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ORIGINAL ARTICLE

Astrophysical constraints and insights on extended relativistic gravity

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Abstract We give precise details to support that observations of gravitational lensing at scales of individual, groups and clusters of galaxies can be understood in terms of non-Newtonian gravitational interactions with a relativistic structure compatible with the Einstein Equivalence Principle. This result is derived on very general grounds without knowing the underlying structure of the gravitational field equations. As such, any developed gravitational theory built to deal with these astrophysical scales needs to reproduce the obtained results of this article.

Keywords Gravitation · Relativistic processes · Gravitational lensing: weak

1 Introduction

The first solid step towards a full development of a non-relativistic theory of gravity was made by Newton in his *PhilosophiæNaturalis Principia Mathematica* book (Newton 1729). The starting point of this non-relativistic theory of gravity began with the third law of planetary motion published by Kepler in his *Harmonices Mundi* book (Kepler et al. 1619). For the known 7 planets back then, this law represents a relation between the mass of the sun M, a planet's particular distance to the sun r and the velocity v of a planet about the sun: $v \propto (M/r)^{1/2}$, for circular orbits. The requirement of centripetal balance during the motion of planets yields:

$$a = -v^2/r = -GM/r^2, \tag{1}$$

where the proportionality factor G is Newton's gravitational constant and the minus sign appears because of the attractive nature of gravity. The acceleration a produced by the sun on a test planet is thus given by a force inversely proportional to its separation from it and linearly depends on the sun's mass. The right hand side of (1) is the simplest form of the mathematical force of gravity introduced by Newton.

In recent years, through dynamical observations of galaxies (e.g. Famaey and McGaugh 2012, and references therein), dwarf spheroidal galaxies (cf. Hernandez et al. 2010), globular clusters (Hernandez and Jiménez 2012; Hernandez et al. 2013) and even wide open binaries (Hernandez et al. 2012), it has became clear that Kepler's third law appears not to hold on all scales in these systems, but rather requires a modification known as the Tully-Fisher law:

$$v \propto M^{1/4}, \tag{2}$$

where v represents the velocity (or dispersion velocity for a dynamically pressure supported astrophysical system) and M is the mass (could be internal mass within a radius r) of the system. Similarly to Newton's approach, the requirement of centripetal balance means that the acceleration $a \propto v^2/r$ at a distance r from the configuration's center and so

$$a = -G_{\mathrm{M}} \frac{M^{1/2}}{r}, \tag{3}$$

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where the constant of proportionality has been written as G_{M} and the minus sign has been introduced in order to manifest the attractive nature of the gravitational force. Equation (3) can be seen as a motivation to suspect that a new theory of gravity needs to be developed in these astrophysical systems, since its right hand side represents a relation between the acceleration felt by a test body of mass M at a distance r. In this sense, the proportionality constant G_{M} can be seen as a new gravitational constant, with dimensions of squared length over squared time by the square root of mass.

This means that, in the same way as G is regarded as a fundamental constant of nature, G_{M} should aspire to the same privileged status. However, in order to gain merits in that direction, G_{M} should play an essential role in the description of relativistic phenomena on its corresponding scales. Nonetheless, it should be noted that the construction of (1) and (3) are completely independent, since they both depend on different and unrelated data sets. As such, the constants G and G_{M} can safely be postulated as independent. Given this independence, one is allowed to think of both as equally fundamental.

Requiring gravity to be described by (1) at some particular scales and behaving at some others according to relation (3), means that the scale invariance of gravity is necessarily broken. One can postulate that at some astrophysical scales gravity is Newtonian and requires modification at some others. The scale is not just a "fixed" distance scale. From the astronomical evidence mentioned above, it follows that the modified regime of gravity appears when the mass of a given astrophysical system divided by its characteristic radius is sufficiently small as compared to the corresponding solar system value, which suggests that the transition scale is dynamical rather than a simple fixed length. A given test particle sufficiently far away from a mass distribution is thus in this modified Kepler's third law of gravity regime.

The approach introduced above for the description of gravitational phenomena departing from standard Newtonian gravity can be connected with the simplest version of the Modified Newtonian Dynamics (MOND) formula by replacing the constant G_{M} with a new constant a_0 introduced by Milgrom (1983) with dimensions of acceleration through the relation:

$$a_0 := G_{\mathrm{M}}^2/G, \tag{4}$$

and so (3) can be written as:

$$a = -\frac{(a_0 G M)^{1/2}}{r}. \tag{5}$$

Since Milgrom's acceleration constant $a_0 \approx 1.2 \times 10^{-10}\ \mathrm{m\,s^{-2}}$ (Famaey and McGaugh 2012) it follows that:

$$G_{\mathrm{M}} \approx 8.94 \times 10^{-11}\ \mathrm{m^2\,s^{-2}\,kg^{-1/2}}. \tag{6}$$

In this regard it is quite important to notice that the formulation of Milgrom (1983) describes a modification on the dynamical sector of Newton's second law and not on the particular form of the gravitational force (cf. Milgrom 2006). This is quite evident from the initial development of the theory, in which the requirement that the squared of the acceleration a^2 proportional to the Newtonian acceleration GM/r^2 implies flattening of rotation curves in spiral galaxies. In this relation, the proportionality constant a_0 with dimensions of acceleration, is required to be a fundamental quantity of nature. By doing so, the Tully-Fisher law is obtained as a consequence of the proposed modification of dynamics.

With the approach made here, it follows that the Tully-Fisher law forces the construction of a full gravitational theory in systems where Newtonian gravity does not work. In its simplest form, the developed theory must converge to (3). As such, no need for modification of Newton's second law needs to be introduced, since only a non-scale invariant character for gravitational interactions is directly inferred from observations.

In our view, the introduction of G_{M} as a fundamental constant of gravity, rather than a_0 as a new fundamental acceleration scale, sheds light onto the strategy to follow to unveil the structure of the underlying theory. In fact, G_{M} points towards a modification on the gravitational sector, whereas a_0 could point towards a break down or possible extensions of special relativity (due to the existence of a universal acceleration scale, similarly as with the speed of light), with potentially dramatic implications even in non-gravitational systems.

If Newtonian gravity breaks at a certain scale, one can legitimately wonder whether the relativistic structure of gravitational interactions remains valid or may require a full reformulation. This is the key point to be addressed in this work. To explore these aspects one should study not only the dynamics of slow massive particles (e.g. (3)) but also the motion of relativistic particles (such as photons) in astrophysical scenarios probing the gravitational field in this new regime with very weak interactions. Being conservative, one may assume that Einstein's insights on the geometrical interpretation of gravity remain valid in this regime and therefore, test particles satisfy the geodesic equation:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0, \tag{7}$$

where $\Gamma^\alpha_{\beta\eta}$ are the Christoffel symbols and summation convention is used over repeated indices (Greek indices vary from 0 to 3 and Latin ones from 1 to 3). The coordinates $x^\alpha = (ct, x, y, z)$ and the interval $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ for a metric tensor $g_{\mu\nu}$ and a velocity of light c.

By knowing Tully-Fisher's modification of Kepler's third law (2) and the geodesic equation (7), one finds that at second order perturbation, the bending of light is completely determined up to a constant (Will 1993), which must be determined observationally. This is due to the fact that, to this order of approximation, coordinates can be chosen such that the motion of photons only depends on the non-relativistic gravitational potential and a parameter $\gamma = \text{const.}$, which measures the proportionality between the leading (second) order corrections of the time and spatial metric components in isotropic coordinates. Note that the motion of massive objects in this regime should, in general, be subject to additional relativistic potentials which, however, do not affect the motion of photons. This property will be exploited in this work to extract useful information on the relativistic structure of gravitation in the very weak, modified non-Newtonian regime.

In this article, therefore, we postulate the Einstein Equivalence Principle to be valid and combine it with the non-relativistic gravitational potential associated to the modified Kepler's third law (Tully-Fisher law). This approach only assumes that gravitation is a geometric phenomenon and, as such, we do not need to know the full details of the underlying relativistic field equations to work out the corresponding predictions for the bending of light. Our approach parallels the strategy followed to understand the relativistic behavior of gravity in the solar system, where traditional Kepler's third law holds.

The article is organised as follows. In Sect. 2 we review the basics of the weak-field, slow-motion regime for massive particles. In Sect. 3 we use gravitational lensing observations in individual, groups and clusters of galaxies combined with the Tully-Fisher law to obtain empirical relations for the metric coefficients of a spherically symmetric space-time at second perturbation order. Finally in Sect. 4 we summarise our main results and discuss them in lights of future theoretical developments in the search for a complete extended theory of gravity not requiring the use of dark matter in the description of astrophysical phenomena

2 Basics of the weak-field, slow-motion regime

In order to understand how the motion of photons can be described when Kepler's third law is adapted to fit the Tully-Fisher law, we must first study the action of gravity on massive particles in the weak field limit. For this purpose, consider a fixed point mass M at the center of coordinates generating a gravitational field. The underlying space-time is thus static and its spherically symmetric line element $\mathrm{d}s$ can be written in spherical Schwarzschild coordinates as:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{00}\, c^2\mathrm{d}t^2 + g_{11}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{8}$$

In the previous equation and in what follows the space-time coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$, where t represents time, r the radial coordinate and the polar and azimuthal angles are given by θ and φ respectively, with an angular displacement $\mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. The symmetry of the problem means that the unknown metric components g_{00} and g_{11} are functions that depend on the radial coordinate r only. For non-relativistic particles, we must consider the limit where the speed of light $c \to \infty$. In this case, the radial component of the geodesic equation (7) is given by:

$$\frac{1}{c^2}\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{1}{2} g^{11}\frac{\partial g_{00}}{\partial r}. \tag{9}$$

The previous relation holds since we have used the following approximation for the weak field limit: $\mathrm{d}s = c\,\mathrm{d}t$, and so, due to the fact that the velocity $v \ll c$ then $v^i \ll \mathrm{d}x^0/\mathrm{d}t$ with $v^i := (\mathrm{d}r/\mathrm{d}t,\ r\mathrm{d}\theta/\mathrm{d}t,\ r\sin\theta\,\mathrm{d}\varphi/\mathrm{d}t)$.

The lowest perturbation order of (9) is obtained when its left-hand side is of order v^2/c^2 and when its right-hand side is of order $g_{00} = 1 + \phi/c^2$ (cf. Will 1993), where ϕ is the non-relativistic gravitational potential. Both are orders $\mathcal{O}(1/c^2)$ of the underlying theory, or simply $\mathcal{O}(2)$.

In this weak-field slow-motion approximation, a particle bounded to a circular orbit about the mass M experiences a centrifugal radial acceleration given by:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{v^2}{r}, \tag{10}$$

for a circular tangential velocity v.

The motion of material and light particles at this lowest perturbation order is such that the metric components are given by (Will 1993, 2006):

$$\begin{aligned}
g_{00} &= {}^{(0)}g_{00} + {}^{(2)}g_{00} + \mathcal{O}(4) = 1 + {}^{(2)}g_{00} + \mathcal{O}(4),\\
g_{11} &= {}^{(0)}g_{11} + {}^{(2)}g_{11} + \mathcal{O}(4) = -1 + {}^{(2)}g_{11} + \mathcal{O}(4),\\
g_{22} &= -\, r^2,\\
g_{33} &= -\, r^2\sin^2\theta,
\end{aligned} \tag{11}$$

where the superscript (p) denotes the order $\mathcal{O}(p)$ at which a particular quantity is approximated. The non-relativistic potential ϕ is defined as (e.g. Landau and Lifshitz 1975; Will 1993; Misner et al. 1973):

$${}^{(2)}g_{00} = \frac{2\phi}{c^2}. \tag{12}$$

From (11) it follows that the contravariant metric components are given by:

$$\begin{aligned}
g^{00} &= {}^{(0)}g^{00} + {}^{(2)}g^{00} + \mathcal{O}(4) = 1 - {}^{(2)}g_{00} + \mathcal{O}(4),\\
g^{11} &= {}^{(0)}g^{11} + {}^{(2)}g^{11} + \mathcal{O}(4) = -1 - {}^{(2)}g_{11} + \mathcal{O}(4),\\
g^{22} &= -\, 1/r^2,\\
g^{33} &= -\, 1/r^2\sin^2\theta.
\end{aligned} \tag{13}$$

At this level of approximation, the motion of non-relativistic massive particles only requires knowledge of the metric component ${}^{(2)}g_{00}$. The motion of photons is fully determined by additionally knowing ${}^{(2)}g_{11}$ (cf. Will 1993).

3 Tully-Fisher's relativistic corrections

Let us take the radial component (9) of the geodesic equations (7) at the lowest relativistic perturbation order. In this limit, the rotation curve for test particles bound to a circular orbit about a mass M with circular velocity v is given by (10) and so:

$$\frac{v^2}{c^2 r} = \frac{1}{2}\frac{\partial\,{}^{(2)}g_{00}}{\partial r}. \tag{14}$$

Since we are interested in the behavior of particles where the modified Kepler's third law (or Tully-Fisher law) holds, (2) can be written as:

$$v = G_{\mathrm{M}}^{1/2}\, M^{1/4}. \tag{15}$$

Substitution of this equation on relation (14) yields:

$$\frac{\partial\,{}^{(2)}g_{00}}{\partial r} = -\frac{2}{r}\left(\frac{v}{c}\right)^2 = -\frac{2G_{\mathrm{M}}M^{1/2}}{c^2 r}, \tag{16}$$

which has a direct analytical solution:

$${}^{(2)}g_{00} = -2\left(\frac{v}{c}\right)^2 \ln\left(\frac{r}{r_\star}\right) = -\frac{2G_{\mathrm{M}}M^{1/2}}{c^2}\ln\left(\frac{r}{r_\star}\right), \tag{17}$$

where $r_\star$ is an arbitrary length.

Having obtained the ${}^{(2)}g_{00}$ component, which determines the non-relativistic motion of massive particles, we now proceed to obtain the ${}^{(2)}g_{11}$ component. It is customary to define a new scalar potential ψ as:

$${}^{(2)}g_{11} = -\frac{2\psi}{c^2}, \tag{18}$$

in complete analogy with (12). The introduction of this potential can be justified considering a more general scenario. Without requiring spherical symmetry, the spatial part of the metric can be written as $g_{ik}\mathrm{d}x^i\mathrm{d}x^k$, with ${}^{(0)}g_{kl} = \delta_{kl}$ being the Minkowskian part. The second order perturbation corrections of g_{kl} could in principle involve other potentials (and not only ϕ or ψ). By a suitable choice of coordinates, one can get rid of the anisotropic contributions at the same perturbation order, which turns g_{kl} into a diagonal form. Given the isotropy of space, there is no preferred direction and so ${}^{(2)}g_{ik} \propto \delta_{ik}$. It is natural to expect that the leading order $\mathcal{O}(2)$ correction must be of the same order of magnitude as the gravitational potential ϕ. Accordingly $g_{kl} = (1 + 2\gamma\phi/c^2)\delta_{kl}$, where γ is a proportionality constant, and so

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t - \left(1 + 2\gamma\phi/c^2\right)\delta_{kl}\mathrm{d}x^k\mathrm{d}x^l. \tag{19}$$

Since spherical Schwarzschild coordinates are widely used in astrophysical literature, let us calculate the metric component g_{11} in such coordinates. The conversion is straightforward since

$$g_{11}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 = \left(1 + 2\gamma\phi/c^2\right)\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\right), \tag{20}$$

for spherical isotropic coordinates $(ct, \tilde{r}, \theta, \varphi)$.

Using (17) and (12) it follows that

$$r = \tilde{r}\left[1 - \gamma\left(\frac{G_{\mathrm{M}}M^{1/2}}{c^2}\right)\ln\left(\frac{r}{r_\star}\right)\right], \tag{21}$$

and so,

$$\mathrm{d}r = \mathrm{d}\tilde{r}\left[1 - \frac{G_{\mathrm{M}}M^{1/2}}{c^2}\ln\left(\frac{\tilde{r}}{r_\star}\right) - \frac{G_{\mathrm{M}}M^{1/2}}{c^2}\right], \tag{22}$$

at perturbation order $\mathcal{O}(2)$. This means that:

$$\psi = -\gamma G_{\mathrm{M}}M^{1/2}, \tag{23}$$

which yields:

$${}^{(2)}g_{11} = -\frac{2\gamma G_{\mathrm{M}}M^{1/2}}{c^2}. \tag{24}$$

As pointed out by Mendoza et al. (2013), recent observations have shown that gravitational lensing on individual (Gavazzi et al. 2007; Koopmans et al. 2006; Barnabè et al. 2011; Suyu et al. 2012; Dutton et al. 2011), groups (More et al. 2012) and clusters of galaxies (Newman et al. 2009; Limousin et al. 2007) can be modelled with the standard Schwarzschild solution of general relativity, assuming the existence of a total dark plus baryonic isothermal halo, where the Tully-Fisher law holds for the baryonic matter. In this dark matter scenario, the bending angle of light can be calculated using the standard lensing equation, finding that it does not depend on the impact parameter and scales with the square root of the total baryonic mass. This bending angle turns out to be an observational constraint and must have the same value in any theory of gravity. For the case of the Tully-Fisher relativistic extension we are dealing with, the lens equation will depend on the ${}^{(2)}g_{00}$ metric component given by (17) and the ${}^{(2)}g_{11}$ unknown metric component. Since the bending angle is already known, by perturbing the lens equation and using the fact that the bending angle is independent of the impact parameter, one finds that the potential ψ is given by (see Mendoza et al. 2013):

$$\psi = -G_{\mathrm{M}}M^{1/2}. \tag{25}$$

Comparison of this result with relation (23) yields:

$$\gamma = 1. \tag{26}$$

We thus conclude that the relativistic structure of the underlying theory of gravity at scales of individual, groups, and cluster of galaxies is compatible with that found in the solar system, where $\gamma = 1$. The difference however, lies on the fact that the gravitational potential ϕ appearing in (19) is not the one associated with Kepler's third law, but the one inferred from the Tully-Fisher law.

4 Discussion

A number of independent astrophysical observations strongly support the view that the scale invariance of Newtonian gravity breaks down at sufficiently large scales that depend on the mass and characteristic sizes of the systems involved (see e.g. Famaey and McGaugh 2012; Hernandez et al. 2010, 2012; Hernandez and Jiménez 2012, and references within). Furthermore, it has recently being noticed by Hernandez et al. (2014) the existence of some astrophysical systems where Newtonian gravity fails to explain the observed phenomenology (usually requiring a Tully-Fisher dynamical scaling). These systems are impossible to account under the standard dark matter paradigm. As such, a relativistic exploration of gravitational phenomena on those regimes is quite important to develop, and has been our main motivation in this article.

In the non-relativistic weak field limit of approximation, the behavior of gravity is Newtonian, and the full relativistic theory that describes objects in this regime is general relativity. At the very weak field limit of approximation, sufficiently far from the masses that produce the gravitational field, Kepler's third law is modified through the Tully-Fisher law and the underlying relativistic theory in this regime is so far unknown.

We have explored some relativistic properties of this modified regime of gravity at the weak field limit of approximation, assuming that gravity is a geometrical phenomenon and that the Einstein Equivalence Principle holds. This is sufficient to build a model independent approach of the relativistic regime at second perturbation order $\mathcal{O}(2)$, in complete analogy to the one used at solar system scales where the dynamics are compatible with Einstein's general relativity. Using this modified Kepler's third law and lensing observations for individual, groups, and clusters of galaxies we have shown that this relativistic approach is in excellent agreement with observations. In isotropic coordinates, the non-relativistic gravitational potentials ϕ and ψ defined in (12) and (23) are proportional to each other, i.e. $\phi \propto \gamma\psi$, with γ the Parameterised Post-Newtonian parameter in this new regime of gravity where Kepler's third law does not hold. As shown in this article, lensing observations require $\gamma = 1$, the same exact value required by general relativity based on Kepler's third law. Note that in spherical Schwarzschild coordinates both gravitational scalar potentials differ from each other as is evident from (17) and (25). For the case of Einstein's general relativity these potentials are coincidentally equal not only in isotropic coordinates, but also in spherical Schwarzschild coordinates.

As shown in this article, the bending of light in regions where the Tully-Fisher law is satisfied can be predicted at second order perturbation without knowledge of the underlying relativistic theory of gravity. This is fully consistent with Einstein's view on the geometrical nature of space-time and relativistic motion. Any viable extended relativistic theory of gravity should be in agreement with the light bending predictions discussed here. The proposal by Bernal et al. (2011) with lensing applications detailed in Mendoza et al. (2013) is an example of such kind of theory.

Before concluding, we would like to note that the results by Hernandez et al. (2012) on the failure of Kepler's third law for wide binary systems can be interpreted as a way to test a key aspect of the mathematical structure of the underlying theory of gravity, namely whether or not external boundary conditions influence the dynamics of local gravitating systems whose internal motions are non-Newtonian, which is sometimes referred to as an *external field effect* (Famaey and McGaugh 2012). This effect means that for example, a gravitating system in the modified Keplerian regime embedded on an external standard Newtonian (or Keplerian) field, would behave in a Newtonian way. Hernandez et al. (2012) studied orbits of wide binary stars $\sim 1\,\mathrm{M}_\odot$ separated by $\gtrsim 7000$ AU. These bound objects are embedded in our galaxy and are subject to its Newtonian gravity. As such, if an external field effect occurs, then these objects would orbit each other in a standard way, following Kepler's third law. However, their analysis shows that a violation of Kepler's third law occurs in these systems. The large statistics and precise astrometry to be obtained with the GAIA probe of the European Space Agency in the near future, should provide a strong test for the validity of Kepler's third law at scales yet to be explored. Furthermore, we have shown that lensing observations strongly support the validity of (7), implying that the effects of external gravitational fields can be removed by a suitable choice of local coordinates (a freely falling frame). To the light of these results, the idea of an external field effect appears as an artificial construction, possibly related to the specific mathematical realisation of particular models.

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