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LETTER

Vector particles tunneling from four-dimensional Schwarzschild black holes

Ge-Rui Chen[1] · Shiwei Zhou[1] · Yong-Chang Huang[1]

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Abstract Vector particles' Hawking radiation from a four-dimensional Schwarzschild black hole is investigated. By applying the WKB approximation and the Hamilton-Jacobi ansatz to the Proca equation, we obtain the tunneling spectrum of vector particles and the expected Hawking temperature.

Keywords Vector particles tunneling · Four-dimensional Schwarzschild black hole · Proca equation

1 Introduction

When Stephen Hawking found Hawking radiation (Hawking 1974, 1975), he described it as a tunneling process triggered by vacuum fluctuations near the horizon. When a virtual particle pair is created inside the horizon, the positive energy virtual particle can tunnel out, and materializes as a real particle, appearing as Hawking radiation. Later on, several derivations of the Hawking radiation were proposed, mostly relying on quantum field theory on a fixed background (Birrel and Davies 1982; Fulling 1989; Frolov and Novikov 1998; Wald 1994). However, in the above calculations, we could not see the tunneling mechanism.

The first semi-classical method-null geodesic method, which models Hawking radiation as a tunneling process, was used by Parikh and Wilczek (Parikh and Wilczek 2000; Parikh 2004) which followed from the work of Kraus and Wilczek (Kraus and Wilczek 1995a, 1995b; Kraus and Keski-Vakkuri 1997). This method uses WKB approximation to calculate the imaginary part of the action for the classically forbidden trajectory across the horizon. The part of the action that contributes an imaginary term is $\operatorname{Im} S_0 = \int_{r_{in}}^{r_{out}} p_r dr$, where p_r is the momentum of the emitted null s-wave. One can use Hamilton's equation and knowledge of the null geodesics to calculate the imaginary part of the action. The other tunneling method is the Hamilton-Jacobi ansatz used by Angheben et al. (2005), Kerner and Mann (2006), which is an extension of the complex path analysis by Padmanabhan et al. (Srinivasan and Padmanabhan 1999; Shankaranarayanan et al. 2001, 2002; Shankaranarayanan 2003). This method applies the WKB approximation to the Klein-Gordon equation, and the lowest order is the Hamilton-Jacobi equation. Then according to the symmetry of the metric, one can pick an appropriate ansatz for the action and put it into the Hamilton-Jacobi equation to solve. This method is often called Hamilton-Jacobi method. The two tunneling methods are both based on the WKB approximation, so the tunneling probability is given by $\Gamma \propto \exp(-\frac{2}{\hbar} \operatorname{Im} S_0)$, where S_0 is the classical action of the trajectory at the leading order in $\hbar$, and the difference is the way to calculate the imaginary part of the action for the classically forbidden trajectory across the horizon. These two methods have been successfully applied to a wide variety of interesting and exotic spacetimes, including Kerr and Kerr-Newman black holes (Jiang et al. 2006; Zhang and Zhao 2005, 2006), Gödel black holes (Kerner and Mann 2007), hot NUT-Kerr-Newman-Kasuya spacetimes (Ali 2007), squashed Kaluza-Klein black holes (Matsuno and Umetsu 2011), rotating AdS black holes in conformal gravity (Wu et al. 2014; Deng 2014), as well as generic weakly isolated horizons (Wu and Gao 2007). In 2008, Kerner and Mann (2008a, 2008b) applied Hamilton-Jacobi method to the Dirac Equation to calculate Dirac particles' Hawking radiation. Then many spacetimes were explored in this way: Kerr black holes (Li et al. 2008), BTZ

✉ G.-R. Chen
chengerui@emails.bjut.edu.cn

1 Institute of Theoretical Physics, Beijing University of Technology, Beijing, 100124, China

black holes (Li and Ren 2008), charged dilatonic black holes (Chen et al. 2008a), rotating black holes in de Sitter spaces (Chen et al. 2008b), GHS and non-extremal D1–D5 black holes (Jiang 2008a), higher-dimensional black holes (Lin and Yang 2009a), higher-dimensional Kerr-Anti-de Sitter black holes (Lin and Yang 2009b), dynamical horizons (Criscienzo and Vanzo 2008), black rings (Jiang 2008b), Vaidya black holes (Li et al. 2008), Reissner-Nordström black holes (Zeng and Yang 2008), accelerating and rotating black holes (Rehman and Saifullah 2011) and so on.

Black holes may radiate different spin particles, not only particles with spin: 0, 1/2. Later on, Yale and Mann investigated the gravitinos tunneling using the tunneling method (Yale and Mann 2009). Recently, Kruglov (2014a, 2014b) applies the WKB approximation and the Hamilton-Jacobi ansatz to the Proca equation for the emission of vector particles from $(1+1)$ and $(1+2)$ dimensional black holes. Vector particles (e.g. $Z, W^{\pm}$) play important role in Standard Model, so it is interesting to study the radiation of vector particles. We extend Kruglov's (2014a, 2014b) method to investigate the vector particles' Hawking radiation from a four-dimensional Schwarzschild black hole which is the most important spacetime in general relativity and astrophysics. We obtain the radiation spectrum of vector particles from a Schwarzschild black hole and the corresponding Hawking temperature. The results show that vector particles radiate with the same Hawking temperature as that of scalar particles, fermions and gravitinos. Like Kerner and Mann (2008a, 2008b), we assume that the change of black hole angular momentum due to the spin of the emitted particle is negligible, which is a good approximation for the black hole with mass much larger than the Planck mass.

The remainder of this paper is organized as follows. In Sect. 2, we calculate the tunneling spectrum of vector particles from four-dimensional Schwarzschild black holes. In Sect. 3, we give some discussions and conclusions.

2 Vector particles tunneling from four-dimensional Schwarzschild black holes

The metric of the Schwarzschild black hole is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right), \tag{1}$$

where M is the ADM mass of the spacetime. The event horizon is given by $r_h = 2M$. We define the following notations for convenience

$$A = 1-\frac{2M}{r}, \qquad B = r^2, \qquad C = \sin^2\theta, \tag{2}$$

so the metric becomes

$$ds^2 = -Adt^2 + \frac{1}{A}dr^2 + Bd\theta^2 + BCd\varphi^2. \tag{3}$$

The determinant of the metric is

$$\sqrt{-g} = B\sqrt{C}. \tag{4}$$

The Proca equations for vector particles are (Kruglov 2014a, 2014b)

$$D_\mu\psi^{\nu\mu} + \frac{m^2}{\hbar^2}\psi^\nu = 0, \tag{5}$$

$$\psi_{\nu\mu} = D_\nu\psi_\mu - D_\mu\psi_\nu = \partial_\nu\psi_\mu - \partial_\mu\psi_\nu, \tag{6}$$

where D_μ is the covariant derivative, and $\psi_\nu = (\psi_0, \psi_1, \psi_2, \psi_3)$. From the definition, $\psi^{\nu\mu}$ is an anti-symmetrical tensor, so using the equation

$$D_\mu\psi^{\nu\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\psi^{\nu\mu}\right), \tag{7}$$

the Proca equations become

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\psi^{\nu\mu}\right) + \frac{m^2}{\hbar^2}\psi^\nu = 0. \tag{8}$$

From the metric (3) and the following relationship

$$\begin{aligned}
&\psi_0 = \psi^0 g_{00} = -A\psi^0 \Rightarrow \psi^0 = -\frac{\psi_0}{A},\\
&\psi^1 = A\psi_1, \qquad \psi^2 = \frac{\psi_2}{B}, \qquad \psi^3 = \frac{\psi_3}{BC},\\
&\psi^{10} = \psi_{10}g^{11}g^{00} = -\psi_{10}, \qquad \psi^{12} = \psi_{12}\frac{A}{B},\\
&\psi^{13} = \frac{A}{BC}\psi_{13}, \qquad \psi^{20} = -\frac{\psi_{20}}{AB},\\
&\psi^{23} = \frac{1}{B^2C}\psi_{23}, \qquad \psi^{30} = -\frac{\psi_{30}}{ABC},
\end{aligned} \tag{9}$$

we obtain the Proca equations in the following explicit form

$$\begin{aligned}
&\frac{1}{B\sqrt{C}}\left\{\partial_r\left[B\sqrt{C}(\partial_t\psi_1-\partial_r\psi_0)\right] + \partial_\theta\left[\frac{\sqrt{C}}{A}(\partial_t\psi_2-\partial_\theta\psi_0)\right]\right.\\
&\quad\left. + \partial_\varphi\left[\frac{1}{A\sqrt{C}}(\partial_t\psi_3-\partial_\varphi\psi_0)\right]\right\} + \frac{m^2\psi_0}{\hbar^2 A} = 0,\\
&\frac{1}{B\sqrt{C}}\left\{\partial_t\left[B\sqrt{C}(\partial_t\psi_1-\partial_r\psi_0)\right]\right.\\
&\quad + \partial_\theta\left[A\sqrt{C}(\partial_r\psi_2-\partial_\theta\psi_1)\right]\\
&\quad\left. + \partial_\varphi\left[\frac{A}{\sqrt{C}}(\partial_r\psi_3-\partial_\varphi\psi_1)\right]\right\} + \frac{m^2A\psi_1}{\hbar^2} = 0,\\
&\frac{1}{B\sqrt{C}}\left\{\partial_t\left[\frac{\sqrt{C}}{A}(\partial_t\psi_2-\partial_\theta\psi_0)\right] - \partial_r\left[A\sqrt{C}(\partial_r\psi_2-\partial_\theta\psi_1)\right]\right.\\
&\quad\left. + \partial_\varphi\left[\frac{1}{B\sqrt{C}}(\partial_\theta\psi_3-\partial_\varphi\psi_2)\right]\right\} + \frac{m^2\psi_2}{\hbar^2 B} = 0,\\
&\frac{1}{B\sqrt{C}}\left\{\partial_t\left[\frac{1}{A\sqrt{C}}(\partial_t\psi_3-\partial_\varphi\psi_0)\right]\right.\\
&\quad - \partial_r\left[\frac{A}{\sqrt{C}}(\partial_r\psi_3-\partial_\varphi\psi_1)\right]\\
&\quad\left. - \partial_\theta\left[\frac{1}{B\sqrt{C}}(\partial_\theta\psi_3-\partial_\varphi\psi_2)\right]\right\} + \frac{m^2\psi_3}{\hbar^2 BC} = 0.
\end{aligned} \tag{10}$$

According to the WKB approximation, the solutions are of the form

$$\psi_\nu = (c_0, c_1, c_2, c_3)\exp\left[\frac{i}{\hbar}S(t, r, \theta, \varphi)\right], \tag{11}$$

where

$$S(t, r, \theta, \varphi) = S_0(t, r, \theta, \varphi) + \hbar S_1(t, r, \theta, \varphi) + \hbar^2 S_2(t, r, \theta, \varphi) + \cdots \tag{12}$$

Putting Eqs. (11), (12) into Eqs. (10), the equations to the leading order in $\hbar$ are

$$\begin{aligned}
&B\sqrt{C}\big[c_0(\partial_r S_0)^2 - c_1(\partial_r S_0)(\partial_t S_0)\big] \\
&\quad + \frac{\sqrt{C}}{A}\big[c_0(\partial_\theta S_0)^2 - c_2(\partial_\theta S_0)(\partial_t S_0)\big] \\
&\quad + \frac{1}{A\sqrt{C}}\big[c_0(\partial_\varphi S_0)^2 - c_3(\partial_\varphi S_0)(\partial_t S_0)\big] \\
&\quad + \frac{m^2 B\sqrt{C}}{A}c_0 = 0, \\
&B\sqrt{C}\big[c_0(\partial_t S_0)(\partial_r S_0) - c_1(\partial_t S_0)^2\big] \\
&\quad + A\sqrt{C}\big[c_1(\partial_\theta S_0)^2 - c_2(\partial_\theta S_0)(\partial_r S_0)\big] \\
&\quad + \frac{A}{\sqrt{C}}\big[c_1(\partial_\varphi S_0)^2 - c_3(\partial_\varphi S_0)(\partial_r S_0)\big] \\
&\quad + m^2 B\sqrt{C}Ac_1 = 0, \\
&\frac{\sqrt{C}}{A}\big[c_0(\partial_t S_0)(\partial_\theta S_0) - c_2(\partial_t S_0)^2\big] \\
&\quad - A\sqrt{C}\big[c_1(\partial_r S_0)(\partial_\theta S_0) - c_2(\partial_r S_0)^2\big] \\
&\quad + \frac{1}{B\sqrt{C}}\big[c_2(\partial_\varphi S_0)^2 - c_3(\partial_\varphi S_0)(\partial_\theta S_0)\big] \\
&\quad + m^2\sqrt{C}c_2 = 0, \\
&\frac{1}{A\sqrt{C}}\big[c_0(\partial_t S_0)(\partial_\varphi S_0) - c_3(\partial_t S_0)^2\big] \\
&\quad - \frac{A}{\sqrt{C}}\big[c_1(\partial_r S_0)(\partial_\varphi S_0) - c_3(\partial_r S_0)^2\big] \\
&\quad - \frac{1}{B\sqrt{C}}\big[c_2(\partial_\theta S_0)(\partial_\varphi S_0) - c_3(\partial_\theta S_0)^2\big] \\
&\quad + \frac{m^2}{\sqrt{C}}c_3 = 0.
\end{aligned} \tag{13}$$

There exists a solution of the form

$$S_0 = -Et + W(r) + J(\theta, \varphi) + K, \tag{14}$$

where $E = -\partial_t S_0$ is the energy of the emitted particle and K is a complex constant. Inserting Eq. (14) into Eqs. (13) we obtain the matrix equation

$$\Lambda(c_0, c_1, c_2, c_3)^T = 0, \tag{15}$$

where Λ is a 4×4 matrix, and its components are expressed as

$$\begin{aligned}
\Lambda_{11} &= B\sqrt{C}\big(W'\big)^2 + \frac{\sqrt{C}}{A}(J_\theta)^2 + \frac{(J_\varphi)^2}{A\sqrt{C}} + \frac{m^2 B\sqrt{C}}{A}, \\
\Lambda_{12} &= B\sqrt{C}EW', \qquad \Lambda_{13} = \frac{\sqrt{C}}{A}EJ_\theta, \\
\Lambda_{14} &= \frac{EJ_\varphi}{A\sqrt{C}}, \qquad \Lambda_{21} = -B\sqrt{C}EW', \\
\Lambda_{22} &= -B\sqrt{C}E^2 + A\sqrt{C}(J_\theta)^2 + \frac{A}{\sqrt{C}}(J_\varphi)^2 + m^2 B\sqrt{C}A, \\
\Lambda_{23} &= -A\sqrt{C}J_\theta W', \qquad \Lambda_{24} = -\frac{A}{\sqrt{C}}J_\varphi W', \\
\Lambda_{31} &= -\frac{\sqrt{C}}{A}EJ_\theta, \qquad \Lambda_{32} = -A\sqrt{C}W'J_\theta, \\
\Lambda_{33} &= -\frac{\sqrt{C}}{A}E^2 + A\sqrt{C}\big(W'\big)^2 + \frac{1}{B\sqrt{C}}(J_\varphi)^2 + m^2\sqrt{C}, \\
\Lambda_{34} &= -\frac{J_\varphi J_\theta}{B\sqrt{C}}, \qquad \Lambda_{41} = -\frac{1}{A\sqrt{C}}EJ_\varphi, \\
\Lambda_{42} &= -\frac{A}{\sqrt{C}}J_\varphi W', \qquad \Lambda_{43} == -\frac{J_\varphi J_\theta}{B\sqrt{C}}, \\
\Lambda_{44} &= -\frac{E^2}{A\sqrt{C}} + \frac{A}{\sqrt{C}}\big(W'\big)^2 + \frac{1}{B\sqrt{C}}(J_\theta)^2 + \frac{m^2}{\sqrt{C}},
\end{aligned} \tag{16}$$

where $W' = \partial_r S_0$, $J_\theta = \partial_\theta S_0$ and $J_\varphi = \partial_\varphi S_0$.

Homogeneous system of linear equations (15) possesses nontrivial solution if the determinant of the matrix Λ equals zero, that is, $\det\Lambda = 0$. After the calculation, we have

$$\frac{m^2\{A(CJ_\theta^2 + J_\varphi^2) + BC[-E^2 + A(m^2 + A(W')^2)]\}^3}{A^3BC^2} = 0. \tag{17}$$

We obtain immediately

$$\big(W'\big)^2 = \frac{E^2 - A(m^2 + \frac{J_\theta^2}{B} + \frac{J_\varphi^2}{BC})}{A^2}, \tag{18}$$

and

$$\begin{aligned}
W_\pm &= \pm\int\sqrt{\frac{E^2 - A(m^2 + \frac{J_\theta^2}{B} + \frac{J_\varphi^2}{BC})}{A^2}}dr \\
&= \pm\int\frac{r\sqrt{E^2 - (1 - \frac{2M}{r})(m^2 + \frac{J_\theta^2}{r^2} + \frac{J_\varphi^2}{r^2\sin^2\theta})}}{r - 2M}dr,
\end{aligned} \tag{19}$$

where $W_+ > 0$ corresponds to vector particles moving away from the black hole and $W_- < 0$ corresponds to vector particles moving toward the black hole. So the probabilities of crossing the horizon each way are

$$\begin{aligned}
P_{out} &\propto \exp\left[-\frac{2}{\hbar}\operatorname{Im} S_0\right] = \exp\left[-\frac{2}{\hbar}(\operatorname{Im} W_+ + \operatorname{Im} K)\right], \\
P_{in} &\propto \exp\left[-\frac{2}{\hbar}\operatorname{Im} S_0\right] = \exp\left[-\frac{2}{\hbar}(\operatorname{Im} W_- + \operatorname{Im} K)\right].
\end{aligned} \tag{20}$$

The incoming particles crossing the horizon should have a 100 % chance of entering the black hole, so we have $\operatorname{Im} K = -\operatorname{Im} W_-$. Considering $W_+ = -W_-$, the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma \propto \exp[-4 \operatorname{Im} W_+], \tag{21}$$

where we set $\hbar$ to unity. Integrating around the pole at the horizon $r_h = 2M$, we obtain

$$\operatorname{Im} W_+ = 2\pi M E, \tag{22}$$

so the tunneling probability is

$$\Gamma \propto \exp(-8\pi M E), \tag{23}$$

and the Hawking temperature is

$$T_H = \frac{1}{8\pi M}. \tag{24}$$

We get the tunneling spectrum of vector particles from four-dimension Schwarzschild black holes. The expected Hawking temperature is recovered.

3 Discussions and conclusions

In summary, we calculate the Hawking temperature of vector particles tunneling from a four-dimensional Schwarzschild black hole by using the Hamilton-Jacobi method. The radiation temperature is the same as that of scalar particles, fermions and gravitinos. Therefore, our result provides further evidence for the universality of black hole radiation.

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