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ORIGINAL ARTICLE

LRS Bianchi type-I cosmological model in $f(R,T)$ theory of gravity

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Abstract The exact solutions of the field equations in respect of LRS Bianchi type-I space time filled with perfect fluid in the framework of $f(R,T)$ gravity (Harko et al., arXiv:1104.2669v2 [gr-qc], 2011) are derived. The physical behavior of the model is studied. In fact, the possibility of reconstruction of the LRS Bianchi type-I cosmology with an appropriate choice of a function $f(T)$ has been proved in $f(R,T)$ gravity.

Keywords $f(R,T)$ gravity · LRS Bianchi type-I space-time · Constant deceleration parameter

1 Introduction

In recent years, there has been a lot of interest in alternative theories of gravitation (Brans and Dicke 1961; Canuto et al. 1977; Saez and Ballester 1985). In view of the late time acceleration of the universe and the existence of the dark matter and dark energy, very recently, modified theories of gravity have been developed. Noteworthy amongst them are $f(R)$ theory of gravity formulated by Nojiri and Odintsov (2003a) and $f(R,T)$ theory of gravity proposed by Harko et al. (2011). Carroll et al. (2004) explained the presence of a late time cosmic acceleration of the universe in $f(R)$ gravity. Caroll et al. proposed dark energy model for specific $(1/R)$ modified gravity. However, this model is non-realistic because it does not pass Newton law. The first $f(R)$ dark energy model which passes the Newton law was proposed by Nojiri and Odintsov (2003a). Nojiri and Odintsov (2003b) demonstrated that phantom scalar in many respects looks like strange effective quantum field theory by introducing a non-minimal coupling of phantom field with gravity.

Nojiri and Odintsov (2004) first proposed a non-linear matter-gravity coupling as asymptotic dark energy and cosmic speed-up. They have shown that the effective quintessence naturally describes current cosmic speed-up. Allemandi et al. (2005) proposed the Palatini formulation of non-linear gravity-matter system. Using a scalar field Lagrangian as matter, it is shown that the emerging FRW cosmology may lead either to an effective quintessence phase (cosmic speed-up) or to an effective phantom phase. Here, the gravity-matter coupling part is assumed to be $f(R)$ theory-like Lagrangian, non-minimally coupled with a scalar field Lagrangian. Nojiri and Odintsov (2006a) developed the general scheme for modified $f(R)$ gravity reconstruction from any realistic FRW cosmology. They proved that the modified $f(R)$ gravity indeed represents the realistic alternative to general relativity, being more consistent in dark epoch. Further, Nojiri et al. (2006) developed the general programme of the unification of matter-dominated era with acceleration epoch for scalar-tensor theory or dark fluid. Nojiri and Odintsov (2003b) reviewed various modified gravities which have been considered as gravitational alternative for dark energy. Specifically, they have considered the versions of $f(R)$, $f(G)$, or $f(R,G)$ gravity, model with non-linear gravitational coupling or string-inspired model with Gauss-Bonnet-dilation coupling in the late universe where they lead to cosmic speed-up. It has been shown that some of such theories may pass the Solar System tests, they may naturally describe the effective (cosmological constant, quintessence or phantom) late-time era with a possible transition from deceleration to acceleration and increase in gravitational terms with scalar curvature decrease. The possible explanation of the coincidence problem as the manifestation

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of the universe expansion and the late time (quintessence or phantom) universe filled with dark fluid with inhomogeneous equation of state are described. Later on, Bertolami et al. (2007) studied the non-minimal $f(R)$-matter theory introduced in Nojiri and Odintsov (2004) and Allemandi et al. (2005). As a result of the coupling the motion of the massive particles is non-geodesic, and an extra force, orthogonal to the four-velocity, arises.

The connections with Modified Newtonian Dynamics (MOND) and the Pioneer anomaly were also explored. This model was extended to the case of the arbitrary couplings in both geometry and matter by Harko (2008). The astrophysical and cosmological implications of the non-minimal coupling matter-geometry coupling were extensively investigated by Harko (2010). The Palatini formulation of the non-minimal geometry-coupling models was considered by Harko et al. (2010). Harko and Lobo (2010) proposed a maximal extension of the Hilbert-Einstein action, by assuming that the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the matter Lagrangian L_m. A specific application of the latter $f(R, L_m)$ gravity was proposed by Poplawski (2006), which may be considered a relativistically covariant model of interacting dark energy, based on the principle of least action. Here, the cosmological constant in the gravitational Lagrangian is a function of the trace of the stress-energy tensor, and consequently the model was denoted "$\Lambda(T)$ gravity". Poplawski (2006) argued that recent cosmological data favor a variable cosmological constant, which is consistent with $\Lambda(T)$ gravity, without the need to specify an exact form of the function $\Lambda(T)$. This $\Lambda(T)$ gravity is more general than the Palatini $f(R)$ gravity and reduces to the latter when we neglect the pressure of the matter. For reviews of $f(R)$ theories of gravity, one can refer Lobo (2010), Capozziello and Faraoni (2010), Nojiri and Odintsov (2010, 2011). Felice and Tsujikawa (2010) as well as Nojiri and Odintsov (2011) presented a detailed reviews of a number of popular models of modified $f(R)$ gravity. Their properties and different representations are also discussed in details. The occurrences of Big Rip and other finite-time future singularities in modified $f(R)$ gravity are reviewed along with their solutions via the addition of higher-derivative gravitational invariants.

Recently, Harko et al. (2011) developed $f(R,T)$ modified theory of gravity, where the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar R and of the trace T of the stress-energy tensor. They have obtained the gravitational field equations in the metric formalism, as well as, the equations of motion for test particles, which follow from the covariant divergence of the stress-energy tensor. Generally, the gravitational field equations depend on the nature of the matter source. They have presented the field equations of several particular models, corresponding to some explicit forms of the function $f(R,T)$.

In $f(R,T)$ gravity theory models, the field equations of this theory are obtained from the Hilbert-Einstein type variational principle.

The action for this modified theory of gravity is given by

$$S = \int \sqrt{-g}\left(\frac{1}{16\pi G} f(R,T) + L_m\right) d^4x. \tag{1.1}$$

Here $f(R,T)$ is an arbitrary function of the Ricci scalar R and of the trace T of the stress-energy tensor of the matter $T_{\mu\nu}$. L_m is the matter Lagrangian.

The stress-energy tensor of matter is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}L_m)}{\delta g^{\mu\nu}}. \tag{1.2}$$

In the present paper, we use the natural system of units with $G = c = 1$ so that the Einstein gravitational constant is defined as $k^2 = 8\pi$.

The corresponding field equations of the $f(R,T)$ gravity are found by varying the action with respect to the metric $g_{\mu\nu}$:

$$\begin{aligned} &f_R(R,T)R_{\mu\nu} - \frac{1}{2}f(R,T)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T) \\ &\quad = 8\pi T_{\mu\nu} - f_T(R,T)T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}, \end{aligned} \tag{1.3}$$

where

$$f_R \equiv \frac{\delta f(R,T)}{\delta R}, \qquad f_T \equiv \frac{\delta f(R,T)}{\delta T}, \qquad \Box \equiv \nabla^\mu\nabla_\mu,$$

∇_μ is the covariant derivative and $T_{\mu\nu}$ is the standard matter energy-momentum tensor derived from the Lagrangian L_m.

One should note that when $f(R,T) \equiv f(R)$ then (1.3) reduces to the field equations of $f(R)$ gravity.

By contracting (1.3), we get

$$\begin{aligned} &f_R(R,T)R + 3\Box f_R(R,T) - 2f(R,T) \\ &\quad = 8\pi T - f_T(R,T)T - f_T(R,T)\Theta. \end{aligned} \tag{1.4}$$

Generally, the field equations also depend on [through the tensor $\Theta_{\mu\nu}$] the physical nature of the matter field. Hence, several theoretical models corresponding to different matter sources in $f(R,T)$ gravity can be obtained.

If we assume that the function $f(R,T)$ is given by

$$f(R,T) = R + 2f(T), \tag{1.5}$$

where $f(T)$ is an arbitrary function of the trace of the stress-energy tensor.

From (1.3), we get the gravitational field equations in this case as

$$\begin{aligned} G_{ij} &\equiv R_{ij} - \frac{1}{2}Rg_{ij} \\ &= 8\pi T_{ij} - 2f'(T)T_{ij} - 2f'(T)\Theta_{ij} - f(T)g_{ij}, \end{aligned} \tag{1.6}$$

where the prime denotes a derivative with respect to the argument.

Harko et al. (2011) have investigated FRW cosmological models in this theory by choosing appropriate function $f(T)$. They have also discussed the case of scalar fields since scalar fields play an vital role in cosmology. The equations of motion of test particles and a Brans-Dicke type formulation of the model are also presented.

Observations of microwave background radiation (CMB) and experimental data suggest that the present day universe is largely homogeneous and isotropic which is represented by FRW model. However, at the early stages of evolution of universe there are reasons to believe that the universe is, in general, spatially homogeneous and anisotropic. It is well known that spatially homogeneous and anisotropic cosmological models play a significant role in describing the large structure and behavior of the universe. Such models have been investigated in the framework of general relativity in search of a realistic picture of universe in early stages. Also, Bianchi type cosmological models are important in the sense that these are homogeneous and anisotropic from which the process of isotropisation of the universe is studied through the passage of time. A complete discussion of Bianchi type models is given in Kramer et al. (1980).

The purpose of present paper is to study different viable cosmological models in the newly established extension of the standard general relativity which is known as the $f(R,T)$ gravity theory. In this paper, we study a spatially homogenous and anisotropic Bianchi type-I model in $f(R,T)$ gravity (proposed by Harko et al. 2011) which is straight forward generalization of the flat Friedman Robertson-Walker (FRW) universe.

2 Metric and field equations

The LRS Bianchi-Type-I line element can be written as

$$ds^2 = dt^2 - A(t)^2 dx^2 - B(t)^2(dy^2 + dz^2), \tag{2.1}$$

where $A(t)$ and $B(t)$ are the scale factors (metric tensors) and functions of the cosmic time t only (non-static case).

The field equations in $f(R,T)$ theory of gravity for the function $f(R,T) = R + f(T)$ when the matter source is perfect fluid are given by (Harko et al. 2011; Poplawski 2006)

$$\begin{aligned} G_{ij} &\equiv R_{ij} - \frac{1}{2}Rg_{ij} \\ &= 8\pi T_{ij} + 2f'(T)T_{ij} + [2pf'(T) + f(T)]g_{ij}, \end{aligned} \tag{2.2}$$

where the prime indicates the derivative with respect to the argument.

The matter tensor for perfect fluid is

$$\Theta_{\mu\nu} = -2T_{ij} - pg_{ij} = (\rho, -p, -p, -p), \tag{2.3}$$

where $T_{ij} = (\rho + p)u_i u_j - pg_{ij}$.

Now, choose the function $f(T)$ of the trace of the stress-energy tensor of the matter so that

$$f(T) = \lambda T, \tag{2.4}$$

where λ is constant.

The corresponding field (2.2) for metric (2.1) with the help of (2.3) and (2.4) can be written as

$$\left(\frac{\dot B}{B}\right)^2 + 2\frac{\dot A \dot B}{AB} = (8\pi + 3\lambda)\rho + 2p\lambda, \tag{2.5}$$

$$\left(\frac{\dot B}{B}\right)^2 + 2\frac{\ddot B}{B} = -8\pi p + \lambda\rho, \tag{2.6}$$

$$\frac{\ddot B}{B} + \frac{\dot A \dot B}{AB} + \frac{\ddot A}{A} = -8\pi p + \lambda\rho, \tag{2.7}$$

where dot $(\cdot)$ indicates the derivative with respect to t.

These are three linearly independent equations (2.5)–(2.7) with four unknowns A, B, ρ and p. In order to solve the system completely, we impose a law of variation for the Hubble's parameter proposed by Bergman (1983). According to this law the variation of the mean Hubble parameter for LRS Bianchi type-I metric may be given by

$$H = k(AB^2)^{-m/3}, \tag{2.8}$$

where $k > 0$ and $m \geq 0$ are constants.

The spatial volume is given by

$$V = a^3 = AB^2, \tag{2.9}$$

where a is the mean scale factor.

The mean Hubble parameter H for LRS Bianchi type-I metric may given by

$$H = \frac{\dot a}{a} = \frac{1}{3}\left(\frac{\dot A}{A} + 2\frac{\dot B}{B}\right). \tag{2.10}$$

The directional Hubble parameters in the x, y and z respectively may be defined as

$$H_x = \frac{\dot A}{A} \quad \text{and} \quad H_y = H_z = \frac{\dot B}{B}. \tag{2.11}$$

The volumetric deceleration parameter is

$$q = -\frac{a\ddot a}{\dot a^2}. \tag{2.12}$$

On integration, after equating (2.8) and (2.10), we get

$$V = AB^2 = c_1 e^{3kt} \quad \text{for } m = 0 \tag{2.13}$$

and

$$V = AB^2 = (mkt + c_2)^{\frac{3}{m}} \quad \text{for } m \neq 0, \tag{2.14}$$

where c_1 and c_2 are positive constants of integration.

Using (2.8) and (2.13) for $m = 0$, and with (2.14) for $m \neq 0$ mean Hubble parameters are

$$H = k \quad \text{for } m = 0 \tag{2.15}$$

$$H = k(mkt + c_2)^{-1} \quad \text{for } m \neq 0. \tag{2.16}$$

Using (2.13), (2.14) and (2.9) in (2.12), we get constant values for the deceleration parameter for mean scale factor as:

$$q = m - 1 \quad \text{for } m \neq 0 \tag{2.17}$$

$$q = -1 \quad \text{for } m = 0. \tag{2.18}$$

In this paper, now we consider the model when $m = 0$ i.e. $q = -1$.

Using the mean Hubble parameter (2.10), and after subtraction of the (2.6) from (2.7), we get

$$\frac{d}{dt}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)3H = 0. \tag{2.19}$$

On integration of (2.19) and considering (2.15), we obtain

$$\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = l e^{-3kt}, \tag{2.20}$$

where l is constant of integration.

On integrating (2.20) and using (2.13) we get the exact expression for the scale factors:

$$A(t) = \kappa^{2/3} c_1^{1/3} e^{kt - \frac{2l}{9k} e^{-3kt}}, \tag{2.21}$$

$$B(t) = \kappa^{-1/3} c_1^{1/3} e^{kt + \frac{l}{9k} e^{-3kt}}, \tag{2.22}$$

where κ is positive constant of integration.

3 Physical properties

The different physical quantities are given below:

The directional Hubble parameter

$$H_x = k + \frac{2l}{3k} e^{-3kt}, \tag{3.1}$$

$$H_y = H_z = k - \frac{\lambda}{3k} e^{-3kt}. \tag{3.2}$$

The anisotropy parameter of the expansion Δ is defined as

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2l^2}{27k^4} e^{-6kt}, \tag{3.3}$$

where H_i $(i = 1, 2, 3)$ represents the directional Hubble parameters in the direction x, y and z respectively.

The expansion scalar θ is defined as $\theta = 3H$ and found as

$$\theta = 3k. \tag{3.4}$$

The shear scalar σ^2 is defined as $\sigma^2 = \frac{3}{2}\Delta H^2$ and found as

$$\sigma^2 = \frac{l^2}{9k^2} e^{-6kt}. \tag{3.5}$$

Using (2.21) and (2.22), we obtain the value of pressure as

$$p = \frac{1}{(-8\pi\chi + 2\lambda)}\left\{3k^2 + \frac{l^2}{9k}(3+\chi)e^{-6kt} + \chi\left[k(k+2) + \frac{2l}{3k}e^{-3kt}\right]\right\} \tag{3.6}$$

and we obtain the energy density as

$$\rho = \frac{6k^2}{3\lambda(1+\chi)} + \frac{(24\pi - 2\lambda)}{(3\lambda(1+\chi))(-8\pi\chi + 2\lambda)}\left\{3k^2 + \frac{l^2}{9k}(3+\chi)e^{-6kt} + \chi\left[k(k+2) + \frac{2l}{3k}e^{-3kt}\right]\right\}. \tag{3.7}$$

4 Discussion and conclusion

Equations (2.21) and (2.22) give solution of LRS Bianchi type-I model with exponential volumetric expansion in $f(R,T)$ gravity.

(i) These scale factors admit constant values at early times of the universe $[t \to 0]$, after that scale factors start increasing with the increase in cosmic time without showing any type of initial singularity and finally diverge to ∞ as $t \to \infty$.

This shows that at the initial epoch, the universe starts with zero volume and expands exponentially approaching to infinite volume.

(ii) Moreover, the expansion scalar for these scale factors exhibits the constant value i.e. $\theta = 3k$.

This shows uniform exponential expansion from $t = 0$ to $t = \infty$ i.e. universe expands homogeneously.

(iii) We have $H = k$, $q = -1$.

This indicates that the mean Hubble parameter is constant whereas directional Hubble parameters are dynamical. As time approaches from zero to infinity, the directional Hubble parameters start reducing towards the constant value of H and becomes equal as $t \to \infty$.

Also, the deceleration parameter appears with negative sign which implies accelerating expansion of the universe as one can expect for exponential volumetric expansion.

(iv) From (3.3), one can observe that at $t = 0$, the anisotropy parameter measures a constant value while it vanishes at infinite time of the universe. This indicates that the universe expands isotropically at later times.

(v) From (3.6) and (3.7), we get that the matter pressure and density are constant at early stages $(t = 0)$ of the universe and show monotonic behavior in the evolving cosmic time.

Here we have studied the spatially homogeneous and anisotropic LRS Bianchi type-I cosmological model with constant deceleration parameter in $f(R,T)$ theory of gravity. We have considered the exponential model for $m=0$ ($q=-1$) with negative deceleration parameter indicating that the universe is accelerating which is consistent with the present day observations. Perlmutter et al. (1999) and Riess et al. (1998, 1999, 2004) proved that the decelerating parameter of the universe is in the range $-1 \le q \le 0$, and the present day universe is undergoing accelerated expansion.

From (2.5) to (2.7), it is interesting to note that for particular choice of $f(R,T)$ the gravitational coupling becomes effective and time dependant coupling which is of the form

$$G = G_{eff} \pm 2f'(T).$$

Thus, the term 2 $f(T)$ in the gravitational action modifies the gravitational interaction between matter and curvature, replacing G by a running gravitational coupling parameter in LRS Bianchi type-I cosmology also.

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