Astrophys Space Sci (2012) 338:245–249
DOI 10.1007/s10509-011-0958-5

ORIGINAL ARTICLE

Note on the single-shock solutions of the Korteweg-de Vries-Burgers equation

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Received: 4 November 2011 / Accepted: 6 December 2011 / Published online: 15 December 2011

Abstract The well-known shock solutions of the Korteweg-de Vries-Burgers equation are revisited, together with their limitations in the context of plasma (astro)physical applications. Although available in the literature for a long time, it seems to have been forgotten in recent papers that such shocks are monotonic and unique, for a given plasma configuration, and cannot show oscillatory or bell-shaped features. This uniqueness is contrasted to solitary wave solutions of the two parent equations (Korteweg-de Vries and Burgers), which form a family of curves parameterized by the excess velocity over the linear phase speed.

Keywords Plasmas · Shock waves

Among the paradigm nonlinear evolution equations cropping up in various domains of physics, the Korteweg-de Vries-Burgers (KdVB) equation,

$$\frac{\partial \varphi_1}{\partial \tau} + A\varphi_1 \frac{\partial \varphi_1}{\partial \xi} + B\frac{\partial^3 \varphi_1}{\partial \xi^3} = C\frac{\partial^2 \varphi_1}{\partial \xi^2}, \tag{1}$$

arises in physical media where nonlinearity, dispersion and damping interact on slow timescales to produce solitary structures. More specifically, in plasma physics (1) typically obtains by reductive perturbation analysis of a multi-fluid model, through the use of coordinate stretching

$$\xi = \varepsilon^{1/2}(x - \lambda t), \qquad \tau = \varepsilon^{3/2} t, \tag{2}$$

combined with expansions of the dependent variables like

$$\varphi = \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots \tag{3}$$

in addition to an appropriate scaling of the damping coefficient, in many cases due to viscosity. Here x and t are the original space and time coordinates, respectively, and φ refers to the electrostatic potential of the solitary waves. In the absence of damping ($C = 0$), the KdVB equation (1) reduces to the KdV equation, whereas in the absence of dispersion ($B = 0$), it recovers the Burgers equation, which bears kink-shaped monotonic shock profile solutions. All this is well known and has been in the literature for a long time, but we will have to come back to these points later.

For a purely mathematical study of the properties of the KdVB equation, (1) is given and its coefficients A, B and C might be regarded as free parameters. However, the moment the KdVB equation is derived for a particular plasma (astro)physical configuration, the precise and often elaborate form of A, B and C has to be computed. Although the intermediate details need not concern us here, we still have to remind ourselves that A, B and C are functions of the plasma compositional parameters, which also determine the linear phase velocity λ, and thus cannot be chosen randomly. Moreover, in the process of deriving (1) one has imposed/used that φ_1 vanishes in the undisturbed medium, upstream of the shock or soliton solutions, translated as $\varphi_1 \to 0$ for $\xi \to +\infty$. All this has important consequences for the discussion which follows.

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Once this is properly kept in mind, there are several ways of deriving the stationary shock structure of (1), by changing to a co-moving frame with coordinate

$$\chi = \kappa(\xi - V\tau), \tag{4}$$

where κ and V are related to the inverse width and the speed of the shock, respectively. Therefore, it is assumed that both κ and V are positive. The shock solutions of the KdVB equation have been in the literature for a long time, and later rederived by the so-called "tanh" method, formalized by Malfliet and Hereman (1996a, 1996b).

However, we have to come back in explicit detail to the shock solution of the KdVB equation, in view of recent misunderstandings about its validity and its applications, as shown below. One also has to remember that for all solitary waves, for which explicit analytical expressions have been obtained, amplitude, width (inversely related to κ) and velocity V are inherently linked. Usually, fixing one of these parameters determines the others.

Now, when looking at several papers in the recent literature (Shah and Saeed 2009, 2011; Saeed and Shah 2010; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011; Shah et al. 2011), one sees that $\kappa = 1$ is taken, whether explicitly stated (Shah and Saeed 2009, 2011; Saeed and Shah 2010; Shah et al. 2011) or only implicitly (Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011), by using the shock solution in the form given by Shah and Saeed (2009). No justification at all is given as to why one would be allowed to put $\kappa = 1$, nor is there any discussion of the consequences. As we will see, taking $\kappa = 1$ is not only needlessly stringent, but also erroneous, and in many cases one is not even able to verify that it holds, given the complexities in the expressions for A, B and C, except for specific numerical choices of all plasma parameters. Some other papers even leave κ undetermined, as if it were a free parameter (Mahmood and Ur-Rehman 2010; Akhtar and Hussain 2011).

When the transformation (4) is applied to (1), one finds

$$-\kappa V \frac{d\varphi_1}{d\chi} + A\kappa\varphi_1 \frac{d\varphi_1}{d\chi} + B\kappa^3 \frac{d^3\varphi_1}{d\chi^3} - C\kappa^2 \frac{d^2\varphi_1}{d\chi^2} = 0. \tag{5}$$

One of the popular methods of finding the shock structure for (1) is through the tanh method, and we will follow the original paper by Malfliet and Hereman (1996a), rather than a vast array of newcomers. We are forced to do so, to point out where the specific restrictions to plasma (astro)physics applications play a role and to correct some uses in the literature which have strayed in this respect from the original solutions (Malfliet and Hereman 1996a) already available. Our treatment here is more general than that of Malfliet and Hereman (1996a), because in their paper $A = 1$ has been taken. While one can always rescale the absolute value of some of the coefficients in (1), one cannot easily do away with the sign, and we keep therefore A as determined by the plasma model under consideration.

Using the transformation $\alpha = \tanh\chi$ in (5) and noting that $d\alpha/d\chi = 1 - \tanh^2\chi$, we obtain

$$\begin{aligned} &-V\frac{d\varphi_1}{d\alpha} + A\varphi_1\frac{d\varphi_1}{d\alpha} \\ &\quad + B\kappa^2 \frac{d}{d\alpha}\left\{(1-\alpha^2)\frac{d}{d\alpha}\left[(1-\alpha^2)\frac{d\varphi_1}{d\alpha}\right]\right\} \\ &\quad - C\kappa\frac{d}{d\alpha}\left[(1-\alpha^2)\frac{d\varphi_1}{d\alpha}\right] = 0. \end{aligned} \tag{6}$$

Here one common factor κ and one common bracket $(1-\alpha^2)$ have already been divided out, to simplify the subsequent computations.

The idea is then to look for solutions φ_1 as a finite power series in α, which in this case (and in many others) will end with the quadratic term (Malfliet and Hereman 1996a), thus

$$\varphi_1 = \beta_0 + \beta_1\alpha + \beta_2\alpha^2. \tag{7}$$

The reason that the power series breaks off comes from a balance between the highest nonlinearity and dispersive terms in (6). Given that the different powers of α are functionally independent, we get a system of algebraic equations,

$$\alpha^0:\quad -V\beta_1 + A\beta_0\beta_1 - 2B\kappa^2\beta_1 - 2C\kappa\beta_2 = 0, \tag{8}$$

$$\begin{aligned}\alpha^1:\quad &-2V\beta_2 + 2A\beta_0\beta_2 + A\beta_1^2 - 16B\kappa^2\beta_2 \\ &+ 2C\kappa\beta_1 = 0,\end{aligned} \tag{9}$$

$$\alpha^2:\quad 3A\beta_1\beta_2 + 6B\kappa^2\beta_1 + 6C\kappa\beta_2 = 0, \tag{10}$$

$$\alpha^3:\quad 2A\beta_2^2 + 24B\kappa^2\beta_2 = 0, \tag{11}$$

determining the as yet unknown coefficients β_0, β_1 and β_2. Solve first (11) for β_2 to find

$$\beta_2 = -\frac{12B\kappa^2}{A}, \tag{12}$$

and substitute this in (10). This allows now to obtain

$$\beta_1 = -\frac{12C\kappa}{5A}. \tag{13}$$

Solving next (8) yields

$$\beta_0 = \frac{V}{A} + \frac{12B\kappa^2}{A}. \tag{14}$$

Although all coefficients needed for (7) have now been determined, there is still one condition to be satisfied before the scheme can work, namely (9). This was apparently overlooked or not deemed important (Shah and Saeed 2009, 2011; Saeed and Shah 2010; Mahmood and Ur-Rehman 2010; Akhtar and Hussain 2011; Pakzad 2011b; Shah et al. 2011), while others (Pakzad 2011a, 2011c, 2011d; Pakzad and Javidan 2011) just copied the erroneous solution, without going through the algebra. Working out (9), one arrives at

$$\kappa = \frac{C}{10B}, \tag{15}$$

where for simplicity we have taken both B and C positive, as they usually are in most examples found in the literature. Adopting other sign conventions can easily be incorporated but would add nothing to the physics. Indeed, it is straightforward to see that minus signs can be handled in the general solution by appropriate space and/or time reversals. Note in passing that $\kappa \tanh[\kappa(\xi - V\tau)] = -\kappa \tanh[-\kappa(\xi - V\tau)]$, for any real κ.

At this stage it is clear how serious a restriction $\kappa = 1$ is, for two separate reasons. First, all solitary wave characteristics show an inherent link between amplitude, width (inversely related to κ) and velocity V of the structure, and arbitrarily fixing one narrows the choices enormously. Second, assuming $\kappa = 1$ means from (15) that $C = 10B$, a relation which usually cannot be obeyed by inserting some numbers in the rather complicated expressions B and C, as a glance at the papers involved (Shah and Saeed 2009, 2011; Mahmood and Ur-Rehman 2010; Saeed and Shah 2010; Akhtar and Hussain 2011; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011; Shah et al. 2011) will immediately reveal. Taken together, this implies that the resulting numerics, graphs and discussions (Shah and Saeed 2009, 2011; Mahmood and Ur-Rehman 2010; Saeed and Shah 2010; Akhtar and Hussain 2011; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011; Shah et al. 2011) cannot be trusted.

Using now (15) in the coefficients (12)–(14) shows that

$$\beta_0 = \frac{V}{A} + \frac{3C^2}{25AB}, \qquad \beta_1 = -\frac{6C^2}{25AB}, \qquad \beta_2 = -\frac{3C^2}{25AB}. \tag{16}$$

At this stage the shock solution is

$$\varphi_1 = \frac{3C^2}{25AB}(1 - \tanh^2\chi) + \frac{V}{A} - \frac{6C^2}{25AB}\tanh\chi. \tag{17}$$

Since B and C are assumed positive, it is the sign of A which will be determining the polarity of the kink solution. However, this only obeys the requirement that $\varphi_1 \to 0$ for $\xi \to +\infty$ provided one takes

$$V = \frac{6C^2}{25B} = 24B\kappa^2. \tag{18}$$

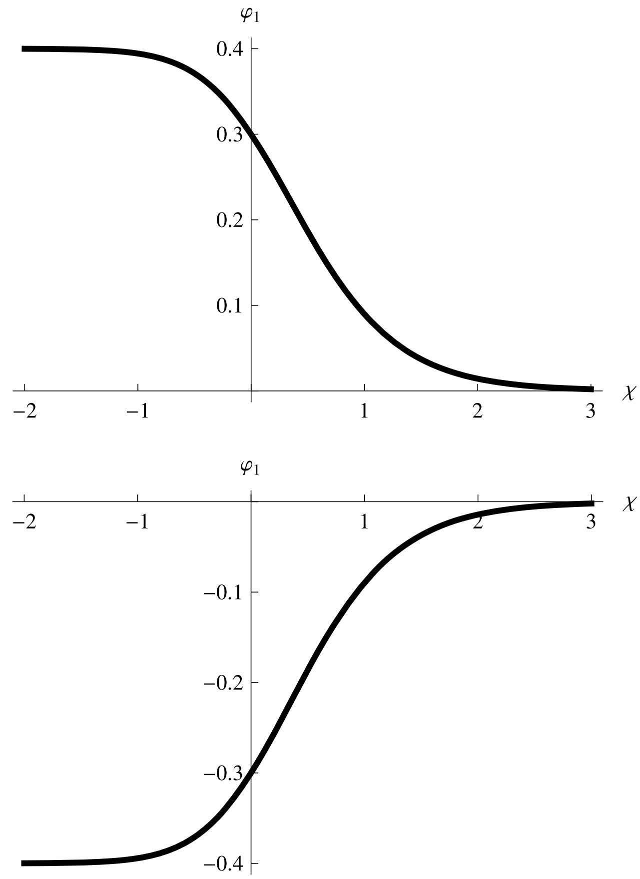

Fig. 1 Typical KdVB shock profile, where the amplitude $3C^2/(25AB) = 0.1$ has been taken for the upper panel and -0.1 for the lower panel

Also this inherent aspect of the correct solution has been overlooked in some of the recent papers (Shah and Saeed 2009; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011). The second expression for V in (18) clearly shows the link between width (through κ) and velocity of the structure, and for right propagating structures V is taken positive, which therefore requires B to be positive.

Finally, we arrive at the shock solution as

$$\varphi_1 = \frac{3C^2}{25AB}[1 - \tanh^2\chi + 2(1 - \tanh\chi)], \tag{19}$$

where in χ we have to insert (15) and (18), giving

$$\chi = \frac{C}{10B}\left(\xi - \frac{6C^2}{25B}\tau\right). \tag{20}$$

The kink structure (19) is unique, since for a given plasma configuration the compositional parameters fully determine A, B and C, and hence there is one and only one shock solution, the generic profile of which we illustrate in Fig. 1, once for a positive (upper panel), once for a negative (lower panel) polarity. This point has already been made before

(Malfliet and Hereman 1996a), in a mathematical discussion, almost in passing, without really stressing its consequences for detailed plasma (astro)physics problems.

Further remarks are in order here. Since (19) can be rewritten as

$$\varphi_1 = \frac{3C^2}{25AB}[4 - (1 + \tanh\chi)^2], \tag{21}$$

the kink is always monotonic, and no oscillatory part nor peak or bell-shaped curve may appear in its graph, contrary to what is found in recent papers (Shah and Saeed 2009; Mahmood and Ur-Rehman 2010; Saeed and Shah 2010; Akhtar and Hussain 2011; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011; Shah et al. 2011). There may be physical situations where shocks including oscillatory trails or precursors are observed, but these cannot be described by the KdVB formalism.

Note that when $C = 0$, the whole shock structure disappears. This is a direct consequence of the very delicate balance needed between a solitary wave (KdV) and a shock wave (Burgers) to form the combined solution (Malfliet and Hereman 1996a). To see this more explicitly, substitute in (19) $1 - \tanh^2\chi = \mathrm{sech}^2\chi$, which is reminiscent of the typical KdV one-soliton solution. In addition, since reductive perturbation analysis requires that φ_1 be small enough to neglect higher-order effects, $3C^2/(25|AB|)$ should be rather smaller than 1.

All this has to be contrasted to what happens when $C = 0$ and (1) reduces to the standard KdV equation, without dissipation through viscosity, or when $B = 0$ and (1) becomes the Burgers equation, in the absence of dispersion. Furthermore, when $C = 0$ the KdV $\mathrm{sech}^2\chi$ soliton cannot be directly recovered, contrary to what is claimed in the literature (Shah and Saeed 2009, 2011; Saeed and Shah 2010; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011; Shah et al. 2011).

To see the differences, let us now first put $C = 0$, return to (8)–(11) and go again through the motions. It turns out that β_2 is still given by (12), but $\beta_1 = 0$ and (14) is replaced here by

$$\beta_0 = \frac{V}{A} + \frac{8B\kappa^2}{A}. \tag{22}$$

Hence, to arrive at the typical KdV soliton solution in $\mathrm{sech}^2\xi = 1 - \tanh^2\xi$, obeying $\varphi_1 \to 0$ when $\xi \to \pm\infty$, it is required that

$$V = 4B\kappa^2, \tag{23}$$

and now for each superacoustic soliton velocity V one finds a soliton of the form

$$\varphi_1 = \frac{3V}{A}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{V}{B}}(\xi - V\tau)\right]. \tag{24}$$

Here $B > 0$ is needed, which is usually the case, and the soliton polarity is given by the sign of A.

Doing a similar exercise for the Burgers equation, with $B = 0$, leads from (10) and (11) to $\beta_2 = 0$, in other words, (7) stops at the linear term (Malfliet and Hereman 1996a). Now (8) and (9) give that

$$\beta_0 = \frac{V}{A}, \qquad \beta_1 = -\frac{2C\kappa}{A}, \tag{25}$$

and the proper solution needs

$$V = 2C\kappa. \tag{26}$$

Taking again V as the free parameter, the shock solution is found as

$$\varphi_1 = \frac{V}{A}\left\{1 - \tanh\left[\frac{V}{2C}(\xi - V\tau)\right]\right\}. \tag{27}$$

With the appropriate changes of notation, the solutions (24) and (27) can be found in the original discussion by Malfliet and Hereman (1996a).

To conclude, we have discussed the intricacies of the proper derivation of the solitary shock structure and its limitations in the context of plasma (astro)physical applications. Although these results and restrictions have been in the literature for a long time (Malfliet and Hereman 1996a, 1996b), it seems to have been forgotten in recent papers (Shah and Saeed 2009, 2011; Mahmood and Ur-Rehman 2010; Saeed and Shah 2010; Akhtar and Hussain 2011; Pakzad 2011a, 2011b, 2011c, 2011d; Pakzad and Javidan 2011; Shah et al. 2011) that a shock modeled by (19) can only be monotonic, without oscillations or peaks, and is, moreover, unique.

This also holds for the coefficients A, B and C, once specific numbers have been assigned to the various compositional parameters in the plasma model under consideration, and therefore A, B and C cannot be treated as free parameters, as they might be in a purely mathematical discussion of the properties of (1). But even then they determine V and κ in a unique way.

One sees that the solitary wave solutions of the two parent nonlinear equations, the KdV and the Burgers equations, are different in character, as they form one-parameter families of curves, dependent on the free choice of the excess velocity V above the linear phase speed λ.

Acknowledgements I.K. and S.S. acknowledge funding from the UK EPSRC (Engineering and Physical Science Research Council) via a Science and Innovation award to Centre for Plasma Physics, Queen's University Belfast (grant No. EP/D06337X/1).

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