Astrophys Space Sci (2014) 352:321–329
DOI 10.1007/s10509-014-1913-z

ORIGINAL ARTICLE

Energy density inhomogeneities with polynomial $f(R)$ cosmology

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Received: 20 March 2014 / Accepted: 27 March 2014 / Published online: 8 April 2014

Abstract In this paper, we study the effects of polynomial $f(R)$ model on the stability of homogeneous energy density in self-gravitating spherical stellar object. For this purpose, we construct couple of evolution equations which relate the Weyl tensor with matter parameters. We explore different factors responsible for density inhomogeneities with non-dissipative dust, isotropic as well as anisotropic fluids and dissipative dust cloud. We find that shear, pressure, dissipative parameters and $f(R)$ terms affect the existence of inhomogeneous energy density.

Keywords Dissipative systems · Relativistic systems · Modified gravity

1 Introduction

Recent cosmological evidences predicted by different measurements indicate transition of our universe from matter dominated epoch to accelerating expansion state (Perlmutter et al. 1999, Riess et al. 2007, Komatsu et al. 2011). The accelerating cosmic expansion has been prompted by an enigmatic ingredient with large negative pressure, dubbed as dark energy. To explain its nature, different models like cosmological constant, phantom, quintessence, Chaplygin gas etc. have been established. The exploration of modified gravity theories obtained by modifying geometric gravitational part of Einstein-Hilbert action has received much attention in mathematical physics. The $f(R)$ gravity (Capozziello 2002; Nojiri and Odintsov 2011) is one of the most viable theories in which Ricci scalar is replaced by its non-linear generic function. Among important features of this theory, the likely one is to present a model that represents early as well as late-time universe expansion in the absence of dark component. Bamba et al. (2012a, 2012b) introduced unified model for inflation as well as late cosmic expansion model in this theory. There exist number of $f(R)$ models (Faraoni and Nadeau 2005; Nojiri and Odintsov 2007; Hu and Sawicki 2007; Bamba et al. 2012a, 2012b) that correspond to cosmological constraints and pass experimental test.

Anisotropic pressure in matter configurations results from several astrophysical factors like pion condensation (Sawyer 1972), various types of phase transitions (Sokolov 1980), presence of a solid core, superfluids (Heiselberg and Jensen 2000) as well as strong magnetic field (Yazadjiev 2012). It is worth stressing that for stable fluid distribution, anisotropy can endorse outwardly increasing density within the core of star (Horvat et al. 2011). After the ground work of Bowers and Liang (1974), there have been number of papers on pressure anisotropy (Herrera and Santos 1997; Bohmer and Harko 2006; Herrera et al. 2008, Sharif and Yousaf 2012a, 2012b, Mimoso et al. 2013, Sharif and Bhatti 2013a, 2013b, 2014b) which assert that anisotropy may have non-negligible consequences on the structure and properties of self-gravitating systems. Thirukkanesh and Ragel (2013) presented spherically symmetric compact star models with anisotropic pressure which help to understand strange quark stars.

Gravitational collapse is the phenomenon in which massive body falls inward due to the action of its own gravity that may lead to stars, star clusters and galaxies from interstellar gas. This occurs due to extremely inhomogeneous initial state thereby showing the importance of energy density inhomogeneities in the collapse process. Penrose and Hawk-

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ing (1979) laid much emphasis on the importance of energy density inhomogeneity in the gravitational time arrow by relating inhomogeneous density with Weyl tensor. Eardley and Smarr (1979) asserted that inhomogeneous spherical dust configuration leads to naked singularities for inhomogeneous collapse. Herrera et al. (1998) discussed the role of density inhomogeneities and local anisotropy of pressure on the structure and evolution of spherically symmetric adiabatic self-gravitating objects through the active gravitational mass. Further, Herrera et al. (2004) investigated density inhomogeneity effects on the evolutionary phases of dissipative anisotropic spherical systems by evaluating a link between the Weyl tensor and local anisotropic pressure. Bamba et al. (2011) discussed matter instability and curvature singularity in the star collapse with $f(R)$ background.

Ziaie et al. (2011) studied collapsing mechanism of a star satisfying barotropic equation of state in $f(R)$ theories and found finite-time singularities. Borisov et al. (2012) analyzed spherical collapse in metric $f(R)$ gravity with the help of time evolution numerical simulations. Guo et al. (2013) studied collapse of spherical star in Einstein $f(R)$ frame and concluded that this may lead to de-Sitter Schwarzschild black hole. We have investigated impact of late and early time cosmic models on the collapse of self-gravitating systems with metric as well as Palatini $f(R)$ theory (Sharif and Yousaf 2013a, 2013b, 2013c, 2013d, 2014a, 2014c).

A great deal of effort has been devoted to study the stability of stellar systems upon fluctuations. Galli and Koshelev (2011) studied a class of late-time cosmic evolution models with perturbations induced by inhomogeneous energy density. Pinheiro and Chan (2011) examined non-adiabatic anisotropic collapse accompanied by inhomogeneous density configuration with and without shearing motion. Sharma and Tikekar (2012) explored shear-free spherical collapse with dissipation through heat to investigate the inhomogeneity effects during evolution. Sharif and his collaborators (Sharif and Yousaf 2012b, 2012c; Sharif and Bhatti 2012a, 2012b, 2014a, 2014c; Sharif and Tahir 2013) studied spherical, cylindrical and planar celestial models and analyzed the role of energy density inhomogeneity in the evolution of fluid parameters that characterize gravitational collapse.

Herrera et al. (2009) explored spherical relativistic fluid configurations through scalar functions, i.e., Y_T, X_T, Y_{TF} and X_{TF}. Herrera et al. (2011) extended their results by invoking cosmological constant to examine the evolution of shear tensor and expansion scalar. They identified X_{TF} as a factor describing inhomogeneity in the energy density. Herrera (2011) discussed stability of inhomogeneous density in anisotropic spherical fluid configuration with diffusion and free-streaming approximations. Recently, we have studied dynamics of spherical matter distribution with structure scalars and ϵR^2 cosmology (Sharif and Yousaf 2014b).

This paper investigates the role of polynomial $f(R)$ gravity on the stability of homogeneous energy density with anisotropic and dissipative spherical matter. The paper is planned as follows. In Sect. 2, we discuss $f(R)$ formulation and relate matter variables with the Weyl scalar. Section 3 is devoted to construct scalar functions with a well-consistent polynomial $f(R)$ model to obtain conservation and Ellis equations. In Sect. 4, we consider various aspects of matter distribution to analyze density inhomogeneity. In the last section, we conclude our results.

2 $f(R)$ formalism

The gravitational part of the Einstein-Hilbert action in $f(R)$ gravity is

$$S_{f(R)}=\frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R), \tag{1}$$

where κ and $f(R)$ are coupling constant and a non-linear generic Ricci scalar function, respectively. The usual GR action can be retrieved by taking $f(R)=R$. The field equations, in metric formalism, are calculated by varying Eq. (1) with respect to $g_{\alpha\beta}$ as follows

$$R_{\alpha\beta}f_R-(\nabla_\alpha\nabla_\beta-g_{\alpha\beta}\Box)f_R-\frac{1}{2}g_{\alpha\beta}f=\kappa T_{\alpha\beta}, \tag{2}$$

where ∇_α and $\Box$ are the covariant derivative and d'Alembert operator, respectively. Equation (2), after some manipulations, can be expressed as

$$G_{\alpha\beta}=\frac{\kappa}{f_R}\big(\overset{(D)}{T}_{\alpha\beta}+T_{\alpha\beta}\big), \tag{3}$$

where

$$\overset{(D)}{T}_{\alpha\beta}=\frac{1}{\kappa}\left\{\nabla_\alpha\nabla_\beta f_R+(f-Rf_R)\frac{g_{\alpha\beta}}{2}-\Box f_R g_{\alpha\beta}\right\},$$

is the stress energy tensor which indicates gravitational contribution due to $f(R)$ terms. Under GR limit, i.e., $f(R)\to R$, $\overset{(D)}{T}_{\alpha\beta}$ disappears identically. The system under consideration is modeled as a sphere with non-static spacetime

$$\begin{aligned}ds_-^2&=A^2(t,r)dt^2-B^2(t,r)dr^2\\&\quad-C^2(t,r)\big(d\theta^2+\sin^2\theta d\phi^2\big),\end{aligned} \tag{4}$$

consisting of locally anisotropic pressure, dissipating in the diffusion (heat) and free streaming (null radiation) approximations. The corresponding stress-energy tensor is

$$\begin{aligned}T_{\alpha\beta}&=(\mu+P_\perp)V_\alpha V_\beta+(P_r-P_\perp)\chi_\alpha\chi_\beta\\&\quad-P_\perp g_{\alpha\beta}+q_\alpha V_\beta+\varepsilon l_\alpha l_\beta+V_\alpha q_\beta,\end{aligned} \tag{5}$$

where μ, $P_\perp$, P_r, q_β and ε, are the energy density, tangential and radial pressures, heat conducting vector and radiation density, respectively. Moreover, l^β, V^β and χ^β are the null four-vector, fluid four-velocity and radial unit four-vector, respectively. These quantities $V^\beta=\frac{1}{A}\delta^\beta_0$, $\chi^\beta=\frac{1}{B}\delta^\beta_1$, $l^\beta=\frac{1}{A}\delta^\beta_0+\frac{1}{B}\delta^\beta_1$, $q^\beta=q(t,r)\chi^\beta$ in comoving coordinates obey

$$V^\alpha V_\alpha=-1,\qquad \chi^\alpha\chi_\alpha=1,\qquad \chi^\alpha V_\alpha=0,$$
$$V^\alpha q_\alpha=0,\qquad l^\alpha V_\alpha=-1,\qquad l^\alpha l_\alpha=0.$$

The expansion and shear scalars for Eq. (1) are given by

$$\Theta A=\left(\frac{2\dot{C}}{C}+\frac{\dot{B}}{B}\right),\quad \sigma A=-\left(\frac{\dot{C}}{C}-\frac{\dot{B}}{B}\right),\tag{6}$$

where dot stands differentiation with respect to t.

The metric $f(R)$ field equations turn out to be

$$\begin{aligned}&\frac{\kappa}{f_R}\bigg[A^2(\mu+\varepsilon)+\frac{A^2}{\kappa}\bigg\{\frac{f_R'}{B^2}\left(\frac{B'}{B}+\frac{2C'}{C}\right)\\&\quad-\frac{f_R}{2}\left(R-\frac{f}{f_R}\right)+\frac{f_R''}{B^2}-\frac{\dot{f}_R}{A^2}\\&\quad\times\left(\frac{\dot{B}}{B}+\frac{2\dot{C}}{C}\right)\bigg\}\bigg]\\&=\left(\frac{\dot{C}}{C}\right)^2+\frac{2\dot{C}\dot{B}}{CB}+\bigg\{\frac{C'}{C}\left(\frac{2B'}{B}-\frac{C'}{C}\right)+\left(\frac{B}{C}\right)^2\\&\quad-\frac{2C''}{C}\bigg\}\left(\frac{A}{B}\right)^2,\end{aligned}\tag{7}$$

$$\begin{aligned}&\frac{\kappa}{f_R}\left[BA(q+\varepsilon)-\frac{1}{\kappa}\left(\dot{f}_R'-\frac{\dot{B}f_R'}{B}-\frac{A'\dot{f}_R}{A}\right)\right]\\&=2\left(\frac{\dot{C}'}{C}-\frac{A'\dot{C}}{CA}-\frac{C'\dot{B}}{BC}\right),\end{aligned}\tag{8}$$

$$\begin{aligned}&\frac{\kappa}{f_R}\bigg[B^2(P_r+\varepsilon)-\frac{B^2}{\kappa}\bigg\{\frac{f_R'}{B^2}\left(\frac{A'}{A}+\frac{2C'}{C}\right)-\frac{f_R}{2}\left(R-\frac{f}{f_R}\right)\\&\quad+\left(\frac{\dot{A}}{A}-\frac{2\dot{C}}{C}\right)\frac{\dot{f}_R}{A^2}-\frac{\ddot{f}_R}{A^2}\bigg\}\bigg]\\&=\bigg\{\left(\frac{2\dot{A}}{A}-\frac{\dot{C}}{C}\right)\frac{\dot{C}}{C}-\frac{2\ddot{C}}{C}\bigg\}\frac{B^2}{A^2}-\frac{B^2}{C^2}\\&\quad+\frac{C'}{C}\left(\frac{C'}{C}+\frac{2A'}{A}\right),\end{aligned}\tag{9}$$

$$\begin{aligned}&\frac{\kappa}{f_R}\bigg[P_\perp C^2-\frac{C^2}{\kappa}\bigg\{\frac{f_R''}{B^2}-\frac{\ddot{f}_R}{A^2}+\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\frac{\dot{f}_R}{A^2}\\&\quad-\frac{f_R}{2}\left(R-\frac{f}{f_R}\right)+\left(\frac{C'}{C}-\frac{B'}{B}+\frac{A'}{A}\right)\frac{f_R'}{B^2}\bigg\}\bigg]\\&=\bigg\{\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)-\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{A}}{BA}-\frac{\ddot{B}}{B}\bigg\}\frac{C^2}{A^2}\\&\quad+\bigg\{\frac{A'}{A}\left(\frac{C'}{C}-\frac{B'}{B}\right)\frac{C''}{C}-\frac{B'C'}{BC}+\frac{A''}{A}\bigg\}\frac{C^2}{B^2},\end{aligned}\tag{10}$$

where prime represents differentiation with respect to r. The Misner-Sharp mass function is given by (Misner and Sharp 1964)

$$m(t,r)=\frac{C}{2}\left(1-g^{\alpha\beta}C_{,\alpha}C_{,\beta}\right)=\left\{1+\left(\frac{\dot{C}}{A}\right)^2-\left(\frac{C'}{B}\right)^2\right\}\frac{C}{2}.\tag{11}$$

The radial and proper derivative operators are defined respectively as follows

$$D_C=\frac{1}{C'}\frac{\partial}{\partial r},\qquad D_T=\frac{1}{A}\frac{\partial}{\partial t}.\tag{12}$$

The proper time rate of change of areal radius of the spherical system is

$$U=D_TC=\frac{\dot{C}}{A}<1\quad\text{(for collapsing bodies).}\tag{13}$$

In terms of collapsing fluid velocity, Eq. (11) can be written as

$$E\equiv\frac{C'}{B}=\left[1+U^2-\frac{2m(t,r)}{C}\right]^{1/2}.\tag{14}$$

The time and radial mass variations can be followed from Eqs. (7)–(9), (11) and (12) as

$$D_Tm=-\frac{\kappa}{2f_R}\left\{U\left(\hat{P}_r+\frac{\overset{(D)}{T}_{11}}{B^2}\right)+E\left(\hat{q}-\frac{\overset{(D)}{T}_{01}}{AB}\right)\right\}C^2,\tag{15}$$

$$D_Cm=\frac{\kappa}{2f_R}\left\{\hat{\mu}+\frac{\overset{(D)}{T}_{00}}{A^2}+\frac{U}{E}\left(\hat{q}-\frac{\overset{(D)}{T}_{01}}{AB}\right)\right\}C^2,\tag{16}$$

where $\hat{P}_r=P_r+\varepsilon$, $\hat{q}=q+\varepsilon$ and $\hat{\mu}=\mu+\varepsilon$. Integration of Eq. (16) provides

$$\begin{aligned}\frac{3m}{C^3}&=\frac{3\kappa}{2C^3}\int_0^r\bigg[\frac{1}{f_R}\bigg\{\hat{\mu}+\frac{\overset{(D)}{T}_{00}}{A^2}\\&\quad+\left(\hat{q}-\frac{\overset{(D)}{T}_{01}}{AB}\right)\frac{U}{E}\bigg\}C^2C'\bigg]dr,\end{aligned}\tag{17}$$

thereby relating mass function and other fluid variables with $f(R)$ terms. The electric component of the Weyl tensor in terms of χ_α and unit four velocity is given by

$$E_{\alpha\beta}=\mathcal{E}\left[\chi_\alpha\chi_\beta-\frac{1}{3}(g_{\alpha\beta}+V_\alpha V_\beta)\right],$$

where

$$\mathcal{E}=\left[\frac{\ddot{C}}{C}+\left(\frac{\dot{B}}{B}-\frac{\dot{C}}{C}\right)\left(\frac{\dot{C}}{C}+\frac{\dot{A}}{A}\right)-\frac{\ddot{B}}{B}\right]\frac{1}{2A^2}-\frac{1}{2C^2}$$
$$-\left[\frac{C''}{C}-\left(\frac{C'}{C}+\frac{B'}{B}\right)\left(\frac{A'}{A}-\frac{C'}{C}\right)-\frac{A''}{A}\right]\frac{1}{2B^2}, \quad (18)$$

which after using Eqs. (7) and (9)–(11) can be expressed as

$$\frac{3m}{C^3}=\frac{\kappa}{2f_R}\left(\hat{\mu}-\hat{\Pi}+\frac{\overset{(D)}{T_{00}}}{A^2}-\frac{\overset{(D)}{T_{11}}}{B^2}+\frac{\overset{(D)}{T_{22}}}{C^2}\right)-\mathcal{E}, \quad (19)$$

where $\hat{\Pi}=\hat{P}_r-P_\perp$. This equation peculiarly relates mass function with Weyl scalar and all the fluid variables in $f(R)$ gravity.

3 Structure scalars and Ellis equations

In this section, we first discuss a viable $f(R)$ model and then construct structure scalars. We also write down conservation laws from the usual as well as effective stress energy tensors and develop the so called Ellis equations. We take a polynomial inflationary model given as follows (Huang 2014)

$$f(R)=R+\epsilon R^2+\frac{\lambda_n(2\epsilon R)^n}{4n\epsilon}, \quad (20)$$

where $\epsilon=\frac{1}{6M^2}$ and λ_n is a dimension-free coupling parameter with $n>2$. Here energy scale M is refined in order to make unit normalization to the higher coefficient of R^2 term. This model corresponds to the model with $f(R)=R+R^n/(3M^2)^n-1$ under $\lambda_n\gg 1$ while $\lambda_n\to 0$ provides Starobinsky model (Starobinsky 1980). In the limit $\lambda_n\ll 1$, R^n terms serve as a small correction to the inflationary $R+\epsilon R^2$ model which of course makes the model expansion around the Starobinsky model. It is interesting to mention here that the inflation induced by $R+R^4$ gravity provides much different platform than that of $R+R^2$ gravity and is close to topological inflation (Saidov and Zhuk 2010). All GR solutions can be found by taking limit $f(R)\to R$.

To formulate $f(R)$ structure scalars, we orthogonally split the Riemann tensor and propose tensors $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ as (Herrera et al. 2011)

$$X_{\alpha\beta}={}^*R^*_{\alpha\mu\beta\nu}V^\mu V^\nu=\frac{1}{2}\eta^{\epsilon\rho}{}_{\alpha\mu}R^*_{\epsilon\rho\beta\nu}V^\mu V^\nu,$$
$$Y_{\alpha\beta}=R_{\alpha\mu\beta\nu}V^\mu V^\nu,$$

where $R^*_{\alpha\beta\mu\nu}=\frac{1}{2}\eta_{\epsilon\rho\mu\nu}R^{\epsilon\rho}{}_{\alpha\beta}$. These equations in terms of trace and trace-less components are given by

$$X_{\alpha\beta}=\frac{1}{3}X_T h_{\alpha\beta}+X_{TF}\left(\chi_\alpha\chi_\beta-\frac{1}{3}h_{\alpha\beta}\right), \quad (21)$$

$$Y_{\alpha\beta}=\frac{1}{3}Y_T h_{\alpha\beta}+Y_{TF}\left(\chi_\alpha\chi_\beta-\frac{1}{3}h_{\alpha\beta}\right). \quad (22)$$

We use Eqs. (7), (9), (10) and (20)–(22) with some manipulations to obtain the following scalar structures

$$X_T=\frac{4\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\left(\hat{\mu}+\frac{\varphi_\mu}{A^2}\right), \quad (23)$$

$$X_{TF}=-\mathcal{E}-\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}$$
$$\times\left(\hat{\Pi}-2\sigma\eta+\frac{\varphi_{P_r}}{B^2}-\frac{\varphi_{P_\perp}}{C^2}\right), \quad (24)$$

$$Y_T=\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}$$
$$\times\left(\hat{\mu}+\frac{\varphi_\mu}{A^2}+\frac{\varphi_{P_r}}{B^2}+\frac{2\varphi_{P_\perp}}{C^2}+3\hat{P}_r-2\hat{\Pi}\right), \quad (25)$$

$$Y_{TF}=\mathcal{E}-\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}$$
$$\times\left(\hat{\Pi}-2\eta\sigma+\frac{\varphi_{P_r}}{B^2}-\frac{\varphi_{P_\perp}}{C^2}\right), \quad (26)$$

where φ_μ, φ_{P_r} and $\varphi_{P_\perp}$ are given in Appendix. It is well established in GR as well as in $f(R)$ gravity that, one of the structure scalars X_T describes matter energy density while its inhomogeneity is discussed with the help of X_{TF} only if the system evolves adiabatically alongwith $\epsilon=0$. The scalar functions Y_{TF} and Y_T incorporating ϵ terms control the evolutionary mechanisms of shearing and expansion rates of the system.

The two independent components of the contracted Bianchi identities are

$$\left(T^{\alpha\beta}+\overset{(D)}{T^{\alpha\beta}}\right)_{;\beta}=0, \qquad \left(T^{\alpha\beta}+\overset{(D)}{T^{\alpha\beta}}\right)_{;\beta}=0, \quad (27)$$

which yield

$$\dot{\hat{\mu}}+\frac{A\hat{q}'}{B}+(\hat{P}_r+\hat{\mu})\frac{\dot{B}}{B}+\frac{2A\hat{q}C'}{BC}$$
$$+2(P_\perp+\hat{\mu})\frac{\dot{C}}{C}+D_0(t,r)=0, \quad (28)$$

$$\frac{A\hat{P}_r'}{B}+\dot{\hat{q}}+(\hat{P}_r+\hat{\mu})\frac{A'}{B}+2\left(\frac{\dot{C}}{C}+\frac{\dot{B}}{B}\right)\hat{q}$$
$$+2\hat{\Pi}\frac{(AC)'}{BC}+D_1(t,r)=0, \quad (29)$$

where D_0 and D_1 are $f(R)$ dark source terms given in Appendix. Now we find two very important differential equations which play a pivotal in the stability analysis of inhomogeneous energy density. These two equations were firstly calculated by Ellis (2009) and then by Herrera et al. (2004) in GR. These equations are obtained by using Eqs. (7)–(10), (15), (16) and (20) as

$$\begin{aligned}&\left[\mathcal{E}-\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\right.\\&\qquad\left.\times\left(\hat{\mu}-\hat{\Pi}+\frac{\varphi_\mu}{A^2}-\frac{\varphi_{P_r}}{B^2}+\frac{\varphi_{P_\perp}}{C^2}\right)\right]_{,0}\\&\quad=\frac{3\dot{C}}{C}\left[\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\right.\\&\qquad\left.\times\left(\hat{\mu}+\hat{P_\perp}+\frac{\varphi_\mu}{A^2}+\frac{\varphi_{P_\perp}}{C^2}\right)-\mathcal{E}\right]\\&\qquad+\frac{6\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\left(\frac{AC'}{BC}\right)\left(\hat{q}-\frac{\varphi_q}{AB}\right),\end{aligned}\tag{30}$$

$$\begin{aligned}&\left[\mathcal{E}-\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\right.\\&\qquad\left.\times\left(\hat{\mu}-\hat{\Pi}+\frac{\varphi_\mu}{A^2}-\frac{\varphi_{P_r}}{B^2}+\frac{\varphi_{P_\perp}}{C^2}\right)\right]_{,1}\\&\quad=-\frac{3C'}{C}\left[\frac{2\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\left(\hat{\mu}+\frac{\varphi_\mu}{A^2}\right)-\frac{3m}{C^3}\right]\\&\qquad-\frac{6\kappa\epsilon R}{4\epsilon R(1+2\epsilon R)+\lambda_n(2\epsilon R)^n}\left(\frac{B\dot{C}}{AC}\right)\left(\hat{q}-\frac{\varphi_q}{AB}\right),\end{aligned}\tag{31}$$

where ψ_q is mentioned in Appendix. Both of the above equations reduce to GR (Herrera 2011) under $\epsilon\to 0$.

4 Stability of homogeneous energy density

In this section, we discuss different factors affecting energy density homogeneity in matter distribution with $f(R)$ framework for different cases. We confine ourselves with present valued cosmological Ricci scalar, i.e., $R=\tilde{R}$.

4.1 Non-dissipative fluids

In this subsection, we perform our analysis with non-dissipative matter distribution with polynomial $f(R)$ gravity model for dust, isotropic and anisotropic fluid configurations.

4.1.1 Dust cloud

Here we take non-dissipative dust fluid with its geodesic motion which gives $\hat{q}=P_\perp=\hat{P}_r=0$ and $A=1$. In this context, Eqs. (30) and (31) give

$$\begin{aligned}&\left[\mathcal{E}-\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right.\\&\qquad\left.\times\left(\mu+\frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n}-\frac{\epsilon\tilde{R}^2}{2\kappa}\right)\right]_{,0}\\&\quad=\frac{3\dot{C}}{C}\left[\frac{2\kappa\epsilon\tilde{R}\mu}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}-\mathcal{E}\right],\end{aligned}\tag{32}$$

$$\begin{aligned}&\left[\mathcal{E}-\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right.\\&\qquad\left.\times\left(\mu+\frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n}-\frac{\epsilon\tilde{R}^2}{2\kappa}\right)\right]'\\&\quad=-\frac{3C'}{C}\mathcal{E}.\end{aligned}\tag{33}$$

By making use of Eqs. (6), (28) and (52)–(55) in Eqs. (32) and (33), we obtain

$$\dot{\mathcal{E}}+\frac{3\dot{C}}{C}\mathcal{E}=\frac{-2\kappa\epsilon A\sigma\mu\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n},\tag{34}$$

$$\mathcal{E}'+\frac{3C'}{C}\mathcal{E}=\frac{2\kappa\epsilon\tilde{R}\mu'}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}.\tag{35}$$

Equation (34) yields

$$\mathcal{E}=\frac{-2\kappa\epsilon\tilde{R}\int_0^t A\sigma\mu C^3dt}{[4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n]C^3},\tag{36}$$

which provides condition for the existence of homogeneity in the dust fluid. This states that non-dissipative homogeneous spherical matter configuration exists only if the system is conformally flat. Similarly, we can identify the Weyl scalar as an inhomogeneity factor from Eq. (35). Equation (36) also asserts that conformal flatness exists if the system evolves with vanishing shear scalar.

4.1.2 Isotropic fluid

Now, we consider adiabatic spherical system with locally isotropic pressure. Under this scenario, Eqs. (30) and (31) become

$$\begin{aligned}&\left[\mathcal{E}-\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right.\\&\qquad\left.\times\left(\mu+\frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n}-\frac{\epsilon\tilde{R}^2}{2\kappa}\right)\right]_{,0}\\&\qquad+\frac{3\dot{C}}{C}\left[\frac{-2\kappa\epsilon\tilde{R}(\mu+P)}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}+\mathcal{E}\right]=0,\end{aligned}\tag{37}$$

$$\left[\mathcal{E}-\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\times\left(\mu+\frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n}-\frac{\epsilon\tilde{R}^2}{2\kappa}\right)\right]'+\frac{3C'}{C}\mathcal{E}=0. \tag{38}$$

It is seen that Eq. (38) turns out to be same as Eq. (33), thus showing that $\mathcal{E}=0$ if and only if $\mu'=0$. Equation (37) after using Eqs. (6) and (28) provides the following differential equation

$$\dot{\mathcal{E}}+\frac{3\dot{C}}{C}\mathcal{E}=\frac{-2\kappa\epsilon A\sigma(\mu+P)\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}, \tag{39}$$

whose solution is

$$\mathcal{E}=\frac{-2\kappa\epsilon\tilde{R}\int_0^t A\sigma(\mu+P)C^3dt}{[4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n]C^3}. \tag{40}$$

This argues that energy density of the system will be homogeneous as long as the system embodies shear-free motion. Thus the condition of locally isotropic pressure in the matter configuration with constant Ricci scalar $f(R)$ model does not affect the stability of homogeneous energy density found in the above case. Let us assume shear-free fluid so that Eq. (39) provides

$$\mathcal{E}=\frac{\omega(r)}{C^3},$$

where $\omega(r)$ is an integration function. If the system is homogeneous initially, i.e., $\mathcal{E}(0,r)=0$, then $\omega=0$ yields $\mathcal{E}(t,r)=0$. Thus the above condition for homogeneous energy density will be valid from $t=0$ to onward. However, if the fluid is expanding such that $\mathcal{E}$ has very small (non-zero) value at the initial stage, then it will remain as it is in all the evolutionary phases. If instead the system is contracting, then the Weyl scalar does not vanish for all time.

4.1.3 Anisotropic fluid

This case corresponds to anisotropic but non-dissipating matter distribution, i.e. $\Pi\neq 0$ and $\hat{q}=0$. In this framework, Eqs. (30) and (31) provide

$$\left[\mathcal{E}-\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\times\left(\mu-\Pi+\frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n}-\frac{\epsilon\tilde{R}^2}{2\kappa}\right)\right]_{,0}=\frac{3\dot{C}}{C}\left[\frac{2\kappa\epsilon\tilde{R}(\mu+P_\perp)}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}-\mathcal{E}\right], \tag{41}$$

$$\left[\mathcal{E}-\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\times\left(\mu-\Pi+\frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n}-\frac{\epsilon\tilde{R}^2}{2\kappa}\right)\right]'=-\frac{3C'}{C}\left[\mathcal{E}+\frac{2\kappa\epsilon\tilde{R}\Pi}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right]. \tag{42}$$

We use Eqs. (6) and (28) in Eqs. (41) and (42) to obtain the following set of evolutionary equations

$$\left[\mathcal{E}+\frac{2\kappa\epsilon\tilde{R}\Pi}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right]_{,0}+\frac{3\dot{C}}{C}\left[\mathcal{E}+\frac{2\kappa\epsilon\tilde{R}\Pi}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right]=\frac{-2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\left\{A\sigma(\mu+P_r)-2\Pi\frac{\dot{C}}{C}\right\},$$

$$\left[\mathcal{E}+\frac{2\kappa\epsilon\tilde{R}\Pi}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right]'-\frac{2\kappa\epsilon\tilde{R}\mu'}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}=-\frac{3C'}{C}\left[\mathcal{E}+\frac{2\kappa\epsilon\tilde{R}\Pi}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\right].$$

These equations in terms of structure scalar (24) can be written as

$$\dot{X}_{TF}+\frac{3X_{TF}\dot{C}}{C}=\frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\times\left\{A\sigma(\mu+P_r)-2\Pi\frac{\dot{C}}{C}\right\},$$

$$X'_{TF}+\frac{3X_{TF}C'}{C}=-\frac{2\kappa\epsilon\tilde{R}\mu'}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n},$$

whose general solutions can be found respectively as

$$X_{TF}=-\frac{2\kappa\epsilon\tilde{R}\int_0^t[2\Pi\dot{C}-AC\sigma(\mu+P_r)]C^2dt}{C^3[4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n]}, \tag{43}$$

$$X_{TF}=-\frac{2\kappa\epsilon\tilde{R}\int_0^r C^3\mu' dr}{C^3[4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n]}. \tag{44}$$

These equations indicate that quantity incorporating stability of inhomogeneous energy density is one of the $f(R)$ structure scalars, i.e., X_{TF}. Equation (44) shows that $\mu'=0$ if and only if X_{TF} vanishes, thereby showing X_{TF} as a factor of controlling inhomogeneity in anisotropic spherical system which is well-consistent with (Herrera et al. 2009, 2011; Herrera 2011). Thus the inclusion of dark matter/energy effects in the evolving system do not disrupt the importance of X_{TF}. Also, the above expressions reduce to GR under the limit $\epsilon \to 0$. However, Eq. (43) asserts that anisotropic pressure, $f(R)$ model and shear scalar are responsible for the emergence of inhomogeneous energy density in the matter distribution.

4.2 Dissipative dust cloud

To see the effects of radiation density and heat conducting vector in the inhomogeneous energy density, we assume geodesic case, i.e., $P_r = P_\perp = 0$ with $A = 1$. Many authors (Herrera et al. 2004; Herrera 2011; Kolassis et al. 1988; Govender et al. 1998; Thirukkanesh and Maharaj 2009; Naidu et al. 2006) discussed spherical dissipative collapsing dust models with geodesics in order to explore dissipation effects through the system. In this case, Eqs. (30) and (31) yield

$$\begin{aligned}&\Bigg[\mathcal{E} - \frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n} \\ &\quad\times\left(\mu + \frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n} - \frac{\epsilon\tilde{R}^2}{2\kappa}\right)\Bigg]_{,0} \\ &\quad= \frac{3\dot{C}}{C}\left[\frac{2\kappa\epsilon\tilde{R}\mu}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n} - \mathcal{E}\right] \\ &\qquad+ \frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\left(\frac{AC'\hat{q}}{BC}\right),\end{aligned} \tag{45}$$

$$\begin{aligned}&\Bigg[\mathcal{E} - \frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n} \\ &\quad\times\left(\mu + \frac{\lambda_n(1-n)(2\epsilon\tilde{R})^n}{8\kappa\epsilon n} - \frac{\epsilon\tilde{R}^2}{2\kappa}\right)\Bigg]' \\ &\quad= -\frac{3C'}{C}\mathcal{E} - \frac{6\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\left(\frac{B\hat{q}\dot{C}}{AC}\right).\end{aligned} \tag{46}$$

Equation (46) gives

$$\Phi \equiv \mathcal{E} - \frac{6\kappa\epsilon\tilde{R}\int_0^r BC^2\tilde{q}\dot{C}dr}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}. \tag{47}$$

It is found that $\mu' = 0$ if and only if $\Phi = 0$, indicating that Φ is responsible for fluid density inhomogeneity in the dust spherical system with free streaming and diffusion approximations. We use Eqs. (6) and (28) in Eq. (45) to obtain Φ evolution equation as follows

$$\begin{aligned}\dot{\Phi} - \frac{\dot{\Psi}}{C^3} &= \frac{2\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n} \\ &\quad\times\left(\frac{\tilde{q}C'}{BC} - \tilde{\mu}\sigma - \frac{\tilde{q}'}{B}\right) - \frac{3\dot{C}}{C}\Phi,\end{aligned} \tag{48}$$

with $\Psi = \frac{6\kappa\epsilon\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}\int_0^r BC^2\tilde{q}\dot{C}dr$, which yields Φ as follows

$$\Phi = \frac{\int_0^t[\dot{\Psi} + \frac{2\kappa\epsilon C^2\tilde{R}}{4\epsilon\tilde{R}(1+2\epsilon\tilde{R})+\lambda_n(2\epsilon\tilde{R})^n}(\frac{\tilde{q}C'}{B} - \tilde{\mu}C\sigma - \frac{\tilde{q}'C}{B})]dt}{C^3}. \tag{49}$$

This indicates that various fluid parameters affect the evolution of Φ in the self-gravitating system. We also see from the above relation that existence of inhomogeneous density depends upon two factors, i.e., dissipative parameters and shear scalar. This describes that shearing scalar, radiation density and heat dissipation hold fundamental importance in the study of inhomogeneous matter distribution leading to gravitational collapse.

5 Summary and discussion

This work analyzes various factors producing inhomogeneity in the energy density of the spherical self-gravitating celestial body in $f(R)$ gravity. We have constructed structure scalars by orthogonally splitting the Riemann curvature tensor to obtain evolution equations using a viable inflationary $f(R)$ model. We have discussed our analysis for non-dissipative dust, isotropic as well as anisotropic fluid configurations and dust cloud dissipating fluid. The results are concluded as follows.

- For non-dissipative dust and locally isotropic ideal matter cloud, it is seen from Eqs. (35) and (38) that the system will encapsulate homogeneous energy density if and only if the system is conformally flat. The extra $f(\tilde{R})$ degrees of freedom terms turn down contribution of $\mathcal{E}$, thus relaxing conformal flatness condition.
- In an adiabatic anisotropic spherical system, the density inhomogeneity is described in terms of pressure anisotropy which in turn controlled by one of the structure scalars, X_{TF} as mentioned in Eq. (24). Equation (44) also establishes X_{TF} as an element of governing inhomogeneity in the system. This result is well-consistent with (Sharif and Yousaf 2014b) under $\lambda_n \to 0$ and (Herrera 2011) under $\epsilon \to 0$ which correspond to solutions in $R+\epsilon R^2$ gravity and GR, respectively.
- The quantity Φ is explored and identified to be responsible for the emergence of inhomogeneity in energy density for geodesic radiating dust fluid. Extra curvature $f(R)$ terms, dissipation parameters and shear scalar affect evolution of Φ as described by Eq. (49).
- All these results correspond to GR under $\epsilon \to 0$ (Herrera 2011). It is worth stressing that structure scalars obtained in Eqs. (23)–(26) hold fundamental importance in the study of self-gravitating system. For $\lambda_n \to 0$, scalar functions reduces for $f(R) = R + \epsilon R^2$ cosmology (Sharif

and Yousaf 2014b) while $\epsilon \to 0$ provides GR results (Herrera et al. 2011).

Acknowledgements We would like to thank the Higher Education Commission, Islamabad, Pakistan for its financial support through the *Indigenous Ph.D. Fellowship for 5K Scholars, Phase-II, Batch-I*.

Appendix

The higher curvature terms D_0 and D_1 of Eqs. (28) and (29) are given as

$$\begin{aligned}
D_0 = {}& \frac{1}{\kappa}\Bigg[\Bigg\{\left(\frac{A'}{A}\dot{f}_R+\frac{\dot{B}}{B}f'_R-\dot{f}'_R\right)\frac{1}{A^2B^2}\Bigg\}_{,1} \\
&+\frac{1}{A^2}\Bigg\{\frac{f_R}{2}\left(\frac{f}{f_R}-R\right)+\left(\frac{2C'}{C}+\frac{B'}{B}\right)\frac{f'_R}{B^2}+\frac{f''_R}{B^2} \\
&-\frac{\dot{f}_R}{A^2}\left(\frac{2\dot{C}}{C}+\frac{\dot{B}}{B}\right)\Bigg\}_{,0}+\frac{1}{A^2}\Bigg\{\frac{\ddot{f}_R}{A^2}-\left(\frac{B'}{B}+\frac{A'}{A}\right)\frac{f'_R}{B^2} \\
&+\frac{f''_R}{B^2}-\left(\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)\frac{\dot{f}_R}{A^2}\Bigg\}\frac{\dot{B}}{B} \\
&+\frac{2}{A^2}\Bigg\{-\left(\frac{A'}{A}-\frac{C'}{C}\right)\frac{f'_R}{B^2}+\frac{\ddot{f}_R}{A^2} \\
&-\frac{\dot{f}_R}{A^2}\left(\frac{3\dot{C}}{C}+\frac{\dot{A}}{A}\right)\Bigg\}\frac{\dot{C}}{C}+\left(\frac{\dot{B}}{B}f'_R-\dot{f}'_R+\frac{A'}{A}\dot{f}_R\right) \\
&\times\frac{1}{A^2B^2}\left(\frac{3A'}{A}+\frac{B'}{B}+\frac{2C'}{C}\right)\Bigg],
\end{aligned} \tag{50}$$

$$\begin{aligned}
D_1 = {}& \frac{1}{\kappa}\Bigg[\Bigg\{\frac{1}{B^2A^2}\left(\frac{A'}{A}\dot{f}_R-\dot{f}'_R+\frac{\dot{B}}{B}f'_R\right)\Bigg\}_{,0} \\
&+\frac{1}{B^2}\Bigg\{\frac{\ddot{f}_R}{A^2}-\frac{f_R}{2}\left(\frac{f}{f_R}-R\right)-\frac{f'_R}{B^2}\left(\frac{A'}{A}+2\frac{C'}{C}\right) \\
&-\frac{\dot{f}_R}{A^2}\left(\frac{\dot{A}}{A}-2\frac{\dot{C}}{C}\right)\Bigg\}_{,1}+\frac{1}{B^2}\Bigg\{\frac{f''_R}{B^2}+\frac{\ddot{f}_R}{A^2} \\
&-\left(\frac{A'}{A}+\frac{B'}{B}\right)\frac{f'_R}{B^2}-\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\frac{\dot{f}_R}{A^2}\Bigg\}\frac{A'}{A} \\
&+\frac{2}{B^2}\Bigg\{-\frac{f'_R}{B^2}\left(\frac{B'}{B}+\frac{C'}{C}\right)+\frac{f''_R}{B^2}-\frac{\dot{f}_R}{A^2}\left(\frac{\dot{B}}{B}+\frac{3\dot{C}}{C}\right)\Bigg\} \\
&\times\frac{C'}{C}-\frac{1}{(AB)^2}\left(\dot{f}'_R-\frac{\dot{B}}{B}f'_R-\frac{A'}{A}\dot{f}_R\right) \\
&\times\left(\frac{3\dot{B}}{B}+\frac{\dot{A}}{A}+\frac{2\dot{C}}{C}\right)\Bigg].
\end{aligned} \tag{51}$$

The quantities φ_μ, φ_{P_r}, $\varphi_{P_\perp}$ and φ_q are

$$\begin{aligned}
\varphi_\mu = {}& \frac{A^2}{\kappa}\Bigg[\frac{2\epsilon R''}{B^2}+\frac{\lambda_n(n-1)(2\epsilon R)^n}{\epsilon(2BR)^2}\Bigg\{\frac{(n-2)R'^2}{R}+R''\Bigg\} \\
&-\left(\frac{\dot{B}}{B}+2\frac{\dot{C}}{C}\right)\Bigg\{\frac{2\epsilon\dot{R}}{A^2}+\frac{\lambda_n(n-1)(2\epsilon R)^n\dot{R}}{2\epsilon R^2A^2}\Bigg\} \\
&-\frac{\epsilon R^2}{2}+\frac{\lambda_n(n-1)(2\epsilon R)^n}{8n\epsilon}-\Bigg\{\frac{2\epsilon R'}{B^2} \\
&+\frac{\lambda_n(n-1)(2\epsilon R)^nR'}{2\epsilon R^2B^2}\Bigg\}\left(\frac{B'}{B}-\frac{2C'}{C}\right)\Bigg],
\end{aligned} \tag{52}$$

$$\begin{aligned}
\varphi_{P_r} = {}& -\frac{B^2}{\kappa}\Bigg[\Bigg\{\frac{2\epsilon\dot{R}}{A^2}+\frac{\lambda_n(n-1)(2\epsilon R)^n\dot{R}}{2\epsilon R^2A^2}\Bigg\}\left(\frac{\dot{A}}{A}-\frac{2\dot{C}}{C}\right) \\
&+\frac{\lambda_n(1-n)(2\epsilon R)^n}{8\epsilon n}-\frac{\epsilon R^2}{2}-\frac{2\epsilon\ddot{R}}{A^2} \\
&+\frac{\lambda_n(1-n)(2\epsilon R)^n}{\epsilon(2AR)^2}\Bigg\{\ddot{R}+\frac{(n-2)\dot{R}^2}{R}\Bigg\}+\Bigg(\frac{A'}{A} \\
&+\frac{2C'}{C}\Bigg)\Bigg\{\frac{2\epsilon R'}{B^2}+\frac{\lambda_n(n-1)(2\epsilon R)^nR'}{2\epsilon R^2B^2}\Bigg\}\Bigg],
\end{aligned} \tag{53}$$

$$\begin{aligned}
\varphi_{P_\perp} = {}& -\frac{C^2}{\kappa}\Bigg[\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\Bigg\{\frac{2\epsilon\dot{R}}{A^2} \\
&+\frac{\lambda_n(n-1)(2\epsilon R)^n\dot{R}}{2\epsilon R^2A^2}\Bigg\}+\frac{\lambda_n(1-n)}{8\epsilon n} \\
&\times(2\epsilon R)^n+\Bigg\{\frac{2\epsilon R'}{B^2}+\frac{\lambda_n(n-1)(2\epsilon R)^nR'}{2\epsilon R^2B^2}\Bigg\} \\
&\times\left(\frac{C'}{C}-\frac{B'}{B}+\frac{A'}{A}\right)+\left(\frac{R''}{B^2}-\frac{\ddot{R}}{A^2}\right)2\epsilon \\
&+\frac{\lambda_n(n-1)(2\epsilon R)^n}{4\epsilon R^2}\Bigg\{\frac{R''}{B^2}-\frac{\ddot{R}}{A^2}+\frac{(n-2)R'^2}{RB^2} \\
&-\frac{(n-2)\dot{R}^2}{RA^2}\Bigg\}-\frac{\epsilon R^2}{2}\Bigg],
\end{aligned} \tag{54}$$

$$\begin{aligned}
\varphi_q = {}& \frac{1}{\kappa}\Bigg[\frac{\lambda_n(n-1)(2\epsilon R)^n}{4\epsilon R^2}\Bigg\{\frac{(n-2)\dot{R}R'}{R}+\dot{R}'\Bigg\} \\
&-2\epsilon\left(\frac{R'\dot{B}}{B}+\frac{\dot{R}A'}{A}\right)-\frac{\lambda_n(n-1)(2\epsilon R)^n}{2\epsilon R^2} \\
&\times\left(\frac{R'\dot{B}}{B}+\frac{\dot{R}A'}{A}\right)+2\epsilon\dot{R}'\Bigg].
\end{aligned} \tag{55}$$

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