected. Then the universe expands as $a=a_0 t^{2/(m+2)(1+\omega)}$. Hence, the Big Rip singularity eventually does not occur.

4 Exact phantom phase power low solution

One may also study the power law solutions where the universe enters a phantom phase in which equation of state (EoS) is smaller than -1, leading to a Big Rip singularity. For this case, the general set of Hubble parameters, shear scalar and cosmological solutions are defined as (Hossienkhani and Pasqua 2014; Saaidi and Hossienkhani 2011; Fayaz et al. 2012, 2013; Fayaz and Hossienkhani 2013)

$$a(t)=a_0(t_s-t)^{-\frac{h(m+2)}{3}}, \quad t\le t_s, \quad (34)$$

$$H=\frac{h(m+2)}{3(t_s-t)}, \qquad \dot{H}=\frac{h(m+2)}{3(t_s-t)^2}, \quad (35)$$

$$\sigma^2=\frac{1}{3}\left(\frac{h(1-m)}{(t_s-t)}\right)^2. \quad (36)$$

Considering the era when phantom phase is dominated i.e., $\omega<-1$, and the smaller redshift, e.g., $z\simeq 0.25$. It is clear Eq. (34) has a Big Rip singularity at $t=t_s$ with $\dot{H}>0$, is that $a(t)\to\infty$. The model is shown that an escape from the Big Rip is possible on making quantum corrections to energy density ρ and pressure p in Bianchi type I space-time. when $t\to t_s$, so the scale factor will diverge for a future value on the world time: t_s and only $0<t<t_s$ is considered due to the fact that H should be real number. Again, using the above solution and working out the analogous calculations we obtain the following results

$$\rho_m=\rho_0(t_s-t)^{h(m+2)(1+\omega)}, \quad (37)$$

$$G=\frac{8mh^3(h(m+2)+3)}{(t_s-t)^4}=\chi'_{hm}(t_s-t)^{-4}, \quad (38)$$

$$\dot{G}=\frac{32mh^3(h(m+2)+3)}{(t_s-t)^5}=4\chi'_{hm}(t_s-t)^{-5}, \quad (39)$$

$$R=\frac{2h}{(t_s-t)^2}\left[h\left(m^2+2m+3\right)+(m+2)\right]. \quad (40)$$

Substituting (37), (38) and (39) into the first BI equation (17) we obtain

$$-\alpha' G^2 f_{GG}+Gf_G-f-\gamma' G^{\frac{1}{2}}+\rho_0\left(\frac{G}{\chi'_{hm}}\right)^{\epsilon}=0, \quad (41)$$

where

$$\alpha' = \frac{96mh^3}{\chi'_{hm}}, \qquad \gamma' = \frac{(1+2m)h^2}{\sqrt{\chi'_{hm}}}. \tag{42}$$

Therefore its solution is obtained by using the same way in (27) as

$$f(G) = -\frac{1+2m}{3-h(m+2)}\sqrt{\frac{h(3+h(m+2))}{2m\kappa^4}}G^{\frac{1}{2}} + C'_1 G + C_2 G^{\frac{1}{\alpha'}} + A'_{hm\omega}G^{-\epsilon}, \tag{43}$$

where

$$A'_{hm\omega} = \frac{4\rho_0(8mh^3)^{\epsilon}[(h(m+2)+3)]^{1+\epsilon}}{[4+h(m+2)(1+\omega)][3+h(m+2)(4+3\omega)]}. \tag{44}$$

Similar to the solutions in the previous section, we assume $C'_1 = C'_2 = 0$. Then, the required form of the function $f(G)$ becomes

$$f(G) = -\frac{1+2m}{3-h(m+2)}\sqrt{\frac{h(3+h(m+2))}{2m\kappa^4}}G^{\frac{1}{2}} + A'_{hm\omega}G^{-\epsilon}. \tag{45}$$

Actually, $h > 0$ leads to a real values function $f(G)$ according to (43). However, demanding a Big Rip that may be during the phantom phase, as the cosmic time t approaches t_s, requires $h \geq 1$ in (34). However, $h = 1$ and $m = 1$ causes a divergence in $f(G)$ through the first term of (43) in the bracket. Moreover, the Gauss-Bonnet term diverges through $A'_{hm\omega}$ for its value of m which the following equation is satisfied

$$\big(4+h(m+2)(1+\omega)\big)\big(3+h(m+2)(4+3\omega)\big) = 0. \tag{46}$$

For $h > 1$ and $m > 1$ one can find the EoS parameter (46) corresponds to the phantom phase regime in which $\omega < -1$. The evolution of the EoS parameter obtained in Eq. (46), is plotted in Fig. 2. Therefore, power-law solutions in the phantom phase of the type $a(t) = a_0(t_s - t)^{-h(m+2)/3}$ exist for the actions, that is as $[R + f(G) + L_m]$ with $f(G)$ given by (43) except for its values of h and m which satisfy (46).

5 De Sitter solutions

de Sitter solutions are well known in the context of cosmology because the current epoch, where in the Universe expansion is being accelerated, can be described approximately with a de Sitter solution. This kind of solution consists of an exponential expansion of the scale factor, which yields a constant Hubble parameter. This kind of solutions consists on an exponential expansion of the scale factor, which yields a constant Hubble parameter. Astrophysical data indicate that ω lies in a very narrow strip close to $\omega \to -1$. In the case of Bianchi type-I and Kantowski-Sachs metrics ($B = C$) in (3), we may assume an exponential expansion for each spatial direction,

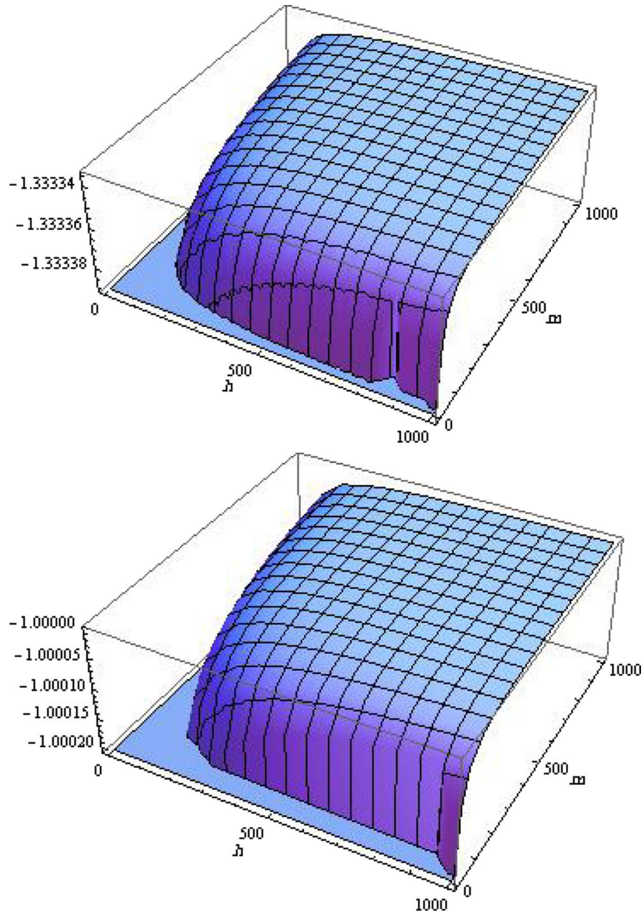

Fig. 2 The evolution of the EoS parameter, Eq. (46), versus h and m

$$A(t) = A_0 e^{\Lambda m t}, \qquad C(t) = B(t) = B_0 e^{\Lambda t}, \tag{47}$$

and the rates of the expansion for each direction can be defined as,

$$H_x = \frac{\dot{A}}{A} = H_{x0}, \qquad H_y = \frac{\dot{B}}{B} = H_{y0}, \qquad H_z = \frac{\dot{C}}{C} = H_{z0}, \tag{48}$$

where $H_{x0} = m\Lambda$ and $H_{y0} = H_{z0} = \Lambda$ are constants. The Gauss-Bonnet invariant defined in (11) is given by,

$$G = H_{x0}^4 + 2H_{y0}^4 + H_{y0}^2 + 2H_{x0}H_{y0}. \tag{49}$$

Then, by assuming $p_x = p_y = p_z = p$ and an equation of state $p = \omega\rho$, the conservation equation can be easily solved for the ansatz (47),

$$\rho_m = \rho_0 e^{(H_{x0}+2H_{y0})(1+\omega)t}. \tag{50}$$

Hence, the field equations (12) become,

$$Gf_G - f(G) - \frac{1}{\kappa^2}\left(H_{y0}^2 + 2H_{x0}H_{y0}\right) + \rho_0 e^{(H_{x0}+2H_{y0})(1+\omega)t} = 0. \tag{51}$$

Note that the only possible solution in the presence of a perfect fluid is one with $\omega = -1$ as the r.h.s. of Eq. (51) is independent of time, according to the expression of the Gauss-Bonnet invariant for a pure de Sitter solution (49), unless $H_{x0} + 2H_{y0} = 0$, which would imply a decelerating expansion in a particular direction, being $H_{i0} < 0$. Moreover, for a particular $f(G)$ action, the system of Eq. (51) reduces to an algebraic system of equations for the variables H_{x0}, H_{y0}, H_{z0}. In the case of BI, (3) and (47), we consider the following form of the scale factor:

$$a = a_0 e^{\frac{\Lambda(2+m)}{3}}, \tag{52}$$

where a_0 is a constant parameter indicating the present day value of $a(t)$, i.e. the value of the scale factor for $t = 0$.

Using the de Sitter scale factor and Eq. (14), we get:

$$H = H_0 = \frac{\Lambda(2+m)}{3}, \qquad \dot{H} = 0, \qquad \sigma^2 = \frac{\Lambda^2(1-m)^2}{3}. \tag{53}$$

For large $H_0 t$ the general solution approaches the isotropic de Sitter solution. This solution describes a Universe model entering the inflationary era at $t = (2H_0)^{-1} = 10^{-35}$ sec as an anisotropic Universe, and terminating this era at $t = 1.3 \times 10^{-33}$ sec as an isotropic de Sitter Universe. The mean expansion anisotropy has decreased during the inflationary era. So, can concluded that a Universe with a large amount of anisotropy will not undergo the inflationary phase. A Universe with only moderate anisotropy will undergo inflation and will be rapidly isotropized. We now want to evaluate the form of $f(G)$ for each of the scale factor mentioned above in the Bianchi type I models. From Eq. (8), we observe that $\omega_{\text{eff}} = -1 - \frac{2}{3}\frac{1+m^2-2m}{1+2m}$ for sufficiently large time t. Note that in the same way, one can construct $f(G)$ action describing other epoch remembering that form of modified gravity is different for different epochs (the inflationary epoch action is different from the form at late-time universe). Moreover, we can determine the expressions of ρ_m, R and G, respectively, as:

$$\rho_m = \rho_0 e^{\Lambda(m+2)(1+\omega)t}, \tag{54}$$

$$R = 2\Lambda^2\left(m^2 + 2m + 3\right), \tag{55}$$

$$G = 8m(m+2)\Lambda^4. \tag{56}$$

Note that the only possible solution in the presence of a perfect fluid is one with $\omega = -1$ as Eq. (54) is independent of time, so, ρ_m is a constant. Substituting Eqs. (53), (54) and (56) into (51), we obtain:

$$Gf_G - f(G) - \frac{(1+2m)}{\kappa^2\sqrt{8m}(m+2)}G^{\frac{1}{2}} = 0, \tag{57}$$

the solution of Eq. (57) is

$$f(G) = C_1 G - \frac{(1+2m)}{\kappa^2\sqrt{2m}(m+2)}\sqrt{G}, \tag{58}$$

where C_1 is integrating constant. Thus, we see that models corresponding to the inflation and the late time accelerated Universe can be reconstructed within KS metrics where the matter content is partially isotropic ($p_x = p_y = p_z$).

6 The stability issue

The stability issue of a large class of modified gravitational models has been discussed with particular emphasis to de Sitter solutions (Capozziello et al. 2006; Cognola et al. 2005, 2008; Cognola and Zerbini 2006; Faraoni 2005a, 2005b). When $\rho_m = 0$, Eq. (12) has a de Sitter Universe solution where H and therefore G are constants. If $H = H_0$ with constant H_0, Eq. (12) looks as

$$G_0 f_0' - f_0 = 3H_0^2 - \sigma_0^2, \tag{59}$$

it has been seen that isotropic Universe, $\sigma_0 = 0$ or $m = 1$, (59) reduces to (Nojiri and Odintsov 2005)

$$G_0 f_0' - f_0 = 3H_0^2. \tag{60}$$

For a large number of choices of the function $f(G)$, Eq. (59) has a non-trivial ($H_0 \neq 0$) real solution for H_0 (de Sitter Universe). The stability issue leads to the following condition (Cognola et al. 2005, 2008; Cognola and Zerbini 2006)

$$R_0^3 f_0'' < 9, \tag{61}$$

where the critical points are defined by

$$R_0 = \frac{18(m^2+2m+3)}{(m+2)^2}H_0^2, \qquad \sigma_0 = \frac{3(1-m)^2}{(m+2)^2}H_0^2, \qquad G_0 = \frac{648m}{(m+2)^3}H_0^4, \tag{62}$$

in the case of phantom phase power law solution, namely (45), the first condition reads as

$$A_{hm\omega}\left(\frac{1+\frac{1}{4}h(m+2)(1+\omega)}{1+2m}\right)G^{-\frac{1}{4}h(m+2)(1+\omega)-\frac{1}{2}} + \frac{1}{2(3-h(m+2))}\sqrt{\frac{h(3+h(m+2))}{2m\kappa^4}} = \sqrt{\frac{1}{8m(m+2)}}, \tag{63}$$

which implies that

$$A_{hm\omega}\left(1+\frac{1}{4}h(m+2)(1+\omega)\right) > 0. \tag{64}$$

By using (64), the second condition (61) reads as

$$-\frac{h(m+2)(1+2m)(1+\omega)(m^2+2m+3)^3}{8(2m(m+2))^{\frac{3}{2}}} \times\left(\frac{1}{\sqrt{2m(m+2)}}-\frac{\sqrt{\frac{h(3+h(m+2))}{2m\kappa^4}}}{3-h(m+2)}\right) + \frac{(1+2m)(m^2+2m+3)^3}{16m^2(3-h(m+2))(m+2)^{\frac{3}{2}}} \times\sqrt{h\big(3+h(m+2)\big)} < 9. \tag{65}$$

Then, the model is stable around de Sitter solution if the arbitrary parameters also satisfy both the conditions (64) and (65).

7 Conclusion

The expansion history of the Universe is thought to have undergone a phase of decelerated power-law expansion followed by late time acceleration. Therefore, power-law solutions play an important role in cosmology as matter dominated phases that later connect to an accelerating phase.

We have considered an $f(G)$ action which describes Einstein's gravity added with a function of the Gauss-Bonnet term in Bianchi type I space-time. A $f(G)$ function corresponding with power law solutions for given scale factor are calculated. Then, it was shown that exact power-law solutions in $f(G)$ gravity only exist for the very special class of models given in (29) and (45). We have derived the gravitational field equations for perfect fluid corresponding to $f(G)$ gravity model. It have been considered three specific $f(G)$ models which are important to describe the late time cosmic acceleration and avoid from the finite time future singularities. The First model have been obtained the field equations using the power-law solution of the type $a = a_0 t^{h(m+2)/3}$. In the second model we have studied, the Universe enters a phantom phase, is that, EoS, $\omega < -1$, given to, the power-law solutions. It is shown that the power-law solution in the phantom phase, with scale factor as $a(t) = a_0(t_s - t)^{-h(m+2)/3}$ there exists for this $f(G)$ except for its values of h and m in which function diverges. In dS solutions, where the scale factor is an exponential function of the cosmic time, has been considered for Bianchi type-I metric by imposing a particular exponential expansion in each direction of the space. We have shown that the only possible solution turns out to the FLRW metric, such that no possible dS anisotropic evolution can be found in $f(G)$, unless one considers an anisotropic fluid. In the case, we shown that the model can realize the early accelerated Universe, characterized by the inflation, and the late time acceleration of our current Universe. The stability issue was studied in $f(G)$ gravity and it was concluded that the model is stable around de Sitter solution. It is interesting to mentioned here that for $\sigma = 0$ or $m = 1$, the results reduce to FRW Universe model.

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