Astrophys Space Sci (2011) 336:357–367
DOI 10.1007/s10509-011-0778-7

ORIGINAL ARTICLE

Families of periodic orbits in the restricted four-body problem

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Received: 16 May 2011 / Accepted: 14 June 2011 / Published online: 30 June 2011

Abstract In this paper, families of simple symmetric and non-symmetric periodic orbits in the restricted four-body problem are presented. Three bodies of masses m_1, m_2 and m_3 (primaries) lie always at the apices of an equilateral triangle, while each moves in circle about the center of mass of the system fixed at the origin of the coordinate system. A massless fourth body is moving under the Newtonian gravitational attraction of the primaries. The fourth body does not affect the motion of the three bodies. We investigate the evolution of these families and we study their linear stability in three cases, i.e. when the three primary bodies are equal, when two primaries are equal and finally when we have three unequal masses. Series, with respect to the mass m_3, of critical periodic orbits as well as horizontal and vertical-critical periodic orbits of each family and in any case of the mass parameters are also calculated.

Keywords Asymptotic orbits · Four-body problem · Periodic orbits · Stability

1 Introduction and equations of motion

The objective of our paper is the investigation, numerically, of the families of simple (crossing the synodical line twice in one period) periodic orbits of a body of negligible mass studding its motion under the Newtonian gravitational attraction of three bodies with masses m_1, m_2 and m_3 which move, with the same angular velocity, in circular orbits around their center of mass fixed at the origin of the coordinate system. In the general problem of three bodies there is a particular solution in which the bodies lie at the vertices of an equilateral triangle, each moving in a Keplerian orbit. This is well known, and was first given by Lagrange in 1772. He found three-body motions in which the mutual distances are constant, and Euler extended them and found solutions in which the ratios of mutual distances are constant. Although Lagrange thought his equilateral triangle solutions were of no great practical significance, it was later realized that the Sun, Jupiter and the Trojan asteroids formed such a configuration in our Solar system. Many scientists, among others, Andoyer (1906), Lindow (1922, 1923), Schaub (1929), MacMillan and Bartky (1932), Hüttenhain (1933), Pedersen (1944, 1952), Brumberg (1957), Palmore (1973), Simo (1978), Majorana (1981), Álvarez-Ramirez and Vidal (2009), Baltagiannis and Papadakis (2011), and references therein, studied the four-body problem.

The same problem, in various versions, has been used by many scientists for practical applications such as, among others, Van Hamme and Wilson (1986), Kloppenborg et al. (2010) (ε Aurigae system), Melita et al. (2008), Robutel and Gabern (2006) (Sun, Jupiter and Saturn system), Schwarz et al. (2009a) (star, two massive planets and a massless Trojan), Schwarz et al. (2009b) (star, brown dwarf, gas giant and a massless Trojan), Ceccaroni and Biggs (2010) (Sun, Jupiter, Trojan Asteroid, Spacecraft), and references therein.

We consider that three particles of masses m_1, m_2 and m_3 always lie at the vertices of an equilateral triangle and one of them, say m_1, is on the positive x-axis at the origin of time. The motion of the system is referred to axes rotating with uniform angular velocity. The three bodies move in the same plane and their mutual distances remain unchanged

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with respect to time. The motion of the primaries consists of circular orbits around their center of gravity.

The equations of motion of the infinitesimal mass of the restricted four-body problem, in the usual dimensionless rectangular rotating coordinate system (x, y, z) are written as Moulton (1900),

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} = x - \sum_{i=1}^{3} \frac{m_i(x - x_i)}{r_i^3},\\ \ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} = y - \sum_{i=1}^{3} \frac{m_i(y - y_i)}{r_i^3},\\ \ddot{z} &= \frac{\partial \Omega}{\partial z} = -\sum_{i=1}^{3} \frac{m_i(z - z_i)}{r_i^3},\end{aligned} \tag{1}$$

where dots denote time derivatives while the gravitational potential Ω in synodic coordinates is defined as

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{m_3}{r_3}, \tag{2}$$

and the distance of the fourth particle from each of the three primaries is:

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2, \quad i = 1, 2, 3. \tag{3}$$

The equations of motion admit a Jacobian type of integral,

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega - C, \tag{4}$$

where C is the Jacobian constant.

If we suppose that the three primary bodies move on the same plane and the axes are so chosen at the origin of time that m_1 is on the x-axis then the coordinates of the three primaries are:

$$\begin{aligned}x_1 &= \frac{|K|\sqrt{m_2^2 + m_2 m_3 + m_3^2}}{K},\\ y_1 &= 0,\\ x_2 &= -\frac{|K|[(m_2 - m_3)m_3 + m_1(2m_2 + m_3)]}{2K\sqrt{m_2^2 + m_2 m_3 + m_3^2}},\\ y_2 &= \frac{\sqrt{3}}{2}\frac{m_3}{m_2^{3/2}}\sqrt{\frac{m_2^3}{m_2^2 + m_2 m_3 + m_3^2}},\\ x_3 &= -\frac{|K|}{2\sqrt{m_2^2 + m_2 m_3 + m_3^2}},\\ y_3 &= -\frac{\sqrt{3}}{2}\frac{1}{m_2}\sqrt{\frac{m_2^3}{m_2^2 + m_2 m_3 + m_3^2}}\end{aligned} \tag{5}$$

where we have abbreviated

$$K = m_2(m_3 - m_2) + m_1(m_2 + 2m_3). \tag{6}$$

2 Case of equal primary bodies

While in the classical restricted three-body problem there are five coplanar equilibrium points, here in the restricted four-body problem (Lagrangian equilateral triangle configuration) the existence as well as the number of the collinear and the non-collinear equilibrium points depends on the particular values of the mass parameters $m_{1,2,3}$ of the primary bodies (Baltagiannis and Papadakis 2011). In our present case where all the primaries have the same mass $m_1 = m_2 = m_3 = 1/3$, the problem admits four collinear (on the x-axis) equilibrium points and six non-collinear (off the x-axis) ones. All these equilibrium points are unstable (for details see Álvarez-Ramirez and Vidal 2009 and Baltagiannis and Papadakis 2011).

In this section we will present the network of ten planar symmetric (with respect to x-axis) simple-periodic orbits of the problem when the masses of the primaries are equal. We have chosen the ten families which correspond (qualitatively) to the most basic families of the classical Copenhagen three-body problem (see Szebehely 1967, p. 455). By symmetric simple-periodic orbits we mean the simplest solutions which have just two perpendicular intersections with the horizontal x-axis.

All calculations reported in this paper were performed using the variable step R-K 8th-order direct integration and settings the allowable energy variation $\Delta C = |C_{start} - C_{end}| < 10^{-12}$ and $|x_0 - x_T| < 10^{-8}$ (initial and final conditions at $t = 0$ and $t = T$).

In Fig. 1 we present the network of ten families of the simple symmetric periodic orbits for equal masses of the primary bodies. The presentation is made in the (x, C) plane, where the initial conditions generating closed solutions reentering after one oscillation, are shown as characteristic periodic family curves. We note here that we have considered the symmetric periodic orbits as represented by their initial conditions x_0, $y_0 = 0$, $\dot{x}_0 = 0$ and $\dot{y}_0 > 0$ i.e. "positive" perpendicular intersections of the x-axis. The position of the primary body m_1 is denoted by a vertical dashed line. Small red circles indicate the positions of the equilibrium points L_i, $i = 1, \dots, 4$. The shaded area is non-accessible to motion due to the Jacobian integral.

At the same time we study the linear stability of the periodic solutions of the problem. The horizontal and vertical isoenergetic stability of each periodic orbit of the ten families was computed using the stability parameters a_h and a_v, defined by Hénon (1965). According to this paper the condition for a symmetric periodic orbit to be horizontal or/and vertical stable is $|a_h| < 1$ or/and $|a_v| < 1$ respectively. For high accuracy (i.e. the accuracy of the numerical integration, which in the present work is 12 significant figures), we integrated the equations of motion simultaneously with equations of variation. In Fig. 1 the stability arcs of the families are presented by red lines.

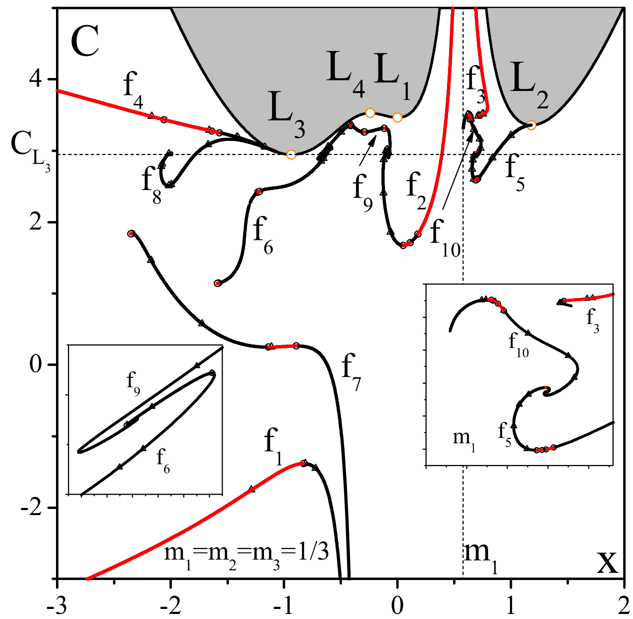

Fig. 1 The characteristic curves of ten families of the simple symmetric periodic orbits for $m_1 = m_2 = m_3 = 1/3$. *Red lines* indicate the horizontal stability arcs of the families. The *small black circles* and *triangles* show the horizontal and the vertical critical periodic orbits correspondingly

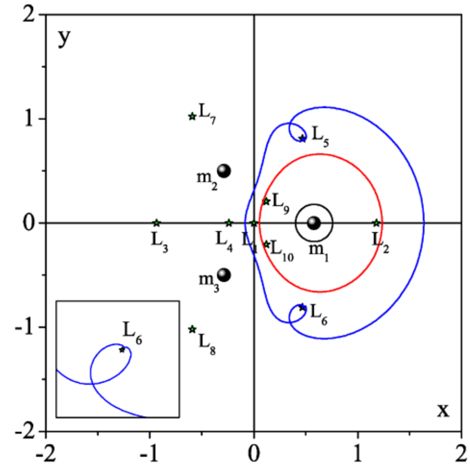

Fig. 2 Three symmetric periodic orbits of family f_2 for $m_1 = m_2 = m_3 = 1/3$. The "end" of the family is a heteroclinic asymptotic orbit on L_5 and L_6 (*blue line*). The *small black points* and the *green stars* denote the positions of the primary bodies and the equilibria of the problem correspondingly

Finally, we calculated the special generating planar orbits i.e. the horizontal and vertical critical periodic orbits of each family. A periodic orbit is a horizontal-critical or vertical-critical periodic orbit if the stability index $|a_h| = 1$ or $|a_v| = 1$ correspondingly (for details see Hénon 1973) and from them we have, generally, the opportunity to calculate other families of planar or three dimensional periodic orbits. The small black circles and triangles indicate the horizontal and the vertical critical periodic orbits of the families correspondingly.

Family f_1 consists of symmetric simple periodic orbits around the three primary bodies. These are retrograde orbits in the rotating as well as in the fixed system like family m of the classical three-body problem. Family f_1, from one side, has orbits where tend to collision with the primaries while on the other side the periodic orbits grow away from the three primary bodies as the Jacobian constant decreases. A large part of the family is stable (red line in Fig. 1) while one horizontal critical periodic orbit and five vertical ones exist in the part of the family where we have calculated (the family exist and for $C < -3$ which is out of the region $-3 \leq C \leq 5$ in which we study the families in this paper).

The periodic orbits of family f_2 are retrograde orbits around the primary body m_1 like family h of the classical three-body problem. The characteristic curve of this family, from one side, evolves inside the funnels of the (x, C) diagram, that are formed between the zero velocity boundaries, and always increases as C increases. From the other side, spirals around point in the (x, C) plane which has ordinate $C = C_{L_{(3,5,6)}} = 2.94673$ (Fig. 1). So, the termination members of the family are asymptotic orbits which intersect the x-axis perpendicularly and tend asymptotically to L_5 for $t \to +\infty$ and to L_6 for $t \to -\infty$, spiralling into (and out of) this point (Fig. 2). The termination of this family has a characteristic behaviour. As the period tends to infinity, the energy of the orbits oscillates in a small shrinking interval around the energy of the asymptotic orbit which is the energy of the equilibrium point $L_{5(6)}$. At each change in the sign of the derivative of the energy, the stability of the orbit changes. This type of termination has been called a "blue sky catastrophe" by Devaney (1977).

In the frames (a) and (b) of Fig. 3 we present the stability of the family f_2 i.e. we plot the horizontal (solid line) and the vertical (dashed line) stability indices versus the Jacobian constant C. The second figure is an enlargement of the first one, around the value of $C = C_{L_{3,5,6}}$ where the "blue sky catastrophe" termination of the family is occurred. Many horizontal (f_2^{ih}) and vertical-critical (f_2^{iv}) periodic orbits of the family are denoted by small circles and triangles correspondingly since the stability of the family changes all the time as the period tends to infinity and the characteristic curve of the family spirals around the point with ordinate $C = C_{L_{3,5,6}}$ in the (x, C) plane. The evolution of the period T of the family f_2 is presented in the third frame of Fig. 3.

Family f_3 has as members direct simple symmetric periodic orbits around the primary body m_1 like family g of the classical three-body problem. As C decreases the orbits increase in size and then change multiplicity. The parts of the families where correspond to higher-multiplicity orbits are not considered in this paper. The majority of orbits in family

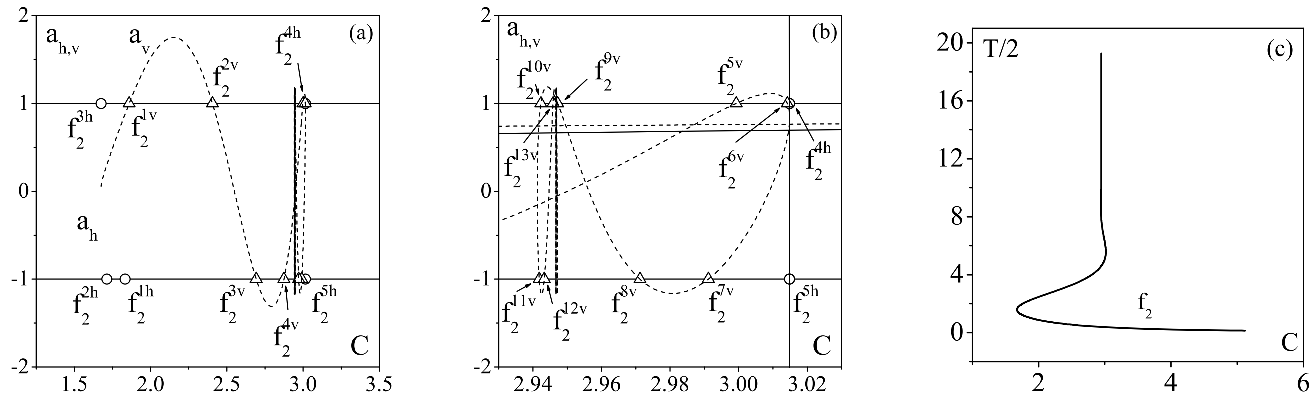

Fig. 3 (**a**) Horizontal and vertical stability diagram of family f_2. The *small circles* and *triangles* indicate the horizontal and the vertical critical periodic orbits correspondingly. (**b**) Zoomed area around the Jacobian constant $C = C_{L_{(3,5,6)}} = 2.94673$ where the stability-instability of the family changes all the time as the period tends to infinity (frame (**c**))

f_3 are stable (red line in Fig. 1) and one horizontal and four vertical-critical periodic orbits exist.

The periodic orbits of family f_4 are retrograde in the rotating system and direct in the fixed system, around the primary bodies m_1, m_2 and m_3 like family l of the classical three-body problem. This family has periodic orbits, from one side, which grow away from the three primary bodies as the Jacobian constant increases while, from the other side, the family "ends" on the zero velocity curve (Fig. 1). Most of the periodic orbits of the family are stable and there are three horizontal and five vertical-critical periodic solutions.

Family f_5 contains retrograde simple symmetric periodic orbits around the equilibrium point L_2 like family a of the classical three-body problem. The infinitesimal elliptic retrograde orbits around L_2 increase in size, as decreasing Jacobian constant, until a heteroclinic asymptotic orbit on equilibrium points L_5 and L_6 is reached (Fig. 4). The majority of the periodic orbits of this family are unstable but there are horizontal and vertical-critical orbits since the family "ends" under the "blue sky catastrophe" termination. The characteristic curve of the family has two small arcs of stable periodic orbits as the Jacobian constant tends to its lower value (red lines in Fig. 1).

Retrograde periodic orbits around the primary body m_1 of the problem are the members of family f_6. From one side the family tends to collision with the primaries m_2 and m_3 while from the other side the family terminates with heteroclinic asymptotic orbits on the equilibrium points L_5 and L_6. The periodic solutions of the family are unstable but two small arcs with stable ones exist (Fig. 1). As the period of the family tends to infinity and the characteristic of the family spirals around a point with $C = C_{L_3}$, horizontal and vertical-critical periodic orbits are created.

Family f_7 has retrograde periodic orbits around the primary bodies m_2 and m_3. The family begins with orbits very close to the two primaries and ends with a collision orbit

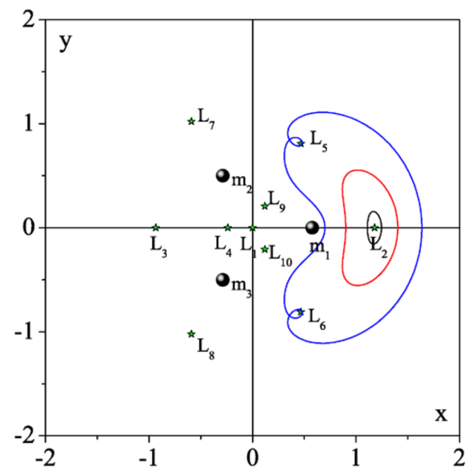

Fig. 4 Three symmetric periodic orbits of family f_5 around the equilibrium point L_2 for $m_1 = m_2 = m_3 = 1/3$. The "end" of the family is a heteroclinic asymptotic orbit on L_5 and L_6 (*blue line*). The *small black points* and the *green stars* denote the positions of the primary bodies and the equilibria of the problem correspondingly

with the primary body m_1. The majority of the periodic solutions of the family are unstable but there are small arcs with stable ones in the middle and close to the end of its characteristic curve (red lines in Fig. 1).

Family f_8 from one side ends on the zero velocity curve while from the other side terminates with heteroclinic asymptotic orbit at L_5 and L_6 and all its members are retrograde periodic orbits around the three primary bodies. The periodic solutions of this family are unstable and we found horizontal and vertical-critical periodic orbits only in its "blue sky catastrophe" termination.

Family f_9 is like family x of the classical three-body problem and consists of retrograde periodic orbits around the primary bodies m_2 and m_3. The termination orbits of both sides of this family are heteroclinic asymptotic orbits at L_5 and L_6. There are small arcs of its characteristic curve with stable periodic solutions.

Finally, family f_{10} consists of direct symmetric periodic orbits around the three primary bodies of the problem and corresponds to family k of the classical three-body problem. The periodic orbits of this family, on both sides, tend to change multiplicity. We continued and calculated the parts of the family with higher-multiplicity and found the termination periodic orbits of this family. Family f_{10} ends, from one side, with a collision periodic orbit with the three primary bodies (black orbit in Fig. 5) and from the other side terminates with asymptotic orbit at the three equilibrium points i.e. the collinear L_3 and the non-collinear L_5 and L_6 (blue orbit in Fig. 5). Family F_{10} has two small arcs of its characteristic curve with stable periodic solutions.

In Table 1 we list the initial conditions of the symmetric periodic orbits which are plotted in Figs. 2, 4 and 5. For each orbit listed in this table, x_0 is the initial position of the particle on the x-axis, $\dot{y}_0$ is the vertical initial velocity ($\dot{x}_0 = 0$), $x_{T/2}$, $\dot{y}_{T/2}$ are denoted the same quantities at the half period, $T/2$ is the half period and C is the Jacobian constant. In the last column we present the horizontal stability of the periodic orbit.

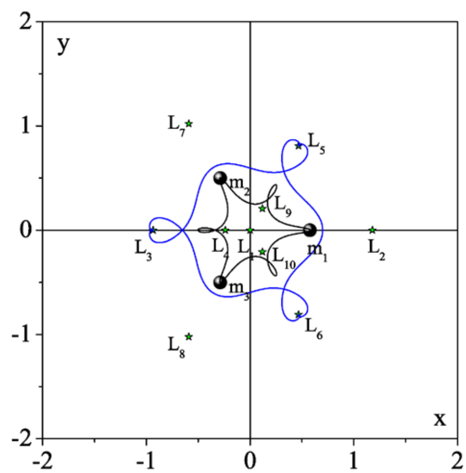

Fig. 5 The termination symmetric periodic orbits of family f_{10} for $m_1 = m_2 = m_3 = 1/3$. The *small black points* and the *green stars* denote the positions of the primary bodies and the equilibria of the problem correspondingly

3 Two primary bodies with equal masses

In this section we will study the simple symmetric periodic solutions of the restricted four-body problem when the two primary bodies m_2 and m_3 have equal masses. The existence of this kind of orbits (symmetric with respect to horizontal x-axis), in the present case, is easily be proved if we insert the transformation $(x, y, \dot{x}, \dot{y}, t) \rightarrow (x, -y, -\dot{x}, \dot{y}, -t)$ in the equation of motion (1). We verify that these equations remain unchanged under the previous transformation.

It is known, that in the Lagrange central configuration the necessary condition for the stability of the configuration is the inequality,

$$\frac{m_1 m_2 + m_2 m_3 + m_3 m_1}{(m_1 + m_2 + m_3)^2} < \frac{1}{27}, \tag{7}$$

where m_1, m_2 and m_3 the three primary bodies (Gascheau 1843). We choose $m_1 = 0.97$ and $m_2 = m_3 = 0.015$ and so the inequality (7) is fulfilled.

And in this case of mass distribution, as in the first one, we calculated the network of the families of the simple symmetric periodic orbits of the problem and we present ten of them. Some of these families, namely $f_{1,2,3,4,5,7}$, are the evolution of the old ones (when $m_1 = m_2 = m_3$) and some of them are new ($f_{6,8,9,10}$) since the old ones do not exist any more. In Fig. 6 we plot the characteristic curves of these ten families.

Comparing the Figs. 1 and 6 we see the differences in the forbidden areas (or the trapped areas), in the absence of the collinear equilibrium points L_1 and L_4 as well as in the characteristic curves of the families. According to the paper Baltagiannis and Papadakis (2011) the restricted four-body problem, when $m_2 = m_3$ admits two or four collinear and four or six non-collinear equilibrium points (depends

Table 1 Symmetric periodic orbits which are plotted in Figs. 2, 4 and 5

Family	x_0	$\dot{y}_0$	$x_{T/2}$	$\dot{y}_{T/2}$	$T/2$	C	Stability
f_2	0.40050000	1.56088315	0.75770121	−1.53644341	0.36611460	3.05967251	S
	0.05350000	1.34205190	1.23459900	−1.30198016	1.56756877	1.67506327	S
	−0.08435450	0.73239406	1.63558530	−1.01441055	19.27221938	2.94672728	U
f_5	1.10950000	0.21881797	1.24602285	−0.20446502	2.32565571	3.33382420	U
	0.90050000	0.93362977	1.40248144	−0.70365211	2.40189387	3.03584294	U
	0.69799469	2.06682108	1.63556565	−1.01438409	14.68467055	2.94673480	U
f_{10}	0.57952380	17.46633001	−0.50246977	0.08137413	4.00510710	3.31516544	U
	0.60309000	4.90830820	−0.36515435	−0.00036434	3.74842644	3.47668129	U

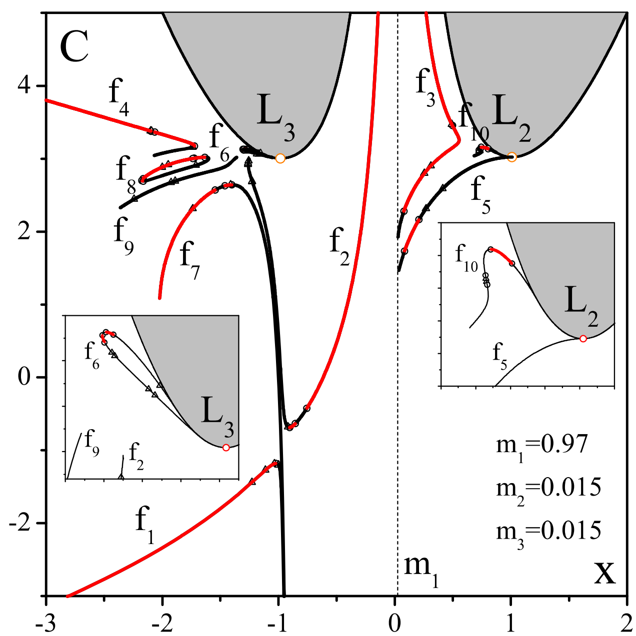

Fig. 6 The network of ten families of the simple symmetric periodic orbits for $m_1 = 0.97$ and $m_2 = m_3 = 0.015$. *Red lines* indicate the horizontal stability arcs of the families. The *small black circles* and *triangles show* the horizontal and the vertical critical periodic orbits correspondingly

on $m_{2,3}$). For $m_2 = m_3 = m_{crit} = 0.2882762$ the equilibrium point L_1 coincides with L_4 and for values less than m_{crit} the equilibria L_1 and L_4 do not exist. So, in the present case the problem has 2 collinear and 6 non-collinear equilibria. All the equilibrium points are unstable except the non-collinear equilibria L_5 and L_6 which are stable.

For $m_1 = 0.97$ and $m_2 = m_3 = 0.015$ we found that families f_1 and f_7 present, qualitatively, the same behaviour as in the case with three equal primary bodies (both families tend to collision with the primaries m_2 and m_3).

Family f_2 now tends to collision with the primaries m_1 and m_2 and it has not simple asymptotic orbits any more.

The evolution of the families $f_{3,4,5}$, as C varies, is different now since f_3 and f_5 tend to collision with m_1 while f_4, as C decreases, changes multiplicity. In the first frame of Fig. 7 the stability diagram of family f_5 is illustrated. The two horizontal and the two vertical critical orbits of this family are marked with small circles and triangles correspondingly.

Family f_6 consists of retrograde symmetric periodic orbits around the three bodies $m_{1,2,3}$ and from both sides terminates on the zero velocity curve (inside zoomed image in Fig. 6). It has two small arcs in its characteristic curve with stable periodic orbits. In this family four horizontal and

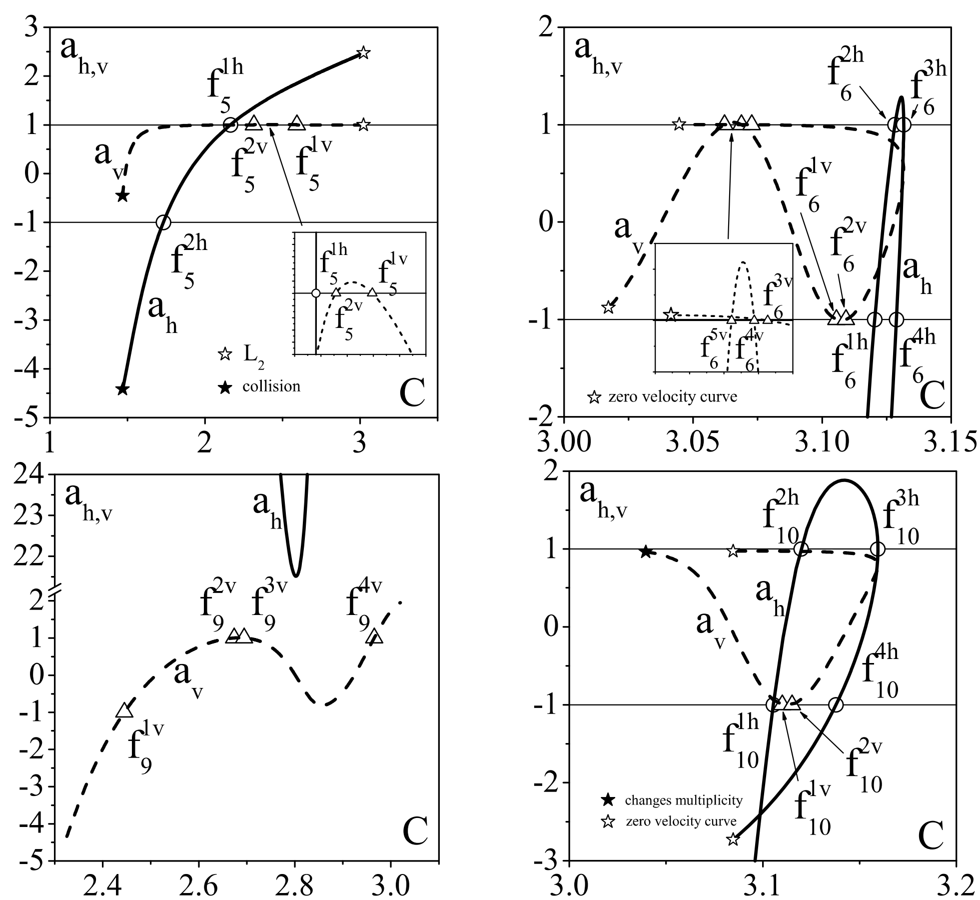

Fig. 7 Horizontal (*solid lines*) and vertical (*dashed lines*) stability diagram of families $f_{5,6,9,10}$ for $m_1 = 0.97$, $m_2 = m_3 = 0.015$. The *small circles* and *triangles* indicate the horizontal and the vertical critical periodic orbits correspondingly

five vertical-critical periodic orbits exist (second frame in Fig. 7).

Family f_8 has retrograde periodic orbits around $m_{1,2,3}$. The characteristic curve of this family is closed. The half periodic orbits of this family are stable (red line in Fig. 6). Six horizontal and four vertical-critical symmetric periodic orbits exist.

The only family where all its periodic orbits are unstable is the family f_9. Consists of retrograde periodic orbits around the three bodies $m_{1,2,3}$. From one side, this family, tends to collision with m_2 and m_3 while from the other side changes multiplicity. Four vertical-critical periodic orbits exist (third frame in Fig. 7).

Finally, the members of the family f_{10} are direct symmetric periodic orbits around the primary m_1. The family has stable periodic orbits, four horizontal and two vertical-critical periodic orbits (last frame in Fig. 7). This family terminates, from one side, on the zero velocity curve while from the other side changes multiplicity (inside zoomed image in Fig. 6).

From the horizontal and vertical stability curves of the families f_5, f_6 and f_{10} it is seen that these families consist mostly of orbits which are vertically stable but horizontally unstable.

4 Primary bodies with unequal masses

Collinear equilibrium points in the restricted four-body problem (Lagrangian configuration) with three unequal masses, do not exist. The problem, in this case, admits eight or ten (depends on the mass parameters) non-collinear equilibrium points (for details see Baltagiannis and Papadakis 2011). In this section we study the families of simple periodic orbits when the mass distribution is $m_1 = 0.97$, $m_2 = 0.02$ and $m_3 = 0.01$. The stability condition (7), for these m_i, is fulfilled. For these values of masses the problem has eight equilibrium points. All equilibria are unstable except L_5 and L_6 which are stable. In this case of the problem, the horizontal x-axis is not a symmetric axis any more (as in the two previous cases in the present paper). So, in this section we will study the families of simple non-symmetric periodic orbits for the specific values of masses of the three primary bodies.

In the two previous cases we presented the characteristic curves of the families of simple symmetric periodic orbits using the initial conditions of the periodic orbits namely $\mathbf{x_0} = \{x_0, y_0 = 0, \dot{x}_0 = 0, \dot{y}_0(C)\}$ while now the characteristic curves of the families of non-symmetric periodic orbits in the (x_0, C) plane do not define the complete set of the initial conditions of the orbits since they do not provide any information about the values of the horizontal ($\dot{x}_0 \neq 0$ now) or the vertical component of the velocity of the fourth infinitesimal body. For comparison reasons we will use the same presentation of the families on the (x_0, C) plane and we will give additional information about the velocity of the fourth body in appropriate tables of data.

While the determination of the initial conditions of an orbit close to a symmetric periodic orbit is possible using the grid method (Markellos et al. 1974), this is not possible for non-symmetric periodic orbits since the velocity of the fourth body is not vertical with respect to horizontal x-axis for $T = 0$. So, we need to know how a symmetric periodic orbit changes as the mass parameters vary. From the previous section we know ten families of simple symmetric periodic orbits with $m_1 = 0.97$ and $m_2 = m_3 = 0.015$. We can obtain a one-parameter set of horizontal or vertical critical periodic orbits by varying the mass parameter (say, m_3). If the parameter is fixed the horizontal (or vertical) critical orbits are isolated. In the present case, we keep constant the mass of the dominant body $m_1 = 0.97$ and we computed monoparametric sets of horizontal or vertical critical orbits for values of the mass parameter of the third primary m_3 varying in the range [0.015, 0.01] (the mass of the second primary body is defined by the equation $m_2 = 1 - m_1 - m_3$). So, we determined critical non-symmetric periodic orbits for $m_1 = 0.97$, $m_3 = 0.01$ and $m_2 = 0.02$. We shall call the set of initial conditions and other quantities describing a critical orbit for a range of values of the mass parameters, a bifurcation series or a series of critical orbits. Once a non-symmetric orbit has been found, we can extend the search to the whole one-parameter family to which that orbit belongs. All the families of symmetric periodic orbits f_i, $i = 1, \ldots, 10$ of the previous case (two equal primaries) have at least one horizontal critical periodic orbit except family f_9. In this case we determined the series of one (it has four) vertical critical periodic orbit of family f_9.

In Table 2 horizontal critical periodic orbits with $S_h = 1$ for the families f_i, $i = 1, \ldots, 8, 10$ and vertical critical ones with $S_v = 1$ for the family f_9 are given. For each family the first critical periodic orbit is the symmetric critical solution for $m_1 = 0.97$ and $m_2 = m_3 = 0.015$ while the second one is the non-symmetric critical solution for $m_1 = 0.97$ $m_2 = 0.02$ and $m_3 = 0.01$ correspondingly. In the last column of Table 2 we present the stability index $S_h = (a_h + d_h)/2$ or $S_v = (a_v + d_v)/2$, where a_h, d_h, a_v and d_v are the isoenergetic stability horizontal (vertical) parameters, of each periodic orbit (for details see Hénon 1965, 1973).

Starting with the initial conditions of Table 2, for $m_1 = 0.97$ $m_2 = 0.02$ and $m_3 = 0.01$, we calculated the ten families of simple non-symmetric periodic orbits. In Fig. 8 we present these ten families with their corresponding stable arcs (red lines). We note again that the characteristic curves presented here do not give any information about the velocity of the non-symmetric periodic orbits of the families f_i, $i = 1, \ldots, 10$.

Table 2 Series, with respect to the mass parameter m_3 ($m_1 = 0.97$ is fixed and $m_2 = 1 - m_1 - m_3$), of critical symmetric and non-symmetric periodic orbits

Family	m_3	x_0	$\dot{x}_0$	$\dot{y}_0$	C	T	Stability index
f_1	0.015	−1.02160447	0.00000000	2.04679983	−1.18104196	2.82991261	$S_h = 1.00000000$
	0.01	−1.02506173	−0.03888367	2.05336268	−1.17807072	2.84078467	S_h
f_2	0.015	−0.75612119	0.00000000	1.89645568	−0.42598534	2.65805735	S_h
	0.01	−0.75507284	0.01748971	1.90269870	−0.42825831	2.66038694	S_h
f_3	0.015	0.49660519	0.00000000	0.97708718	3.45614347	3.19119978	S_h
	0.01	0.50001684	0.00161738	0.97122228	3.44521168	3.25098464	S_h
f_4	0.015	−2.09768018	0.00000000	1.40690658	3.37872619	9.39826718	S_h
	0.01	−2.09833939	−0.00119843	1.40823351	3.37878217	9.39762923	S_h
f_5	0.015	0.20901246	0.00000000	2.92071648	2.16398820	6.24319928	S_h
	0.01	0.26503026	0.26031156	2.42511743	2.30234921	6.23835093	S_h
f_6	0.015	−1.29832006	0.00000000	0.34444543	3.12037989	14.34282809	S_h
	0.01	−1.31371897	0.08828457	0.40643256	3.10118167	15.15805445	S_h
f_7	0.015	−1.54435005	0.00000000	1.05782952	2.57088764	5.64506825	S_h
	0.01	−1.55780271	0.01646369	1.08000666	2.55924268	5.70518828	S_h
f_8	0.015	−1.72477906	0.00000000	1.06562373	3.00644003	18.17937805	S_h
	0.01	−1.70207170	0.00374697	1.03334713	3.01572943	18.04783658	S_h
f_9	0.015	−2.24258103	0.00000000	1.86532152	2.44520826	17.63176140	$S_v = 1.00000000$
	0.01	−2.21748105	0.03965238	1.83278958	2.46324526	17.58806770	S_v
f_{10}	0.015	0.73304548	0.00000000	0.46045805	3.10542112	9.37317852	S_h
	0.01	0.73452874	−0.14568935	0.47286556	3.07075149	10.71055379	S_h

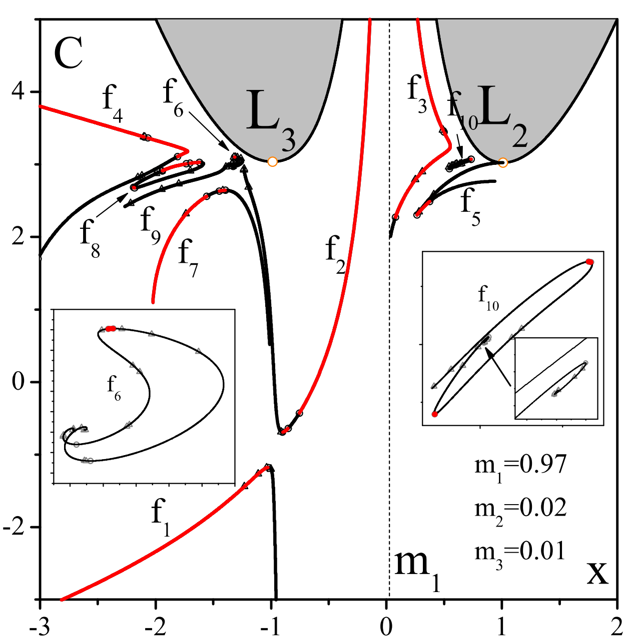

Fig. 8 The characteristic curves of ten families of the simple non-symmetric periodic orbits for $m_1 = 0.97$, $m_2 = 0.02$ and $m_3 = 0.01$. *Red lines* indicate the horizontal stability arcs of the families. The *small black circles* and *triangles show* the horizontal and the vertical critical periodic orbits correspondingly

The network of the ten families of periodic orbits we found is quite close to this network of families in the case where the two primaries are equal ($m_2 = m_3 = 0.015$) but with a significant difference i.e. in the present case all the periodic solutions are non-symmetric. Namely, families $f_{1,2,3,4,7,8,9}$ have the same, qualitatively, behaviour as in the previous case while families $f_{5,6,10}$ are evolved different.

Family f_5 consists of simple retrograde non-symmetric periodic orbits which some of them are Trojan type orbits (generally not crossing the synodical line). From one side has solutions around the equilibrium point L_2 and from the other side has as members periodic orbits around the stable equilibrium point L_6 (f_5 in Fig. 9). The evolution of this family is like the short-period asymmetric family emanating from the Lagrangian equilibrium point L_5 and terminates on a symmetrical periodic orbit belonging to the family b, which family b emanates from the collinear equilibrium point L_3 in the classical restricted three-body problem (Deprit et al. 1967). Family f_5 has stable periodic orbits and three horizontal and two vertical critical non-symmetric periodic solutions.

Family f_6 has retrograde non-symmetric periodic orbits around the three primaries and now, in this mass distribution, from both sides terminates with solutions where are asymptotic orbits at the equilibrium point L_3 (Fig. 8, inside

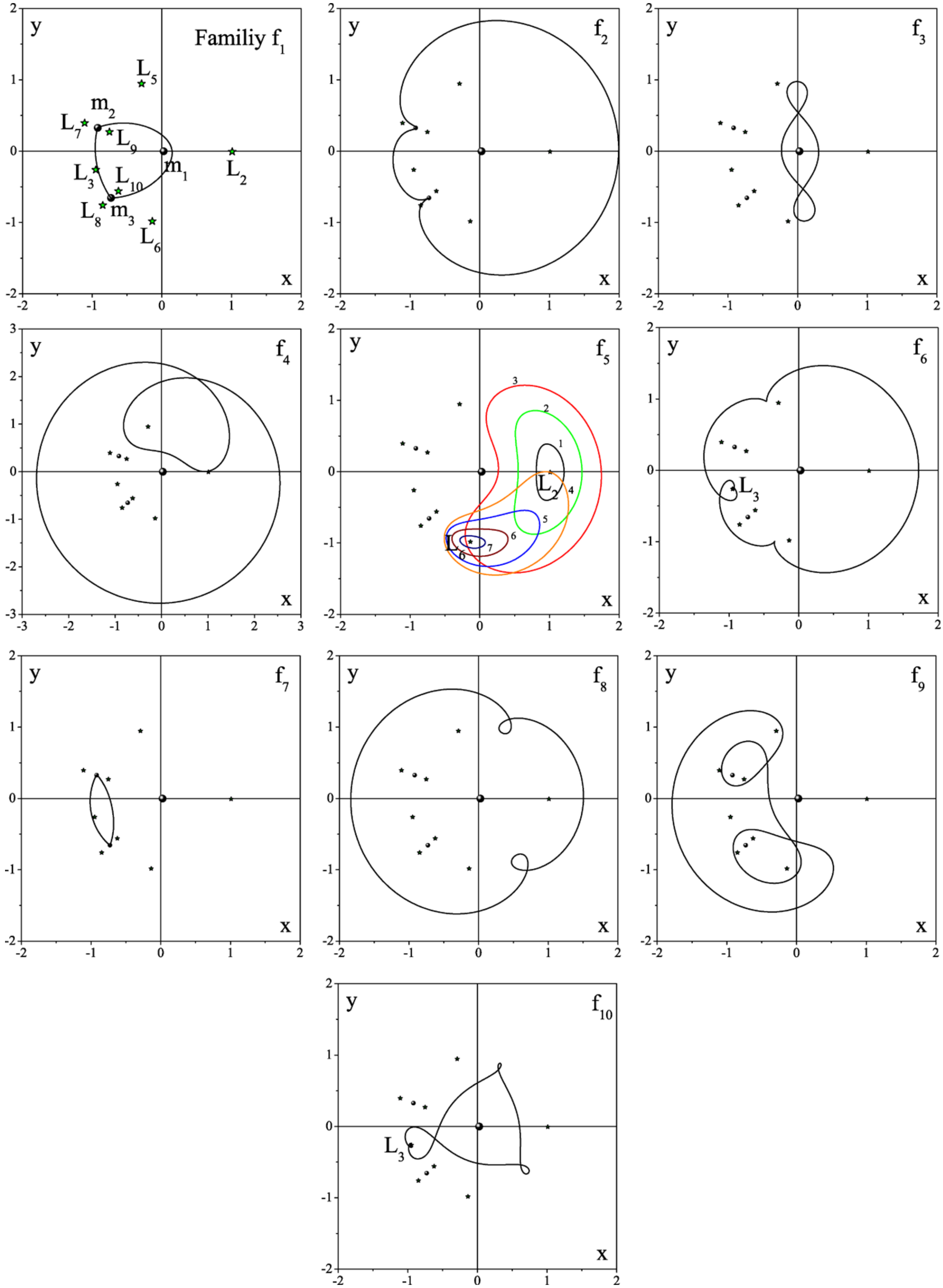

Fig. 9 Non-symmetric periodic orbits from the ten families f_i, $i = 1, \dots, 10$ for $m_1 = 0.97$, $m_2 = 0.02$ and $m_3 = 0.01$. The *small black points* and the *green stars* denote the positions of the primary bodies and the equilibria of the problem correspondingly

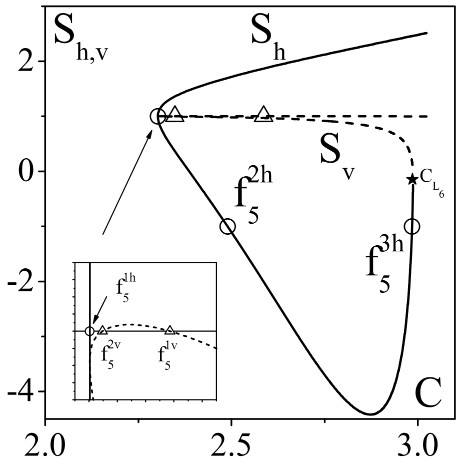

Fig. 10 Horizontal (*solid line*) and vertical (*dashed line*) stability diagram of the family f_5 of the simple non-symmetric periodic orbits for $m_1 = 0.97$, $m_2 = 0.02$ and $m_3 = 0.01$. The *small circles* and *triangles* indicate the horizontal and the vertical critical periodic orbits of this family correspondingly

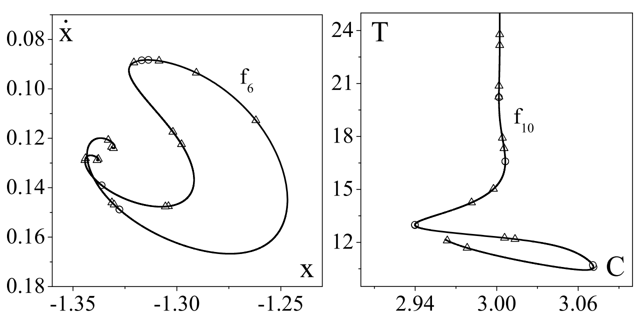

Fig. 11 *Left*: x versus $\dot{x}$ from the family f_6 of the simple non-symmetric periodic orbits for $m_1 = 0.97$, $m_2 = 0.02$ and $m_3 = 0.01$. *Right*: C versus T from the family f_{10}. The *small circles* and *triangles* indicate the horizontal and the vertical critical periodic orbits of these families correspondingly

left window and frame f_6 in Fig. 9). It has a small arc of stable periodic orbits (red line in Fig. 8) and we have calculated many horizontal and vertical periodic solutions close to its termination (small circles and triangles in left frame of Fig. 11).

Finally, family f_{10} consists of direct simple non-symmetric periodic orbits around the dominant primary body m_1. This family from one side has periodic orbits where change multiplicity and from the other side has asymptotic orbits at the equilibrium point L_3 (f_{10} in Fig. 9). Many horizontal and vertical periodic orbits are calculated (small circles and triangles in right frame of Fig. 11).

In Fig. 9 we plot samples of simple non-symmetric periodic orbits from the ten determined families. In frames f_6 and f_{10} asymptotic orbits at L_3 from families f_6 and f_{10} are presented correspondingly. In frame f_5 we illustrate seven non-symmetric periodic orbits where they present the evolution of family f_5 which we have described previously.

In Fig. 10 the stability diagrams from the family f_5 is illustrated. The positions of the horizontal and vertical critical periodic orbits from this family are also presented. In Fig. 11 we plot two characteristic curves of families f_6 and f_{10} correspondingly. In the second figure we denote the positions of the critical periodic orbits as the period T tends to infinity.

5 Conclusions and remarks

Our goal of this paper was the study of the families of the simple periodic orbits of the restricted four-body problem (Lagrangian equilateral triangle configuration), when the primary bodies of the problem have equal or not masses. We found ten families of periodic orbits of each case i.e. when the primary bodies are equal, when the two primaries m_2 and m_3 are equal and finally when $m_1 \neq m_2 \neq m_3$. We chosen, in the last two cases, one dominant primary body (m_1) and two small ($m_{2,3}$) in order the stability inequality (7) of the configuration to be valid.

We found a large number of various types of orbits, such as symmetric and non-symmetric, with respect to the horizontal x-axis, orbits around one, two or three primaries, orbits around an equilibrium position, collision orbits, Trojan orbits, as well as asymptotic orbits at one, two or three equilibria. The orbits are manly retrograde but there are families where consist of direct periodic orbits.

We have also studied the stability of each periodic orbit and we found the special generating planar horizontal and vertical critical solutions of each family.

In any case of the mass distribution, the families f_1, f_2, f_3 and f_4 have the majority of their periodic orbits to be stable (like the basic families m, h, g and l of the classical three-body problem). The majority of the periodic orbits of the rest families are unstable.

The network of the families of the periodic orbits changes substantially as the mass parameters vary and some families disappear while some news appear. More specifically, our main results can be summarized, in each case, as follows:

5.1 Three equal masses

- There is not family without stable periodic orbits. The majority of the periodic orbits of the families f_1, f_2, f_3 and f_4 are stable. The rest families have small arcs of their characteristic curves with stable periodic orbits.
- Six of the ten families, under consideration, have as terminating members, asymptotic orbits. Namely, families f_2, f_5, f_6, f_8 and f_9 terminate with heteroclinic asymptotic orbits at the equilibrium points L_5 and L_6. One family, f_{10}, has as terminating member an asymptotic orbit at the three equilibrium points L_3, L_5 and L_6.

- Family f_5 emanates from the collinear equilibrium point L_2 and terminates with asymptotic orbits at the non-collinear equilibria L_5 and L_6.
- All the families have horizontal and vertical-critical periodic orbits as members.
- All the symmetric periodic orbits we found are asymmetric with respect to the vertical y-axis.
- All the families consist of retrograde periodic orbits except families f_3 and f_{10} where have direct ones.

5.2 Two equal masses

- Only one family, namely f_9, is entirely unstable. All the others families have stable periodic solutions.
- Simple asymptotic orbits, in the ten families under consideration, do not exist.
- All the symmetric periodic orbits we found are asymmetric with respect to the vertical y-axis.
- The characteristic curve of family f_8 is closed.
- All the families consist of retrograde periodic orbits except families f_3 and f_{10} where have direct ones.

5.3 Unequal masses

- All the families determined here consist of simple non-symmetric periodic orbits (both in horizontal and in vertical axis).
- All the families have stable periodic orbits.
- Two (of ten) families, namely, f_3 and f_{10} have terminating orbits asymptotic solutions at the equilibrium point L_3.
- Family f_5 emanates (or terminates) from the non-collinear equilibrium point L_2 and terminates (or emanates) at the non-collinear equilibria L_6.
- The characteristic curve of family f_8 is closed.
- All the families consist of retrograde periodic orbits except families f_3 and f_{10} where have direct ones.

Closing this article we would like to express our intention to apply these results in the Sun-Jupiter-Asteroid system. Families of periodic orbits have already been calculated and we will present them in a future article.

Acknowledgements During this work the first Author was in receipt of a "K. Karatheodory" research grant.

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