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ORIGINAL ARTICLE

Some FRW models of accelerating universe with dark energy

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Abstract The paper deals with a spatially homogeneous and isotropic FRW space-time filled with perfect fluid and dark energy components. The two sources are assumed to interact minimally, and therefore their energy momentum tensors are conserved separately. A special law of variation for the Hubble parameter proposed by Berman (Nuovo Cimento B 74:182, 1983) has been utilized to solve the field equations. The Berman's law yields two explicit forms of the scale factor governing the FRW space-time and constant values of deceleration parameter. The role of dark energy with variable equation of state parameter has been studied in detail in the evolution of FRW universe. It has been found that dark energy dominates the universe at the present epoch, which is consistent with the observations. The physical behavior of the universe has been discussed in detail.

Keywords FRW space-time · Hubble parameter · Deceleration parameter · Dark energy

1 Introduction

Recent observations of type Ia supernovae (SN Ia) (Perlmutter et al. 1997, 1998, 1999; Riess et al. 1998, 2004), galaxy redshift surveys (Fedeli et al. 2009), cosmic microwave background radiation (CMBR) data (Caldwell and Doran 2004; Huang et al. 2006) and large scale structure (Daniel et al. 2008) strongly suggest that the observable universe is undergoing an accelerated expansion. Observations also suggest that there had been a transition of the universe from the earlier deceleration phase to the recent acceleration phase (Caldwell et al. 2006). The cause of this sudden transition and the source of the accelerated expansion is still unknown. Measurements of CMBR anisotropies, most recently by the WMAP satellite, indicate that the universe is very close to flat. For a flat universe, its energy density must be equal to a certain critical density, which demands a huge contribution from some unknown energy stuff. Thus, the observational effects like the cosmic acceleration, sudden transition, flatness of universe and many more, need explanation. It is generally believed that some sort of 'dark energy' (DE) is pervading the whole universe. It is a hypothetical form of energy that permeates all of space and tends to increase the rate of expansion of the universe (Peebles and Ratra 2003). The most recent WMAP observations indicate that DE accounts for 72% of the total mass energy of the universe (Hinshaw et al. 2009). However, the nature of DE is still a mystery.

Many cosmologists believe that the simplest candidate for the DE is the cosmological constant (Λ) or vacuum energy since it fits the observational data well. During the cosmological evolution, the Λ-term has the constant energy density and pressure $p^{(de)} = -\rho^{(de)}$, where the superscript (de) stands for DE. However, one has the reason to dislike the cosmological constant since it always suffers from the theoretical problems such as the "fine-tuning" and "cosmic coincidence" puzzles (Copeland et al. 2006). That is why, the different forms of dynamically changing DE with an effective equation of state (EoS), $\omega^{(de)} = p^{(de)}/\rho^{(de)} < -1/3$, have been proposed in the literature. Other possible forms of DE include quintessence ($\omega^{(de)} > -1$) (Steinhardt et al. 1999), phantom ($\omega^{(de)} < -1$) (Caldwell 2002) etc. While the possibility $\omega^{(de)} \ll -1$ is ruled out by current cosmological data from SN Ia (Supernovae Legacy Survey, Gold sam-

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ple of Hubble Space Telescope) (Riess et al. 2004; Astier et al. 2006), CMBR (WMAP, BOOMERANG) (Eisentein et al. 2005; MacTavish et al. 2006) and large scale structure (Sloan Digital Sky Survey) (Komatsu et al. 2009) data, the dynamically evolving DE crossing the phantom divide line (PDL) ($\omega^{(de)} = -1$) is mildly favored. SN Ia data collaborated with CMBR anisotropy and galaxy clustering statistics suggest that $-1.33 < \omega^{(de)} < -0.79$ (see, Tegmark et al. 2004).

In 1983, Berman proposed a special law of variation of Hubble parameter in FRW space-time, which yields a constant value of deceleration parameter (DP). Such a law of variation for Hubble's parameter is not inconsistent with the observations and is also approximately valid for slowly time-varying DP models. The law provides explicit forms of scale factors governing the FRW universe and facilitates to describe accelerating as well as decelerating modes of evolution of the universe. Models with constant DP have been extensively studied in the literature in different contexts (see, Kumar and Singh 2007 and references therein). Most of the models with constant DP have been studied by considering perfect fluid or ordinary matter in the universe. But the ordinary matter is not enough to describe the dynamics of an accelerating universe as mentioned earlier. This motivates the researchers to consider the models of the universe filled with some exotic type of matter such as the DE along with the usual perfect fluid. Recently, some dark energy models with constant DP have been investigated by Akarsu and Kilinc (2010a, 2010b, 2010c), Yadav (2010) and Yadav and Yadav (2010).

In this paper, we have considered minimally interacting perfect fluid and DE energy components with constant DP within the framework of a FRW space-time in general relativity. The paper is organized as follows. In Sect. 2, the model and field equations have been presented. Section 3 deals with the exact solutions of the field equations and physical behavior of the model. Finally, concluding remarks have been given in Sect. 4.

2 Model and field equations

In standard spherical coordinates $(x^i) = (t, r, \theta, \phi)$, a spatially homogeneous and isotropic FRW line element has the form (in units $c = 1$)

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{1}$$

where $a(t)$ is the cosmic scale factor, which describes how the distances (scales) change in an expanding or contracting universe, and is related to the redshift of the 3-space; k is the curvature parameter, which describes the geometry of the spatial section of space-time with closed, flat and open universes corresponding to $k = -1, 0, 1$, respectively. The coordinates r, θ and ϕ in the metric (1) are 'comoving' coordinates. The FRW models have been remarkably successful in describing the observed nature of universe.

The Einstein's field equations in case of a mixture of perfect fluid and DE components, in the units $8\pi G = c = 1$, read as

$$R_{ij} - \frac{1}{2}g_{ij}R = -T_{ij}, \tag{2}$$

where $T_{ij} = T_{ij}^{(m)} + T_{ij}^{(de)}$ is the overall energy momentum tensor with $T_{ij}^{(m)}$ and $T_{ij}^{(de)}$ as the energy momentum tensors of ordinary matter and DE, respectively. These are given by

$$\begin{aligned} T_j^{(m)\,i} &= \mathrm{diag}\,[-\rho^{(m)},\ p^{(m)},\ p^{(m)},\ p^{(m)}] \\ &= \mathrm{diag}\,[-1,\ \omega^{(m)},\ \omega^{(m)},\ \omega^{(m)}]\rho^{(m)} \end{aligned} \tag{3}$$

and

$$\begin{aligned} T_j^{(de)\,i} &= \mathrm{diag}\,[-\rho^{(de)},\ p^{(de)},\ p^{(de)},\ p^{(de)}] \\ &= \mathrm{diag}\,[-1,\ \omega^{(de)},\ \omega^{(de)},\ \omega^{(de)}]\rho^{(de)} \end{aligned} \tag{4}$$

where $\rho^{(m)}$ and $p^{(m)}$ are, respectively the energy density and pressure of the perfect fluid component or ordinary baryonic matter while $\omega^{(m)} = p^{(m)}/\rho^{(m)}$ is its EoS parameter. Similarly, $\rho^{(de)}$ and $p^{(de)}$ are, respectively the energy density and pressure of the DE component while $\omega^{(de)} = p^{(de)}/\rho^{(de)}$ is the corresponding EoS parameter.

In a comoving coordinate system, the field equations (2), for the FRW space-time (1), in case of (3) and (4), read as

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\omega^{(m)}\rho^{(m)} - \omega^{(de)}\rho^{(de)}, \tag{5}$$

$$3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} = \rho^{(m)} + \rho^{(de)}. \tag{6}$$

Here the over dot denotes derivative with respect to t.

The energy conservation equation $T^{(de)\,ij}_{\ ;j} = 0$ yields

$$\dot{\rho}^{(m)} + 3(1+\omega^{(m)})\rho^{(m)}H + \dot{\rho}^{(de)} + 3(1+\omega^{(de)})\rho^{(de)}H = 0, \tag{7}$$

where $H = \dot{a}/a$ is the Hubble parameter.

3 Solution of field equations

The field equations (5) and (6) involve five unknown variables, viz., a, $\omega^{(m)}$, $\omega^{(de)}$, $\rho^{(m)}$ and $\rho^{(de)}$. Therefore, to find a deterministic solution of the equations, we need three suitable assumptions connecting the unknown variables.

Following Akarsu and Kilinc (2010a), first we assume that the perfect fluid and DE components interact minimally. Therefore, the energy momentum tensors of the two sources may be conserved separately.

The energy conservation equation $T^{(m)ij}_{\ \ \ ;j} = 0$, of the perfect fluid leads to

$$\dot{\rho}^{(m)} + 3(1+\omega^{(m)})\rho^{(m)} H = 0, \tag{8}$$

whereas the energy conservation equation $T^{(de)ij}_{\ \ \ ;j} = 0$, of the DE component yields

$$\dot{\rho}^{(de)} + 3(1+\omega^{(de)})\rho^{(de)} H = 0. \tag{9}$$

Next, we assume that the EoS parameter of the perfect fluid to be a constant, that is,

$$\omega^{(m)} = \frac{p^{(m)}}{\rho^{(m)}} = \text{const.}, \tag{10}$$

while $\omega^{(de)}$ has been allowed to be a function of time since the current cosmological data from SN Ia, CMBR and large scale structures mildly favor dynamically evolving DE crossing the PDL as discussed in Sect. 1.

Integration of (8) leads to

$$\rho^{(m)} = c_0 a^{-3(1+\omega^{(m)})}, \tag{11}$$

where c_0 is a positive constant of integration.

Finally, we constrain the system of equations with a law of variation for the Hubble parameter proposed by Berman (1983), which yields a constant value of DP. The law reads as

$$H = Da^{-n}, \tag{12}$$

where $D > 0$ and $n \geq 0$ are constants. In the following subsections, we discuss the DE cosmology for $n \neq 0$ and $n = 0$ by using the law (12).

3.1 DE cosmology for $n \neq 0$

In this case, integration of (12) leads to

$$a(t) = (nDt + c_1)^{\frac{1}{n}}, \tag{13}$$

where c_1 is a constant of integration. Therefore, the model (1) becomes

$$ds^2 = -dt^2 + (nDt + c_1)^{\frac{2}{n}} \times \left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{14}$$

The Hubble parameter (H), energy density ($\rho^{(m)}$) of perfect fluid, DE density ($\rho^{(de)}$) and EoS parameter ($\omega^{(de)}$) of DE, for the model (14) are found to be

$$H = D(nDt + c_1)^{-1}, \tag{15}$$

$$\rho^{(m)} = c_0(nDt + c_1)^{\frac{-3(1+\omega^{(m)})}{n}}, \tag{16}$$

$$\rho^{(de)} = 3D^2(nDt + c_1)^{-2} + 3k(nDt + c_1)^{\frac{-2}{n}} - c_0(nDt + c_1)^{\frac{-3(1+\omega^{(m)})}{n}}, \tag{17}$$

$$\omega^{(de)} = \frac{1}{\rho^{(de)}}\Big[(2n-3)D^2(nDt + c_1)^{-2} - k(nDt + c_1)^{\frac{-2}{n}} - c_0\omega^{(m)}(nDt + c_1)^{\frac{-3(1+\omega^{(m)})}{n}}\Big]. \tag{18}$$

The above solutions satisfy (9) identically, as expected.

The spatial volume (V) and expansion scalar (θ) of the model read as

$$V = a^3 = (nDt + c_1)^{\frac{3}{n}}, \tag{19}$$

$$\theta = 3H = 3(nDt + c_1)^{-1}. \tag{20}$$

The density parameter $\Omega^{(m)}$ of perfect fluid and the density parameter $\Omega^{(de)}$ of DE are given by

$$\Omega^{(m)} = \frac{\rho^{(m)}}{3H^2} = \frac{c_0}{3D^2}(nDt + c_1)^{\frac{2n-3(1+\omega^{(m)})}{n}}, \tag{21}$$

$$\Omega^{(de)} = \frac{\rho^{(de)}}{3H^2} = 1 + \frac{3k}{D^2}(nDt + c_1)^{\frac{-2(1-n)}{n}} - \frac{c_0}{3D^2}(nDt + c_1)^{\frac{2n-3(1+\omega^{(m)})}{n}}. \tag{22}$$

The value of DP (q) is found to be

$$q = -\frac{a\ddot{a}}{a^2} = n - 1, \tag{23}$$

which is a constant. The sign of q indicates whether the model inflates or not. A positive sign of q, i.e., $n > 1$ corresponds to the standard decelerating model whereas the negative sign of q, i.e., $0 < n < 1$ indicates inflation. The expansion of the universe at a constant rate corresponds to $n = 1$, i.e., $q = 0$. Also, recent observations of SN Ia (Perlmutter et al. 1997, 1998, 1999; Riess et al. 1998, 2004; Tonry et al. 2003; Knop et al. 2003; John 2004) reveal that the present universe is accelerating and value of DP lies somewhere in the range $-1 < q < 0$. It follows that in the derived model, one can choose the values of DP consistent with the observations.

We observe that at $t = -c_1/nD$, the spatial volume vanishes while all other parameters diverge. Therefore, the model has a big bang singularity at $t = -c_1/nD$, which can be shifted to $t = 0$ by choosing $c_1 = 0$. The cosmological evolution in FRW space-time is expansionary, since the

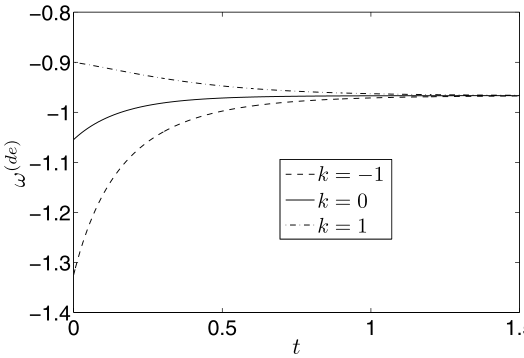

Fig. 1 $\omega^{(de)}$ versus t with $n = 0.05$, $D = 2$, $c_0 = c_1 = 1$, $\omega^{(m)} = 0$

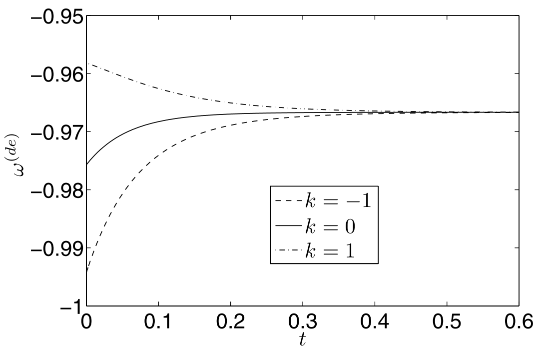

Fig. 2 $\omega^{(de)}$ versus t with $n = 0.05$, $D = 6$, $c_0 = c_1 = 1$, $\omega^{(m)} = 0$

scale factor monotonically increases with the cosmic time. So, the universe starts expanding with a big bang singularity in the derived model. The parameters H, $\rho^{(m)}$, $\rho^{(de)}$ and θ start off with extremely large values, and continue to decrease with the expansion of the universe. The spatial volume grows with the cosmic time.

The EoS parameter $\omega^{(de)}$ of DE asymptotically approaches to $\frac{2n}{3} - 1$ provided $n < 1$. Thus, the dynamics of $\omega^{(de)}$ depends on n for sufficiently large values of t. Figures 1 and 2 show the variation of $\omega^{(de)}$ during the cosmic evolution of closed ($k = -1$), flat ($k = 0$) and open ($k = 1$) universes for $D = 2$ and $D = 6$, respectively. From Fig. 1, we observe that for closed and flat universes, $\omega^{(de)}$ starts in phantom region ($\omega^{(de)} < -1$), crosses the PDL ($\omega^{(de)} = -1$) and finally varies in the quintessence region ($\omega^{(de)} > -1$). However, it evolves within the quintessence region only for open universe. For $D = 6$ (see Fig. 2), $\omega^{(de)}$ varies within the quintessence region throughout the evolution of the three universes. Thus, the nature of DE depends on the constants involved in the expression (18) of $\omega^{(de)}$. Also, observations predict that $-1.33 < \omega^{(de)} < -0.79$ (see, Tegmark et al. 2004). Therefore, Figs. 1 and 2 suggest that the derived model is consistent with the observations.

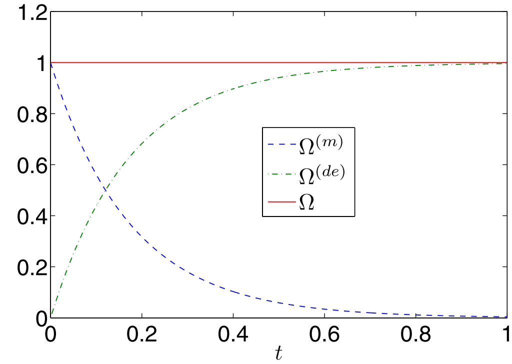

Fig. 3 Density parameters versus t with $n = 0.05$, $D = 2$, $c_0 = 12$, $c_1 = 1$, $\omega^{(m)} = 0$

Adding (21) and (22), we get the overall density parameter

$$\Omega = \Omega^{(m)} + \Omega^{(de)} = 1 + \frac{3k}{D^2}(nDt + c_1)^{\frac{-2(1-n)}{n}}. \tag{24}$$

Equation (24) gives $\Omega < 1$, $\Omega = 1$ and $\Omega > 1$ according as $k = -1$ (closed universe), $k = 0$ (flat universe) and $k = 1$ (open universe) respectively, as expected. We find that $\Omega \approx 1$ in the cases $k = -1$ and $k = 1$ for sufficiently large values of t provided $n < 1$. Thus, the model predicts a flat universe for large times. Since the present-day universe is very close to flat, so the derived model is consistent with the observations.

Figure 3 demonstrates the behavior of density parameters in the evolution of a flat universe ($k = 0$). We observe that initially the ordinary matter density dominates the universe. But later on, the DE dominates the evolution, which is probably the possible cause of acceleration of the present universe.

3.2 DE cosmology for $n = 0$

In this case, integration of (12) yields

$$a(t) = c_2 e^{Dt}, \tag{25}$$

where c_2 is a positive constant of integration. Therefore, the model (1) becomes

$$ds^2 = -dt^2 + c_2^2 e^{2Dt}\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{26}$$

The Hubble parameter, energy density of perfect fluid, DE density and EoS parameter of DE, for the model (26) are obtained as

$$H = D, \tag{27}$$

$$\rho^{(m)} = c_0 c_2^{-3(1+\omega^{(m)})} e^{-3D(1+\omega^{(m)})t}, \tag{28}$$

$$\rho^{(de)} = 3D^2 + 3kc_2^{-2}e^{-2Dt} - c_0 c_2^{-3(1+\omega^{(m)})} e^{-3D(1+\omega^{(m)})t}, \quad (29)$$

$$\omega^{(de)} = \frac{1}{\rho^{(de)}}\Big[-3D^2 - kc_2^{-2}e^{-2Dt} - c_0 c_2^{-3(1+\omega^{(m)})}\omega^{(m)} e^{-3D(1+\omega^{(m)})t}\Big]. \quad (30)$$

The above solutions satisfy (9) identically, as expected.

The spatial volume and expansion scalar of the model read as

$$V = c_2^3 e^{3Dt}, \quad (31)$$

$$\theta = 3D. \quad (32)$$

The density parameters of perfect fluid and DE are given by

$$\Omega^{(m)} = \frac{c_0 c_2^{-3(1+\omega^{(m)})}}{3D^2} e^{-3D(1+\omega^{(m)})t}, \quad (33)$$

$$\Omega^{(de)} = 1 + \frac{3kc_2^{-2}}{3D^2}e^{-2Dt} - \frac{c_0 c_2^{-3(1+\omega^{(m)})}}{3D^2} e^{-3D(1+\omega^{(m)})t}. \quad (34)$$

The DP is given by

$$q = -1. \quad (35)$$

Recent observations of SN Ia (Perlmutter et al. 1997, 1998, 1999; Riess et al. 1998, 2004; Tonry et al. 2003; Knop et al. 2003; John 2004) suggest that the universe is accelerating in its present state of evolution. It is believed that the way universe is accelerating presently; it will expand at the fastest possible rate in future and forever. For $n = 0$, we get $q = -1$; incidentally this value of DP leads to $dH/dt = 0$, which implies the greatest value of Hubble's parameter and the fastest rate of expansion of the universe. Therefore, the derived model can be utilized to describe the dynamics of the late time evolution of the actual universe. So, in what follows, we emphasize upon the late time behavior of the derived model.

Figure 4 depicts the behavior of the EoS parameter $\omega^{(de)}$ of DE in the closed, flat and open universes. We observe that for sufficiently large time, $\omega^{(de)} \approx -1$ in each case. Therefore, the so called cosmological constant is a suitable candidate to represent the behavior of DE in the derived model at late times.

Further, at late times, we have

$$\rho^{(m)} \approx 0$$

and

$$\rho^{(de)} \approx 3D^2.$$

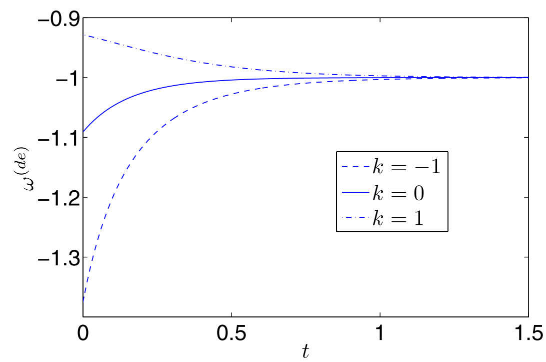

Fig. 4 $\omega^{(de)}$ versus t with $D = 2$, $c_0 = c_2 = 1$, $\omega^{(m)} = 0$

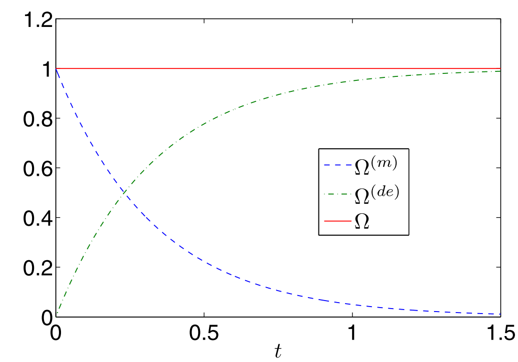

Fig. 5 Density parameters versus t with $D = 1$, $c_0 = 3$, $c_2 = 1$, $\omega^{(m)} = 0$

This shows that the ordinary matter density becomes negligible whereas the accelerated expansion of the universe continues with non-zero and constant DE density at late times, as predicted by the observations. Figure 5 illustrates the behavior of the density parameters during the evolution of the universe in the derived model. It is observed that the DE component dominates the universe at late times and $\Omega = \Omega^{(m)} + \Omega^{(de)} \approx 1$.

4 Concluding remarks

The special law of variation for Hubble's parameter proposed by Berman in FRW space-time yields constant value of DP given by $q = n - 1$, which provides accelerating models of the universe for $n < 1$ and decelerating ones for $n > 1$. In the present work, we have emphasized that decelerating models can be described by the usual perfect fluid, while dynamics of accelerating universe can be described by considering some exotic type of matter such as the DE. We have investigated the role of DE with variable EoS parameter in the evolution of universe within the framework of a spatially homogeneous FRW space-time by taking into account the special law of variation of Hubble parameter. DE cosmologies have been discussed in two different cases, viz., power-law ($n \neq 0$) and exponential-law ($n = 0$). The analysis of the models reveals that the present-day universe is dominated by DE, which can successfully describe the accelerating nature of the universe consistent with the observations.

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