

The motion of axisymmetric satellite with drag and radiation pressure

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Abstract The axisymmetric satellite problem including radiation pressure and drag is treated. The equations of motion of the satellite are derived. The energy-like and Laplace-like invariants of motion have been derived for a general drag force function of the polar angle, and the Laplace-like invariant is used to find the orbit equation in the case of a spherical satellite. Then using the small parameter, the orbit of the satellite is determined for an axisymmetric satellite.

Keywords Artificial satellite · Drag effect · Radiation pressure

1 Introduction

The classical Kepler problem with additional forces due to the resistance of the medium surrounding the attracting centre, the radiation pressure, or both forces together has been discussed by many authors such as Jezewski and Mittleman (1983), Mittleman and Jezewski (1982), Danby (1962), Leach (1987), Brouwer and Hori (1961), Mavraganis (1991), Gorringer and Leach (1988), McMahon and Scheeres (2010) etc. The two body problem with a radiating major body in the existence of Danby's drag force has been studied by Mavraganis and Michalakis (1994).

In the present work we consider the problem of axisymmetric satellite in the gravitational field of a radiating body with drag. We obtain the equation of the satellite's orbit. First the Laplace-like invariant of motion has been derived and then it is used to get the orbit for a spherical satellite. Finally the orbit of the axisymmetric satellite is determined using the method of the small parameter. In addition the energy-like invariant of motion has been derived also. The drag force function is taken to be generally depending on the polar angle of the orbit.

2 The equation of motion and the integral of angular momentum

Now, we deal with motion of an axisymmetric satellite under the gravitational force of a spherical body with an additional force due to the resistance force and radiation pressure. The air resistance is taken as a general function of the polar angle θ . Then the equation of motion is, Bhatnagar (1986), Mavraganis and Michalakis (1994),

$$\ddot{\mathbf{r}} + \frac{f(\theta)}{r^2} \dot{\mathbf{r}} + \frac{(1-\beta)\mu}{r^3} \mathbf{r} + \frac{3\mu(C-A)(1-3\gamma^2)}{2r^5} \mathbf{r} = 0, \quad (1)$$

where μ, β are constants, and C, A are the principal moments of inertia of the satellite (C about the symmetric axis). While γ is the direction cosine of the radius vector w.r.t. the axis of the satellite. The range of physically possible values of β , for a repulsive force, is $0 < \beta < 1$. For $\beta = 0$ the attracting centre does not radiate at all, hence the satellite is acted upon only by two forces, the gravitation and drag effect.

We shall show that at any time the center of the satellite lies in a plane. Taking the vector product of Eq. (1) with the

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vector \underline{L}

$$\mathbf{r} \times \ddot{\mathbf{r}} + \frac{f(\theta)}{r^2} \mathbf{r} \times \dot{\mathbf{r}} + \frac{(1-\beta)\mu}{r^3} \mathbf{r} \times \mathbf{r} + \frac{3\mu(C-A)(1-3\gamma^2)}{2r^5} \mathbf{r} \times \mathbf{r} = 0, \tag{2}$$

and recalling the definition of the angular momentum, $\mathbf{H} = \mathbf{r} \times \dot{\mathbf{r}}$, we achieve the relation

$$\dot{\mathbf{H}} + \frac{f(\theta)}{r^2} \mathbf{H} = 0 \tag{3}$$

from which we readily obtain

$$\mathbf{H} \times \dot{\mathbf{H}} = \underline{0}.$$

This expression admits a first vector which is the constant direction $\mathbf{e}_H = \mathbf{H}/H$ of the angular momentum \mathbf{H} . Therefore the motion is planar. This enables us to simplify Eq. (3) by writing

$$\dot{H} + \frac{f(\theta)}{r^2} H = 0. \tag{3a}$$

We see that the magnitude of the angular momentum is not conserved, although its direction is. For this magnitude we find, after solving Eq. (3a),

$$H = r^2 \dot{\theta} = h - F(\theta), \tag{4}$$

where, h is the constant angular momentum in the absence of the drag force, and $F(\theta) = \int_{\theta_0}^{\theta} f(\theta) d\theta$.

Now, if we consider $C - A \ll 1$, we can solve Eq. (1) on two steps first for order zero, and then for the first order terms which contain $C - A$. The zero order equation is,

$$\ddot{\mathbf{r}} + \frac{f(\theta)}{r^2} \dot{\mathbf{r}} + \frac{(1-\beta)\mu}{r^3} \mathbf{r} = 0. \tag{5}$$

The above equation can be solved using the Laplace-like invariant. This will be given in the following section.

3 The Laplace-like invariant of motion for a spherical satellite

Now, we cross product Eq. (5) by $\Gamma(\theta)\mathbf{H}$,

$$\Gamma \ddot{\mathbf{r}} \times \mathbf{H} + \frac{\Gamma f(\theta)}{r^2} \dot{\mathbf{r}} \times \mathbf{H} + \frac{\Gamma(1-\beta)\mu}{r^3} \mathbf{r} \times \mathbf{H} = 0. \tag{6}$$

Using Eq. (3) together with,

$$\frac{d}{dt}(\Gamma \dot{\mathbf{r}} \times \mathbf{H}) = \dot{\Gamma} \dot{\mathbf{r}} \times \mathbf{H} + \Gamma \ddot{\mathbf{r}} \times \mathbf{H} + \Gamma \dot{\mathbf{r}} \times \dot{\mathbf{H}}.$$

We get,

$$\begin{aligned} & \frac{d}{dt}(\Gamma \dot{\mathbf{r}} \times \mathbf{H}) + \left(2\Gamma \frac{f(\theta)}{r^2} - \dot{\Gamma}\right) \dot{\mathbf{r}} \times \mathbf{H} \\ & + \frac{\Gamma(1-\beta)\mu}{r^3} \mathbf{r} \times \mathbf{H} = 0. \end{aligned} \tag{7}$$

We require the second term to vanish. This is guaranteed if,

$$\frac{d\Gamma}{d\theta} = \frac{2\Gamma f(\theta)}{r^2 \dot{\theta}}$$

which, using Eq. (4) has the solution

$$\Lambda(\theta) = \left[\frac{1}{h - F(\theta)} \right]^2.$$

Equation (7) becomes

$$\frac{d}{dt} \left(\frac{\dot{\mathbf{r}} \times \mathbf{H}}{[h - F(\theta)]^2} \right) + \frac{(1-\beta)\mu}{[h - F(\theta)]^2} \frac{\mathbf{r} \times \mathbf{H}}{r^3} = 0. \tag{8}$$

Recalling that

$$\frac{\mathbf{r} \times \mathbf{H}}{r^3} = -\dot{\theta} \mathbf{e}_\theta$$

where \mathbf{e}_θ is the unit vector along the direction of increase of the polar angle θ , we get,

$$\frac{\dot{\mathbf{r}} \times \mathbf{H}}{[h - F(\theta)]^2} = (1-\beta)\mu \int \frac{\mathbf{e}_\theta d\theta}{[h - F(\theta)]^2}. \tag{9}$$

To integrate the above integral, we define the function $g(\theta)$ such that

$$g'' + g = \frac{1}{[h - F(\theta)]^2}. \tag{10}$$

The above equation has the solution,

$$g(\theta) = \int_{\theta_0}^{\theta} \frac{\sin(\theta - \eta)}{[h - F(\theta)]^2} d\eta. \tag{11}$$

Now, substituting from Eq. (10) in Eq. (9), we get

$$\frac{\dot{\mathbf{r}} \times \mathbf{H}}{[h - F(\theta)]^2} = (1-\beta)\mu \int (g'' + g) \mathbf{e}_\theta d\theta. \tag{12}$$

After integrating, we get the Laplace-like vector invariant

$$\frac{\dot{\mathbf{r}} \times \mathbf{H}}{[h - F(\theta)]^2} - (1-\beta)\mu (g' \mathbf{e}_\theta + g \mathbf{e}_r) = \mathbf{J} \tag{13}$$

where, \mathbf{J} is a constant vector.

Substituting for

$$\dot{\mathbf{r}} \times \mathbf{H} = r^2 \dot{\theta} (r \dot{\theta} \mathbf{e}_r - \dot{r} \mathbf{e}_\theta), \quad h - F(\theta) = r^2 \dot{\theta}$$

we get

$$\mathbf{J} = \left(\frac{1}{r} - (1-\beta)\mu g \right) \mathbf{e}_r - \left(\frac{r'}{r^2} + (1-\beta)\mu g' \right) \mathbf{e}_\theta. \tag{14}$$

Equation (14) gives the Laplace-like invariant of motion for a spherical satellite.

4 The orbit equation for a spherical satellite

Now, if we measure the polar angle θ from the constant direction of the vector \mathbf{J} , we dot product Eq. (14) by \mathbf{e}_r to get the orbit equation

$$r = \frac{1}{J \cos \theta + (1 - \beta)\mu g}. \quad (15)$$

Equations (15) and (11) give the orbit equation for a spherical satellite under the effect of radiation pressure and an air drag force given as a general function of the polar angle θ . It is clear that the result will be reduced to the case of Danby's drag force when $f(\theta) = \text{Const}$.

5 The energy-like invariant of motion for a spherical satellite

Now, we get the energy like integral by dot product Eq. (5) by $\gamma(\theta)\dot{\mathbf{r}}$ which gives

$$\gamma\ddot{\mathbf{r}}\dot{\mathbf{r}} + \frac{\gamma f(\theta)}{r^2}\dot{\mathbf{r}}\dot{\mathbf{r}} + \frac{\gamma(1-\beta)\mu}{r^3}\mathbf{r}\dot{\mathbf{r}} = 0, \quad (16)$$

which can be written in the form,

$$\frac{1}{2} \frac{d}{dt} (\gamma\dot{\mathbf{r}}\dot{\mathbf{r}}) - \left(\frac{\dot{\theta}}{2} \frac{d\gamma}{d\theta} - \frac{\gamma f(\theta)}{r^2} \right) \dot{\mathbf{r}}\dot{\mathbf{r}} + \frac{\gamma(1-\beta)\mu}{r^3} \mathbf{r}\dot{\mathbf{r}} = 0. \quad (17)$$

Again, we require the second term to vanish. This is guaranteed if,

$$\frac{d\gamma}{d\theta} = \frac{2\gamma f(\theta)}{r^2 \dot{\theta}}, \quad (18)$$

which, using Eq. (4) has the solution

$$\gamma(\theta) = \left[\frac{1}{h - F(\theta)} \right]^2. \quad (19)$$

Therefore Eq. (17) becomes,

$$\frac{1}{2} \frac{d}{dt} (\gamma\dot{\mathbf{r}}\dot{\mathbf{r}}) + \frac{\gamma(1-\beta)\mu}{r^2} \dot{r} = 0, \quad (20)$$

which enables us to change the independent variable from t to θ by the substitution,

$$\frac{d}{dt} = \dot{\theta} \frac{d}{d\theta}.$$

Then we get

$$\frac{1}{2} \frac{d}{d\theta} \left(\frac{\dot{\mathbf{r}}\dot{\mathbf{r}}}{[h - F(\theta)]^2} \right) = \frac{(1-\beta)\mu}{[h - F(\theta)]^2} \frac{du}{d\theta}, \quad (21)$$

where, $u = \frac{1}{r}$.

Thus,

$$\frac{1}{2} \left(\frac{\dot{\mathbf{r}}\dot{\mathbf{r}}}{[h - F(\theta)]^2} \right) = (1-\beta)\mu \int \frac{u'}{[h - F(\theta)]^2} d\theta. \quad (22)$$

Using the function $g(\theta)$ defined before, we get

$$\begin{aligned} & \frac{1}{2} \left(\frac{\dot{\mathbf{r}}\dot{\mathbf{r}}}{[h - F(\theta)]^2} \right) \\ &= (1-\beta)\mu \int \{ [ug + u'g']' - g'(u'' + u) \} d\theta \\ &= (1-\beta)\mu \left\{ ug + u'g' - \int g'(u'' + u) d\theta \right\}. \end{aligned}$$

Now, using Eq. (4), with the equation of motion in the \mathbf{e}_θ direction written in terms of $u = \frac{1}{r}$ and θ , we can show that

$$u'' + u = \frac{(1-\beta)\mu}{[h - F(\theta)]^2}$$

which gives,

$$\begin{aligned} & \frac{1}{2} \left(\frac{\dot{\mathbf{r}}\dot{\mathbf{r}}}{[h - F(\theta)]^2} \right) \\ &= (1-\beta)\mu \left\{ ug + u'g' - (1-\beta)\mu \int g'(g'' + g) d\theta \right\}. \end{aligned}$$

We finally arrive at

$$\begin{aligned} & \frac{1}{2} \left(\frac{\dot{\mathbf{r}}\dot{\mathbf{r}}}{[h - F(\theta)]^2} \right) \\ &= (1-\beta)\mu \left\{ ug + u'g' - \frac{(1-\beta)\mu}{2} (g'^2 + g^2) \right\} + E. \end{aligned} \quad (23)$$

Equation (23) defines the energy-like invariant of motion of a spherical satellite.

6 The orbit equation for an oblate satellite

Now, we solve the equations of motion by taking into consideration the oblateness of the satellite. The equations of motion in the directions of \mathbf{e}_r and \mathbf{e}_θ are thus,

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 + \frac{f(\theta)}{r^2}\dot{r} + (1-\beta)\frac{\mu}{r^2} \\ + \frac{3}{2} \frac{\mu(C-A)(1-3\gamma^2)}{r^4} = 0, \end{aligned} \quad (24a)$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} + \frac{f(\theta)}{r^2}r\dot{\theta} = 0. \quad (24b)$$

It may easily be shown that the latter equation admits the integral given by Eq. (4). The former equation, under substi-

tution $u = \frac{1}{r}$, becomes

$$-\frac{u''}{u^2}\dot{\theta}^2 - \frac{u'}{u}\left(-2\frac{u'}{u^2}\dot{\theta}^2 + \frac{\ddot{\theta}}{u} + \alpha u\dot{\theta}\right) - \frac{\dot{\theta}^2}{u} + (1 - \beta)\mu u^2 + \frac{3}{2}\mu(C - A)(1 - 3\gamma^2)u^4 = 0 \tag{25}$$

(the prime denotes differentiation w.r.t. the angle θ). Writing Eq. (24b) in terms of u and u' , we see that it is exactly the same with the coefficient of the ratio $\frac{u'}{u}$ in Eq. (25). Therefore we have

$$u'' + u = \frac{(1 - \beta)\mu}{[h - F(\theta)]^2} + \frac{3}{2}\frac{\mu(C - A)(1 - 3\gamma^2)}{[h - F(\theta)]^2}u^2$$

which can be written as

$$u'' + u = \frac{(1 - \beta)}{[h_1 - F_1(\theta)]^2} + \frac{3}{2}\frac{(C - A)(1 - 3\gamma^2)}{[h_1 - F_1(\theta)]^2}u^2, \tag{26.1}$$

$$h_1 = \frac{h}{\sqrt{\mu}}, \quad F_1(\theta) = \frac{F(\theta)}{\sqrt{\mu}}. \tag{26.2}$$

Let ε be a small parameter then u can be written as

$$u = u_0 + \varepsilon u_1 \tag{27}$$

where $\varepsilon = \frac{3}{2}(C - A)$, and u_0 is the zero order term of the solution corresponding to the case of a spherical satellite, and u_1 is the first order term arising from the asphericity of the satellite. Equation (26.1), will be spited to,

$$u_0 + u_0 = \frac{(1 - \beta)}{[h_1 - F_1(\theta)]^2}, \tag{28}$$

$$u_1 + u_1 = \frac{1 - 3\gamma^2}{[h_1 - F_1(\theta)]^2}u_0^2. \tag{29}$$

It is clear that Eq. (28) is the equation of motion for a spherical satellite, written in terms of $u = \frac{1}{r}$ and θ , which has the solution given in Eq. (15),

$$u = J \cos \theta + (1 - \beta)\mu g. \tag{15a}$$

In case of the stationary rotational motion of the satellite there exist three positions, Bhatnagar (1986),

$$(i) \quad \gamma = 0, \quad (ii) \quad \gamma = 1, \quad (iii) \quad \gamma = \cos\left(\frac{\pi}{2} + \phi\right),$$

where ϕ is the angle between the radius vector and the normal to the tangent. In the case (i) the axis of the satellite is normal to the plane of motion. In the case (ii) the axis of the satellite takes the direction of the radius vector. But in the case (iii) the axis of the satellite takes the direction of the tangent of the orbit.

Now, we consider three cases of Eq. (29)

(a) $\gamma = 0$, then Eq. (29) takes the form

$$u''_1 + u_1 = \frac{u_0^2}{[h_1 - F_1(\theta)]^2}, \tag{29a}$$

which has the solution,

$$u_1 = \int_{\theta_0}^{\theta} \sin(\theta - \eta) \left(\frac{J \cos \eta + (1 - \beta)\mu g(\eta)}{h_1 - F_1(\eta)} \right)^2 d\eta. \tag{30a}$$

(b) $\gamma = 1$, then we have

$$u''_1 + u_1 = \frac{-2u_0^2}{[h_1 - F_1(\theta)]^2}, \tag{29b}$$

which has the solution,

$$u_1 = -2 \int_{\theta_0}^{\theta} \sin(\theta - \eta) \left(\frac{J \cos \eta + (1 - \beta)\mu g(\eta)}{h_1 - F_1(\eta)} \right)^2 d\eta. \tag{30b}$$

(c) $\gamma = \cos(\frac{\pi}{2} + \phi)$, then we get

$$u''_1 + u_1 = \frac{1}{[h_1 - F_1(\theta)]^2} \left(\frac{u_0^2 - 2u_0^2}{u_0^2 + u_0^2} \right) \tag{29c}$$

which has the solution,

$$u_1 = \int_{\theta_0}^{\theta} \sin(\theta - \eta) \left(\frac{J \cos \eta + \mu(1 - \beta)g(\eta)}{h_1 - F_1(\eta)} \right)^2 \times W(\eta) d\eta, \tag{30c.1}$$

where,

$$W(\eta) = \frac{(J \cos \eta + h_1 g(\eta))^2 - 2(h_1 g'(\eta) - J \sin \eta)^2}{(J \cos \eta + h_1 g(\eta))^2 + (h_1 g'(\eta) - J \sin \eta)^2}. \tag{30c.2}$$

7 Conclusion

The motion of an axisymmetric satellite about a radiating body in the presence of air drag has been studied. The work can be divided to three parts: in the first part the invariants of motion of the energy-like and Laplace-like have been determined for a general drag function of the polar angle θ . Second the Laplace-like integral has been used to determine the orbit equation in a closed form terms of the parameters of the different forces of gravitation, radiation and the drag force. Finally, the oblateness of the satellite has been introduced and studied through the small parameter $C - A$ considered $\ll 1$, and the orbit is determined in terms of $C - A$.

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