

New class of Well behaved exact solutions of relativistic charged *white-dwarf* star with perfect fluid

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Abstract The paper presents a class of interior solutions of Einstein–Maxwell field equations of general relativity for a static, spherically symmetric distribution of the charged fluid. This class of solutions describes well behaved charged fluid balls. The class of solutions gives us wide range of parameter K ($0.3277 \leq K \leq 0.49$), for which the solution is well behaved hence, suitable for modeling of super dense star. For this solution the mass of a star is maximized with all degree of suitability and by assuming the surface density $\rho_b = 2 \times 10^{14} \text{ g/cm}^3$. Corresponding to $K = 0.3277$ with $X = -0.15$, the maximum mass of the star comes out to be $M = 0.92M_\odot$ with radius $r_b \approx 17.15 \text{ km}$ and the surface red shift $Z_b \approx 0.087187$. It has been observed that under well behaved conditions this class of solutions gives us the mass of super dense object within the range of *white-dwarf*.

Keywords Charge fluid · Reissner–Nordstrom · General relativity · Exact solution

1 Introduction

Exact solutions with well behaved nature of Einstein–Maxwell field equations are of vital importance in relativistic astrophysics. Such solutions may be used to make a suitable model of super dense object with charge matter. Eventually, these exact solutions of the Einstein–Maxwell field equations joining smoothly to the Nordstrom solution at the pressure free interface. It is interesting to observe that, in the presence of charge, the gravitational collapse of a spherical

symmetric distribution of the matter to a point singularity may be avoided. In this scenario gravitational attraction is counterbalanced by the Coulombian repulsive force together with the pressure gradient. On account of the nonlinearity of Einstein–Maxwell field equations, not many realistic well behaved, analytic solutions are known for the description of relativistic charge fluid spheres. For well behaved model of relativistic star with charged and perfect fluid matter, following conditions should be satisfied (Pant et al. 2010):

- (i) The solution should be free from physical and geometrical singularities i.e. finite and positive values of central pressure, central density and non zero positive values of e^λ and e^ν .
- (ii) The solution should have positive and monotonically decreasing expressions for pressure and density (p and ρ) with the increase of r . The solution should have positive value of ratio of pressure-density and less than 1 (weak energy condition) and less than 1/3 (strong energy condition) throughout within the star.
- (iii) The solution should have positive and monotonically decreasing expression for fluid parameter $\frac{p}{\rho c^2}$ with the increase of r .
- (iv) The solution should have positive and monotonically decreasing expression for velocity of sound ($\frac{dp}{d\rho}$) with the increase of r and causality condition should be obeyed at the centre i.e. $\frac{dp}{c^2 d\rho} < 1$.
- (v) The red shift Z should be positive, finite and monotonically decreasing in nature with the increase of r .
- (vi) Electric intensity E is positive and monotonically increasing from centre to boundary and at the centre the Electric intensity is zero.

Under these well behaved conditions, one has to assume the gravitational potential and electric field intensity in such a way that the field equation can be integrated and solution

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should be well behaved. Keeping in view of this aspect, several authors obtained the parametric class of exact solutions Cataldo and Mitskievic (1992), Ivanov (2002), Pant et al. (2010, 2011), Gupta and Maurya (2010a, 2010b), Maurya and Gupta (2011), Pant (2010), etc. These coupled solutions are well behaved with some positive values of charge parameter K and completely describe interior of the super-dense astrophysical object like White dwarf with charge matter. In fact, a White Dwarf is an astrophysical object whose mass is less than $1.42M_{\odot}$ (Chandrasekhar limit). Further, the mass of the such modeled super dense object can be maximized by assuming surface density is $\rho_b = 2 \times 10^{14} \text{ g/cm}^3$.

In the present paper we have obtained yet another new parametric class of well behaved exact solutions of Einstein–Maxwell field equations, which is compatible within the range of White dwarf.

2 Einstein–Maxwell equation for charged fluid distribution

Let us consider a spherical symmetric metric in curvature coordinates

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) + e^{\nu} dt^2 \tag{1}$$

where the functions $\lambda(r)$ and $\nu(r)$ satisfy the Einstein–Maxwell equations

$$\begin{aligned} -\frac{8\pi G}{c^4} T_j^i &= R_j^i - \frac{1}{2} R \delta_j^i \\ &= -\frac{8\pi G}{c^4} \left[(c^2 \rho + p) v^i v_j - p \delta_j^i \right. \\ &\quad \left. + \frac{1}{4\pi} \left(-F^{im} F_{jm} + \frac{1}{4} \delta_j^i F_{mn} F^{mn} \right) \right] \end{aligned} \tag{2}$$

where ρ , p , v^i , F_{ij} denote energy density, fluid pressure, velocity vector and skew-symmetric electromagnetic field tensor respectively.

In view of the metric (1), the field equation (2) gives,

$$\frac{v'}{r} e^{-\lambda} - \frac{(1 - e^{-\lambda})}{r^2} = \frac{8\pi G}{c^4} p - \frac{q^2}{r^4} \tag{3a}$$

$$\begin{aligned} \left(\frac{v''}{2} - \frac{\lambda' v'}{4} + \frac{v'^2}{4} + \frac{v' - \lambda'}{2r} \right) e^{-\lambda} \\ = \frac{8\pi G}{c^4} p + \frac{q^2}{r^4} \end{aligned} \tag{3b}$$

$$\frac{\lambda'}{r} e^{-\lambda} + \frac{(1 - e^{-\lambda})}{r^2} = \frac{8\pi G}{c^2} \rho + \frac{q^2}{r^4} \tag{3c}$$

where, prime (') denotes the differentiation with respect to r and $q(r)$ represents the total charge contained within the sphere of radius r .

By using the transformation

$$e^{\nu} = B(1 + x)^{-1/2}, \quad x = c_1 r^2 \quad \text{and} \quad e^{-\lambda} = Y. \tag{4}$$

Where B being positive constants. Now putting (4) into (3a)–(3c), we have

$$\frac{-Y}{1+x} - \frac{(1-Y)}{x} + \frac{c_1 q^2}{x^2} = \frac{1}{c_1} \frac{8\pi G}{c^4} p \tag{5}$$

$$\frac{(1-Y)}{x} - 2 \frac{dY}{dx} - \frac{c_1 q^2}{x^2} = \frac{1}{c_1} \frac{8\pi G}{c^2} \rho \tag{6}$$

and Y satisfying the equation

$$\frac{dY}{dx} + \frac{x^2 - 8x - 4}{2x(1+x)(2+x)} Y = \frac{2(1+x)}{x(2+x)} \left(\frac{2c_1 q^2}{x} - 1 \right) \tag{7}$$

Our task is to explore the solutions of (7) and obtain the fluid parameters p and ρ from (5) and (6).

3 New class of solutions

In order to solve the differential equation (7), we consider the electric intensity E of the following form

$$E^2 = \frac{c_1^2 q^2}{x^2} = \frac{4Kc_1 x}{(2+x)^3} \tag{8}$$

where K is a positive constant. The electric intensity is so assumed that the model is physically significant and well behaved, i.e. E remains regular and positive throughout the sphere. In addition, E vanishes at the centre of the star.

In view of (8) differential equation (7) yields the following solution

$$\begin{aligned} Y &= e^{-\lambda} \\ &= -32K \frac{x(1+x)^2}{(2+x)^4} \\ &\quad + \frac{16}{(2+x)^4} \left[Ax(1+x)^{\frac{5}{2}} + \frac{(4+4x-x^2)(1+x)^2}{4} \right] \end{aligned} \tag{9}$$

where A is an arbitrary constant of integration.

4 Properties of the new class of solutions

Using (4), (8), (9) into (5) and (6), we get the following expressions for pressure and energy density

$$\frac{1}{c_1} \frac{8\pi G}{c^4} p = 4K \frac{(x^2 - 6x - 8)}{(2+x)^4}$$

$$+ \frac{1}{(2+x)^4} [16A(1+x)^{\frac{3}{2}} + (96 + 120x - 60x^2 - 90x^3 - 5x^4)] \tag{11}$$

$$- x(12 + 12x + x^2)] \tag{10} \quad \frac{8\pi G}{c^4} p_0 = (-2K + A)c_1 \tag{12}$$

$$\frac{1}{c_1} \frac{8\pi G}{c^2} \rho = 4K \frac{(-9x^3 + 60x^2 + 116x + 48)}{(2+x)^5} \quad \frac{8\pi G}{c^2} \rho_0 = (6K - 3A - 3)c_1 \tag{13}$$

$$- \frac{1}{(2+x)^4} [16A(6 + 11x)(1+x)^{\frac{3}{2}}] \quad \text{Differentiating (10) and (11) w.r.t. } x, \text{ we get:}$$

$$\frac{1}{c_1} \frac{8\pi G}{c^4} \frac{dp}{dx} = \frac{-8K(x^2 - 11x - 10) - 8A(2 + 5x)(1+x)^{\frac{1}{2}} - 24 - 12x + 18x^2 + x^3}{(2+x)^5} \tag{14}$$

$$\frac{1}{c_1} \frac{8\pi G}{c^2} \frac{d\rho}{dx} = \frac{4K(18x^3 - 234x^2 - 224x - 8)}{(2+x)^6} + \frac{-50x^3 + 420x^2 + 600x + 144 - 8A(1+x)^{\frac{1}{2}}\{-33x^2 + 14x + 32\}}{(2+x)^5} \tag{15}$$

$$\left(\frac{8\pi G}{c^4} \frac{dp}{dx}\right)_{x=0} = \left(\frac{5K}{2} - \frac{A}{2} - \frac{3}{4}\right)c_1 \tag{16}$$

$$\left(\frac{8\pi G}{c^2} \frac{d\rho}{dx}\right)_{x=0} = \left(-\frac{K}{2} - 8A + \frac{9}{2}\right)c_1 \tag{17}$$

and hence the velocity of sound v is given by the following expression

$$v^2 = \frac{dp}{d\rho} \tag{18}$$

$$\frac{1}{c^2} \frac{dp}{d\rho} = \frac{-8K(x^2 - 11x - 10) - 8A(2 + 5x)(1+x)^{\frac{1}{2}} - 24 - 12x + 18x^2 + x^3}{4K(18x^3 - 234x^2 - 224x - 8)(2+x)^{-1} + -50x^3 + 420x^2 + 600x + 144 - 8A(1+x)^{\frac{1}{2}}\{-33x^2 + 14x + 32\}}$$

The expression for gravitational red-shift (z) is given by

$$z = \frac{(1+x)^{1/4}}{\sqrt{B}} - 1 \tag{19}$$

The central value of gravitational red shift to be non zero positive finite and monotonically decreasing with the increase of radial coordinate r , we have

$$1 > \sqrt{B} > 0 \tag{20}$$

$$x < 0 \quad \text{or} \quad c_1 < 0 \tag{21}$$

5 Boundary conditions

The solutions so obtained are to be matched over the boundary with Reissner–Nordstrom metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r_b} + \frac{e^2}{r_b^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$+ \left(1 - \frac{2GM}{c^2 r_b} + \frac{e^2}{r_b^2}\right) dt^2 \tag{22}$$

which requires the continuity of e^λ, e^ν and q across the boundary $r = r_b$

$$e^{-\lambda(r_b)} = 1 - \frac{2GM}{c^2 r_b} + \frac{e^2}{r_b^2} \tag{23}$$

$$e^{\nu(r_b)} = 1 - \frac{2GM}{c^2 r_b} + \frac{e^2}{r_b^2} \tag{24}$$

$$q(r_b) = e \tag{25}$$

$$p(r_b) = 0 \tag{26}$$

On setting $x_{r=r_b} = X = c_1 r_b^2$ (r_b being the radius of the charged sphere).

Table 1 The march of pressure, density, pressure–density ratio, square of adiabatic sound speed, gravitational red shift and electric field intensity within the ball corresponding to $K = 0.49$ with $X = -0.005$.

By assuming surface density; $\rho_b = 2 \times 10^{14}$ g/cm³. The resulting well behaved model has the mass $M = 0.0023M_\odot$ with radius $r_b \approx 3.48$ km

r/r_b	$\frac{8\pi G}{c^4} \rho r_b^2$	$\frac{8\pi G}{c^2} \rho r_b^2$	$(\frac{p}{\rho c^2})$	$\frac{1}{c^2} (\frac{dp}{d\rho})$	Z	$E \times r_b$
0	3.4507×10^{-7}	0.044698	7.72×10^{-6}	0.00835	0.00377	0
0.1	3.41×10^{-7}	0.044697	7.64×10^{-6}	0.004168	0.003757	0.000247
0.2	3.3×10^{-7}	0.044695	7.39×10^{-6}	0.004145	0.003719	0.000495
0.3	3.12×10^{-7}	0.04469	6.98×10^{-6}	0.004107	0.003657	0.000743
0.4	2.86×10^{-7}	0.044684	6.4×10^{-6}	0.004054	0.003569	0.000991
0.5	2.53×10^{-7}	0.044677	5.67×10^{-6}	0.003986	0.003456	0.001239
0.6	2.14×10^{-7}	0.044667	4.8×10^{-6}	0.003903	0.003318	0.001487
0.7	1.69×10^{-7}	0.044656	3.78×10^{-6}	0.003805	0.003154	0.001736
0.8	1.18×10^{-7}	0.044643	2.63×10^{-6}	0.003694	0.002966	0.001985
0.9	0.611×10^{-7}	0.044629	1.3×10^{-6}	0.003568	0.002752	0.002234
1.0	0	0.044612	0	0.003427	0.002513	0.002484

Table 2 The march of pressure, density, pressure–density ratio, square of adiabatic sound speed, gravitational red shift and electric field intensity within the ball corresponding to $K = 0.3277$ with $X = -0.15$.

By assuming surface density; $\rho_b = 2 \times 10^{14}$ g/cm³. The resulting well behaved model has the mass $M = 0.92M_\odot$ with radius $r_b \approx 17.15$ km

r/r_b	$\frac{8\pi G}{c^4} \rho r_b^2$	$\frac{8\pi G}{c^2} \rho r_b^2$	$(\frac{p}{\rho c^2})$	$\frac{1}{c^2} (\frac{dp}{d\rho})$	Z	$E \times r_b$
0	0.00567	1.160909	0.004884	0.79237	0.132269	0
0.1	0.005616	1.160771	0.004838	0.38201	0.131844	0.006079
0.2	0.005454	1.160335	0.0047	0.344872	0.130567	0.012198
0.3	0.005182	1.159531	0.004469	0.296506	0.128428	0.018401
0.4	0.004799	1.158244	0.004144	0.247409	0.125413	0.024731
0.5	0.004302	1.156305	0.003721	0.203429	0.121501	0.031233
0.6	0.003688	1.15349	0.003198	0.166488	0.116664	0.037957
0.7	0.002954	1.149513	0.00257	0.136379	0.110864	0.044958
0.8	0.002096	1.144014	0.001832	0.112096	0.104058	0.052294
0.9	0.001112	1.136558	0.000978	0.092491	0.096188	0.060032
1.0	0	1.126615	0	0.076535	0.087187	0.06825

Pressure at $p(r=r_b) = 0$ gives

$$A = -\frac{K(X^2 - 6X + 8)}{4(1+X)^{\frac{3}{2}}} + \frac{X(X^2 + 12X + 12)}{16(1+X)^{\frac{3}{2}}} \tag{27}$$

In view of (23) and (24) we get,

$$B = \left\{ -32K \frac{X(1+X)^2}{(2+X)^4} + \frac{16}{(2+X)^4} \left[AX(1+X)^{\frac{5}{2}} + \frac{(4+4X-X^2)(1+X)^2}{4} \right] \right\} (1+X)^{\frac{1}{2}} \tag{28}$$

The expression for mass can be written as

$$\frac{GM}{c^2} = \frac{r_b}{2} \left[1 + \frac{4Kc_1X^2}{(2+X)^3} - B(1+X)^{-\frac{1}{2}} \right] \tag{29}$$

The surface density is given by

$$\frac{8\pi G}{c^2} \rho_b r_b^2 = X \left\{ 4K \frac{(-9X^3 + 60X^2 + 116X + 48)}{(2+X)^5} - \frac{1}{(2+X)^4} [16A(6+11X)(1+X)^{\frac{3}{2}} + (96 + 120X - 60X^2 - 90X^3 - 5X^4)] \right\}$$

Centre red shift is given by

$$z_0 = B^{-1/2} - 1 \tag{30}$$

In view of Table 1 and we observe that pressure, density, pressure–density ratio, square of adiabatic sound speed and gravitational red shift decrease monotonically with the in-

crease of radial coordinate however, the charge distribution is increasing in nature.

6 Discussion

In view of and Tables 1 and 2, it has been observed that all the physical parameters (p , ρ , $\frac{p}{\rho c^2}$, $\frac{dp}{d\rho}$, z and E) are positive at the centre and within the limit of realistic equation of state and well behaved conditions for all values of K satisfying the inequalities $0.3277 \leq K \leq 0.49$. However, corresponding to any value of $K < 0.3277$, there exist no value of X for which causality principle is obeyed at the centre and for $K > 0.49$, the central pressure is negative for all values of X .

It has been observed that under well behaved conditions this class of solutions gives us the mass of super dense object within the range of white-dwarf.

We now present here a model of super dense star based on the particular solution discussed above corresponding to $K = 0.3277$ with $X = -0.15$, by assuming surface density; $\rho_b = 2 \times 10^{14}$ g/cm³. The resulting well behaved model has

the mass $M = 0.92M_\odot$ with radius $r_b \approx 17.15$ km and the surface red shift $Z_b \approx 0.087187$.

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References

- Cataldo, M., Mitskievic, N.V.: *Class. Quantum Grav.* **9**, 545 (1992)
- Gupta, Y.K., Maurya, S.K.: *Astrophys. Space Sci.* doi:[10.1007/s10509-010-0503-y](https://doi.org/10.1007/s10509-010-0503-y) (2010a)
- Gupta, Y.K., Maurya, S.K.: *Astrophys. Space Sci.* doi:[10.1007/s10509-010-0541-5](https://doi.org/10.1007/s10509-010-0541-5) (2010b)
- Ivanov, B.V.: *Phys. Rev. D* **65**, 104001 (2002)
- Maurya, S.K., Gupta, Y.K.: *Astrophys. Space Sci.* doi:[10.1007/s10509-011-0616-y](https://doi.org/10.1007/s10509-011-0616-y) (2011)
- Pant, N.: *Astrophys. Space Sci.* doi:[10.1007/s10509-010-0-525](https://doi.org/10.1007/s10509-010-0-525) (2010)
- Pant, N. et al.: *Astro. Space Sci.* doi:[10.1007/s10509-010-0509-5](https://doi.org/10.1007/s10509-010-0509-5) (2010)
- Pant, N. et al.: *Astro. Space Sci.* doi:[10.1007/s10509-011-0607-z](https://doi.org/10.1007/s10509-011-0607-z) (2011)