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ORIGINAL ARTICLE

# Periodic orbits of the generalized Friedmann–Robertson–Walker Hamiltonian systems

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**Abstract** The averaging theory of first order is applied to study a generalization of the Friedmann–Robertson–Walker Hamiltonian systems with three parameters. We provide sufficient conditions on the three parameters of the generalized system to guarantee the existence of continuous families of periodic orbits parameterized by the energy, and these families are given up to first order in a small parameter.

**Keywords** Periodic orbit · Averaging theory · Friedmann–Robertson–Walker Hamiltonian system

## 1 Introduction

The dynamics of the universe is an area of the astrophysics where the application of modern results coming from dynamical systems has been revealed very fruitful, specially in galactic dynamics see for instance the articles (Belmonte et al. 2007; Merritt and Valluri 1996; Papaphilippou and Laskar 1996, 1998; Zhao et al. 1999) and the references quoted there.

Calzeta and Hasi (1993) present analytical and numerical evidence of the existence of chaotic motion for the simplified Friedmann–Robertson–Walker Hamiltonian

$$H=\frac{1}{2}\left(p_Y^2-p_X^2\right)+\frac{1}{2}\left(Y^2-X^2\right)+\frac{b}{2}X^2Y^2, \tag{1}$$

which modelates a universe, filled with a conformally coupled but massive real scalar field. Although this model is too simplified to be considered realistic, its simplicity itself makes it an interesting testing ground for the implications of chaos in cosmology, either classical, semiclassical or quantum, see for more details (Calzeta and Hasi 1993). Similar models have been used by Hawking (1985) and Page (1991) to discuss the relationship between the cosmological and thermodynamic arrow of time, in the framework of quantum cosmology.

In problems of galactic dynamics it is usual to consider potentials of the form $V(x^2, y^2)$, i.e. potentials exhibiting a reflection symmetry with respect to both axes, see Pucacco et al. (2008) and the previous articles mentioned on galactic dynamics. For this reason here we generalize the Calzeta–Hasi's model as follows

$$H=\frac{1}{2}\left(p_Y^2-p_X^2\right)+\frac{1}{2}\left(Y^2-X^2\right)+\frac{a}{4}X^4+\frac{b}{2}X^2Y^2+\frac{c}{4}Y^4. \tag{2}$$

A general result of the qualitative theory of differential systems states that any orbit or trajectory of a differential system is homeomorphic either to a point, or to a circle, or to a straight line. The orbits homeomorphic to a point are the equilibrium points, and the ones homeomorphic to circles are the periodic orbits. It is well known that these two types of orbits play a relevant role in the dynamics of a differential system, and in general they are easier to study than the orbits homeomorphic to straight lines which sometimes can exhibit a very complicate dynamics. In short, the first analysis for understanding the dynamics of a differential system is to start studying its equilibrium points, its periodic solutions and their kind of stability.

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In this work we study the periodic orbits and their kind of stability of the simplified Friedmann–Robertson–Walker Hamiltonian (2) in every non-zero energy level, showing that in each of such levels there is at least one, two or three periodic orbits depending on the parameters of the system. So the modeled galaxy for this Hamiltonian system can exhibit periodic motion.

We shall provide sufficient conditions on the three parameters $a$, $b$ and $c$ of the Hamiltonian system with Hamiltonian (2) to guarantee the existence of continuous families of periodic orbits parameterized by the energy, and these families are given explicitly up to first order in a small parameter.

Our objective is to study analytically the periodic orbits of the two degree of freedom Hamiltonian system defined by Hamiltonian (2). We shall use the averaging theory for computing an explicit analytic approximation of four families of periodic orbits parameterized by the energy level $H=h$. The Hamiltonian system associated to Hamiltonian (2) is

$$\begin{aligned}\dot{X}&=-p_X,\\ \dot{Y}&=p_Y,\\ \dot{p}_X&=X-\big(aX^3+bXY^2\big),\\ \dot{p}_Y&=-Y-\big(bX^2Y+cY^3\big).\end{aligned}\tag{3}$$

Doing the rescaling of the variables

$$X=\sqrt{\varepsilon}x,\qquad Y=\sqrt{\varepsilon}y,\qquad p_X=\sqrt{\varepsilon}p_x,$$
$$p_Y=\sqrt{\varepsilon}p_y,$$

the Hamiltonian system (3) becomes the new Hamiltonian system

$$\begin{aligned}\dot{x}&=-p_x,\\ \dot{y}&=p_y,\\ \dot{p}_x&=x-\varepsilon\big(ax^3+bxy^2\big),\\ \dot{p}_y&=-y-\varepsilon\big(bx^2y+cy^3\big),\end{aligned}\tag{4}$$

with Hamiltonian

$$\begin{aligned}H=&\frac{1}{2}\big(p_y^2-p_x^2\big)+\frac{1}{2}\big(y^2-x^2\big)\\ &+\varepsilon\left(\frac{a}{4}x^4+\frac{b}{2}x^2y^2+\frac{c}{4}y^4\right).\end{aligned}\tag{5}$$

Periodic orbits are the most simple non-trivial solutions of a differential system. Their study is of special interest because the motion in their neighborhood can be determined by their kind of stability. We shall use the averaging theory of first order as it is stated in Sect. 2 for studying the periodic orbits of the Hamiltonian system (4) in every energy level $H=h$. Our main result on the periodic orbits is the next one.

**Theorem 1** *At every energy level $H=h$ with $h\neq 0$ the generalized Friedmann–Robertson–Walker Hamiltonian system (4) has at least one, two or three periodic solutions if one, two or three of the following conditions hold*:

(1) $h(b+c)(a+2b+c)<0$, $h(a+b)(a+2b+c)>0$ *and* $b\neq 0$, *the corresponding periodic solution is unstable if* $b(a+2b+c)>0$, *and linear stable if* $b(a+2b+c)<0$;
(2) $h(b+3c)(3a+2b+3c)<0$, $h(3a+b)(3a+2b+3c)>0$ *and* $b\neq 0$, *the corresponding periodic solution is unstable if* $b(3a+2b+3c)<0$, *and linear stable if* $b(3a+2b+3c)>0$;
(3) $h<0$, $b\neq 0$ *and* $(a+b)(3a+b)\neq 0$, *the corresponding periodic solution is unstable if* $(a+b)(3a+b)<0$, *and linear stable if* $(a+b)(3a+b)>0$; *and*
(4) $h>0$, $b\neq 0$ *and* $(b+c)(b+3c)\neq 0$, *the corresponding periodic solution is unstable if* $(b+c)(b+3c)<0$, *and linear stable if* $(b+c)(b+3c)>0$.

For the Hamiltonian (1) studied by Calzeta and Hasi (1993) we have the following result, which follows directly from Theorem 1.

**Corollary 2** *At every energy level $H=h$ with $bh\neq 0$ the Friedmann–Robertson–Walker Hamiltonian system (4) with $a=c=0$ has at least one periodic solution.*

We can be more precise than in the statement of Theorem 1. Thus we consider the following seven hyperplanes

$$h=0,\qquad a+b=0,\qquad 3a+b=0,\qquad b+c=0,$$
$$b+3c=0,\qquad a+2b+c=0,\qquad 3a+2b+c=0,$$

in the 4-dimensional space of parameters $(h,a,b,c)\in\mathbb{R}^4$. These seven hyperplanes separate the parameter space $\mathbb{R}^4$ into $2^7=128$ open regions. Every one of these open regions will be denoted by the seven signs of the seven hyperplanes. Thus when we write $+++++++$, 4 this means that in the region

$$h>0,\qquad a+b>0,\qquad 3a+b>0,\qquad b+c>0,$$
$$b+3c>0,\qquad a+2b+c>0,\qquad 3a+2b+3c>0,$$

there is only one periodic orbit provided by the fourth condition of Theorem 1. On the other hand, $++++-++$, 2, 4 this means that in the region

$$h>0,\qquad a+b>0,\qquad 3a+b>0,\qquad b+c>0,$$
$$b+3c<0,\qquad a+2b+c>0,\qquad 3a+2b+3c>0,$$

there are two periodic orbits provided by the second and fourth conditions of Theorem 1.

Note that either condition (3) or (4) of Theorem 1 always occurs in every one of the 128 open regions.

Now we summarize the number of periodic orbits in every one of the 128 open regions:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| +++++++ | 4 | −−−−−−− | 3 | −++++++ | 3 | +−+++++ | 4 |
| ++−++++ | 4 | +++−+++ | 1, 4 | ++++−++ | 2, 4 | +++++−+ | 4 |
| ++++++− | 4 | −−+++++ | 1, 3 | +−−++++ | 4 | ++−−+++ | 1, 4 |
| +++−−++ | 1, 2, 4 | ++++−−+ | 2, 4 | +++++−− | 4 | −+−++++ | 2, 3 |
| +−+−+++ | 4 | ++−+−++ | 4 | +++−+−+ | 4 | ++++−+− | 4 |
| −++−+++ | 3 | +−++−++ | 2, 4 | ++−++−+ | 4 | +++−++− | 1, 4 |
| −+++−++ | 1, 3 | +−+++−+ | 1, 4 | ++−+++− | 2, 4 | −++++−+ | 3 |
| +−++++− | 4 | −+++++− | 3 | ++−−−−− | 4 | −++−−−− | 1, 2, 3 |
| −−++−−− | 2, 3 | −−−++−− | 3 | −−−−++− | 3 | −−−−−++ | 3 |
| +−+−−−− | 4 | −+−+−−− | 3 | −−+−+−− | 3 | −−−+−+− | 1, 3 |
| −−−−+−+ | 2, 3 | +−−+−−− | 1, 4 | −+−−+−− | 1, 3 | −−+−−+− | 2, 3 |
| −−−+−−+ | 3 | +−−−+−− | 2, 4 | −+−−−+− | 3 | −−+−−−+ | 3 |
| +−−−−+− | 4 | −+−−−−+ | 1, 3 | +−−−−−+ | 4 | −−−++++ | 1, 2, 3 |
| −−+−+++ | 3 | −−++−++ | 1, 3 | −−+++−+ | 2, 3 | −−++++− | 1, 3 |
| −+−−+++ | 2, 3 | −+−+−++ | 3 | −+−++−+ | 2, 3 | −+−+++− | 3 |
| −++−−++ | 3 | −++−+−+ | 1, 3 | −++−++− | 3 | −+++−−+ | 3 |
| −+++−+− | 2, 3 | −++++−− | 3 | +−−−+++ | 4 | +−−+−++ | 4 |
| +−−++−+ | 1, 4 | +−−+++− | 2, 4 | +−+−−++ | 2, 4 | +−+−+−+ | 4 |
| +−+−++− | 4 | +−++−−+ | 1, 2, 4 | +−++−+− | 4 | +−+++−− | 1, 4 |
| ++−−−++ | 1, 4 | ++−−+−+ | 4 | ++−−++− | 1, 2, 4 | ++−+−−+ | 4 |
| ++−+−+− | 4 | ++−++−− | 2, 4 | +++−−−+ | 2, 4 | +++−−+− | 1, 4 |
| +++−+−− | 4 | ++++−−− | 4 | +++−−−− | 4 | ++−+−−− | 4 |
| ++−−+−− | 2, 4 | ++−−−+− | 4 | ++−−−−+ | 4 | +−++−−− | 1, 4 |
| +−+−+−− | 4 | +−+−−+− | 4 | +−+−−−+ | 2, 4 | +−−++−− | 1, 4 |
| +−−+−+− | 4 | +−−+−−+ | 1, 4 | +−−−++− | 2, 4 | +−−−+−+ | 4 |
| +−−−−++ | 4 | −+++−−− | 2, 3 | −++−+−− | 1, 3 | −++−−+− | 2, 3 |
| −++−−−+ | 1, 3 | −+−++−− | 3 | −+−+−+− | 3 | −+−+−−+ | 3 |
| −+−−++− | 3 | −+−−+−+ | 1, 2, 3 | −+−−−++ | 3 | −−+++−− | 3 |
| −−++−+− | 1, 2, 3 | −−++−−+ | 3 | −−+−++− | 3 | −−+−+−+ | 3 |
| −−+−−++ | 3 | −−−+++− | 1, 3 | −−−++−+ | 2, 3 | −−−+−++ | 1, 3 |
| −−−−+++ | 2, 3 | +−−−−−− | 4 | −+−−−−− | 1, 3 | −−+−−−− | 2, 3 |
| −−−+−−− | 3 | −−−−+−− | 3 | −−−−−+− | 3 | −−−−−−+ | 3 |

Theorem 1 is proved in Sect. 3.

## 2 The averaging theory

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst (1996).

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{6}$$

with $\mathbf{x} \in D$, where $D$ is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F_1(t, \mathbf{x})$ and $F_2(t, \mathbf{x}, \varepsilon)$ are $T$-periodic in $t$. We also consider in $D$ the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{x}_0, \tag{7}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T} \int_0^T F_1(t, \mathbf{y}) dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T$-periodic solutions of (6).

**Theorem 3** *Consider the two initial value problems* (6) *and* (7). *Suppose*:

(i) *$F_1$, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, $F_2$ and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of $\varepsilon$ in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$.*
(ii) *$F_1$ and $F_2$ are $T$-periodic in $t$ ($T$ independent of $\varepsilon$).*

*Then the following statements hold.*

(a) *If $p$ is an equilibrium point of the averaged (7) and*

$$\det\left(\frac{\partial f_1}{\partial \mathbf{y}}\right)\bigg|_{\mathbf{y}=p} \neq 0,$$

*then there exists a $T$-periodic solution $\varphi(t,\varepsilon)$ of (6) such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*

(b) *The stability or instability of the limit cycle $\varphi(t,\varepsilon)$ is given by the stability or instability of the equilibrium point $p$ of the averaged system (7). In fact the singular point $p$ has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$.*

We point out the main facts in order to prove Theorem 3(b), for more details see Sects. 6.3 and 11.8 in Verhulst (1996).

## 3 Proof of Theorem 1

Periodic orbits of a Hamiltonian system of more than one degree of freedom are generically on cylinders filled with periodic orbits in the phase space (for more details see Abraham and Marsden 1978), then we will not be able to apply directly the Averaging Theorem of Sect. 2 to a Hamiltonian system because the Jacobian of the corresponding function $f_1$ at the fixed point $a$ will be always zero. This problem will be solved by fixing an energy level, where the periodic orbits generically are isolated.

In the variables $(x, y, p_x, p_y)$ we consider the Hamiltonian system (3).

Let $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{S}^1$ the circle. We do the change of variables $(x, y, p_x, p_y) \to (r,\theta,s,\alpha) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1$ defined by

$$x = r\cos\theta, \qquad p_x = r\sin\theta, \qquad y = s\cos(\alpha-\theta),$$
$$p_y = s\sin(\alpha-\theta).$$

Note that it is not canonical, so we loss the Hamiltonian structure of the differential equations. The differential system in the new variables become

$$\begin{aligned}
\dot r &= -\varepsilon r\sin\theta\cos\theta\big(ar^2\cos^2\theta + bs^2\cos^2(\alpha-\theta)\big),\\
\dot\theta &= 1 - \varepsilon\cos^2\theta\big(ar^2\cos^2\theta + bs^2\cos^2(\alpha-\theta)\big),\\
\dot s &= -\varepsilon\frac{1}{4}s\big(br^2 + cs^2 + cs^2\cos\big(2(\alpha-\theta)\big)\\
&\quad + br^2\cos(2\theta)\big)\sin\big(2(\alpha-\theta)\big),\\
\dot\alpha &= \varepsilon\big(-cs^2\cos^4(\alpha-\theta) - b\big(r^2+s^2\big)\\
&\quad \times\cos^2\theta\cos^2(\alpha-\theta) - ar^2\cos^4\theta\big),
\end{aligned}\tag{8}$$

having the first integral

$$\begin{aligned}
H &= \frac{1}{2}\big(s^2-r^2\big) + \frac{\varepsilon}{4}\big(cs^4\cos^4(\alpha-\theta)\\
&\quad + 2br^2s^2\cos^2(\alpha-\theta)\cos^2\theta + ar^4\cos^4\theta\big).
\end{aligned}\tag{9}$$

In order that the right hand side of the differential system (8) be periodic with respect to the independent variable, we change the old independent variable $t$ by the new independent variable $\theta$, for obtaining the periodicity necessary for applying the averaging theory. Dividing system (8) by $\dot\theta$ omitting the $\dot\theta$ equation, system (8) goes over to

$$\begin{aligned}
r' &= -\varepsilon r\sin\theta\cos\theta\big(ar^2\cos^2\theta + bs^2\cos^2(\alpha-\theta)\big)\\
&\quad + O\big(\varepsilon^2\big),\\
s' &= -\frac{\varepsilon}{4}s\big(br^2 + cs^2 + cs^2\cos\big(2(\alpha-\theta)\big)\\
&\quad + br^2\cos(2\theta)\big)\sin\big(2(\alpha-\theta)\big) + O\big(\varepsilon^2\big),\\
\alpha' &= \varepsilon\big(-cs^2\cos^4(\alpha-\theta) - b\big(r^2+s^2\big)\\
&\quad \times\cos^2\theta\cos^2(\alpha-\theta) - ar^2\cos^4\theta\big) + O\big(\varepsilon^2\big),
\end{aligned}\tag{10}$$

where the prime denotes the derivative with respect to the new independent variable $\theta$. System (10) is $2\pi$-periodic in the variable $\theta$. However as the differential system (10) comes from a Hamiltonian system, as we mentioned before, its periodic orbits are not isolated in the set of all periodic orbits of system (10). Consequently, in order to use the averaging theory for studying its periodic orbits, we restrict the differential system (10) to every fixed energy level $H(r,\theta,s,\alpha) = h$. Then in such energy levels, we can put $s$ in function of $h$, $\theta$, $r$ and $\alpha$ and substitute $s$ in (10), and we will be able to apply Theorem 3. For $s$ we get

$$s = \sqrt{2h + r^2} + O(\varepsilon).$$

As we will apply averaging of first order, we do not need more information on $s$. Substituting $s$ in (10), this becomes

$$\begin{aligned}
r' &= -\varepsilon r\sin\theta\cos\theta\big(ar^2\cos^2\theta + b\big(2h+r^2\big)\cos^2(\alpha-\theta)\big)\\
&\quad + O\big(\varepsilon^2\big),\\
\alpha' &= \varepsilon\big(-c\big(2h+r^2\big)\cos^4(\alpha-\theta)\\
&\quad - 2b\big(h+r^2\big)\cos^2(\alpha-\theta)\cos^2\theta - ar^2\cos^4\theta\big)\\
&\quad + O\big(\varepsilon^2\big).
\end{aligned}$$

If we write the previous system as a Taylor series of first order in $\varepsilon$ we get

$$\begin{aligned}
r' &= \varepsilon F_{11}(\theta,r,\alpha) + O\big(\varepsilon^2\big),\\
\alpha' &= \varepsilon F_{12}(\theta,r,\alpha) + O\big(\varepsilon^2\big).
\end{aligned}\tag{11}$$

We see that system (11) has the canonical form (6) for applying the averaging theory and satisfies the assumptions of Theorem 3 for $|\varepsilon| > 0$ sufficiently small, with $T = 2\pi$ and $F_1 = (F_{11}, F_{12})$ which are analytical functions.

Averaging the function $F_1$ with respect to the variable $\theta$ we obtain

$$f_1(r,\alpha) = \big(f_{11}(r,\alpha), f_{12}(r,\alpha)\big) = \frac{1}{2\pi}\int_0^{2\pi}\big(F_{11}(\theta,r,\alpha), F_{12}(\theta,r,\alpha)\big)d\theta, \tag{12}$$

where

$$\begin{aligned} f_{11}(r,\alpha) &= -\frac{1}{8}br\big(2h+r^2\big)\sin(2\alpha), \\ f_{12}(r,\alpha) &= \frac{1}{8}\big(-6ch-3(a+c)r^2-4b\big(h+r^2\big) \\ &\quad -2b\big(h+r^2\big)\cos(2\alpha)\big). \end{aligned} \tag{13}$$

We have to find the zeros $(r^*,\alpha^*)$ of the function $f_1(r,\alpha)$, and to check that the Jacobian determinant at these points is not zero, i.e.

$$\det\left(\frac{\partial(f_{11},f_{12})}{\partial(r,\alpha)}\bigg|_{(r,\alpha)=(r^*,\alpha^*)}\right)\neq 0. \tag{14}$$

From $f_{11}(r,\alpha)=0$ we obtain that either

$$\begin{aligned} &\alpha = 0, \quad \pm\pi/2, \quad \pi, \quad \text{or} \\ &r=\sqrt{-2h} \quad \text{when } h<0. \end{aligned} \tag{15}$$

We look for the solutions of $f_{12}(r,\alpha)=0$ at these solutions of (15). We obtain seven possible solutions $(\alpha^*, r^*, s^*)$ with $r^*\geq 0$ and $s^*\geq 0$, namely

$$\begin{aligned} &\left(0, \sqrt{\frac{-2h(b+c)}{a+2b+c}}, \sqrt{\frac{2h(a+b)}{a+2b+c}}\right), \\ &\left(\pm\frac{\pi}{2}, \sqrt{\frac{-2(b+3c)h}{3a+2b+3c}}, \sqrt{\frac{2(3a+b)h}{3a+2b+3c}}\right), \\ &\left(\pm\frac{1}{2}\arccos\left(\frac{-3a-2b}{b}\right), \sqrt{-2h}, 0\right), \\ &\left(\pm\frac{1}{2}\arccos\left(\frac{-3a-2b}{b}\right), 0, \sqrt{2h}\right). \end{aligned} \tag{16}$$

But the solutions with $\pm$ provide two different initial conditions of the same periodic orbit. So we only have four different periodic orbits.

Finally we calculate the determinant (14) of the Jacobian matrix

$$\begin{pmatrix} -\frac{1}{8}b(2h+3r^2)\sin(2\alpha) & -\frac{1}{4}br(2h+r^2)\cos(2\alpha) \\ -\frac{1}{4}r(3a+4b+3c+2b\cos(2\alpha)) & \frac{1}{2}b(h+r^2)\sin 2\alpha \end{pmatrix} \tag{17}$$

at the four solutions $(r^*,\alpha^*,s^*)$ given in (16). The determinants are respectively given by

$$\begin{aligned} &\frac{3b(a+b)(b+c)h^2}{4(a+2b+c)}, \qquad -\frac{b(3a+b)(b+3c)h^2}{4(3a+2b+3c)}, \\ &\frac{3}{4}(a+b)(3a+b)h^2, \qquad \frac{3}{8}(b+c)(b+3c)h^2. \end{aligned} \tag{18}$$

To have the solutions (16) defined and the above determinants different from zero, we must have one of the following four conditions

(1) $h(b+c)(a+2b+c)<0$, $h(a+b)(a+2b+c)>0$ and $b\neq 0$;
(2) $h(b+3c)(3a+2b+3c)<0$, $h(3a+b)(3a+2b+3c)>0$ and $b\neq 0$;
(3) $h<0$, $b\neq 0$ and $(a+b)(3a+b)\neq 0$; and
(4) $h>0$, $b\neq 0$ and $(b+c)(b+3c)\neq 0$.

We conclude that under each of the four cases, the solutions $(r^*,\alpha^*,s^*)$ of (16) provide a periodic solution of system (11), and consequently of system (4).

According to Theorem 3(b), for completing the proof of Theorem 1 we need to study the kind of stability of the found periodic orbits. For this we only need to study the eigenvalues of the Jacobian matrix (17) at the different solutions $(r^*,\alpha^*,s^*)$ of (16), which are respectively

(1) $\pm\sqrt{3}h\sqrt{-B(A+B)(B+C)/(A+2B+C)}/2$;
(2) $\pm h\sqrt{B(3A+B)(B+3C)/(3A+2B+3C)}/2$;
(3) $\pm\sqrt{3}h\sqrt{-(A+B)(3A+B)}/2$; and
(4) $\pm\sqrt{3}h\sqrt{-(B+C)(B+3C)}/4$.

according with the previous four conditions. Then, from statement (b) of Theorem 3, it follows the stability of the periodic orbits described in Theorem 1.

## 4 Conclusions

We have used one important tools of the area of dynamical systems, the averaging theory for studying analytically the existence of periodic orbits and their stability adapted to Hamiltonian systems. The main results on the periodic orbits of the Hamiltonian system (4) are summarized in Theorem 1.

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